Quantization of the LTB Cosmological Equation

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Abstract

The Hamiltonian H of the Kepler-like equation of the LTB cosmological model is considered. The Schrödinger operator $\hat{H}$ obtained by canonical quantization of H , is assumed to govern the Schrödinger equation for the wave function ψ of the LTB universe. The scheme, that still depends on the arbitrary integration functions $E(r)$, $m(r)$ of the cosmological model, is integrated exactly. The eigenvalues of $\hat{H}$ are interpreted to give the possible values of E . For $E < 0$ one finds $m \propto n\sqrt{-E}$, $n = 1, 2, 3, \dots$ with E arbitrary. Consistently the solution ψ can be interpreted as superposition of pure states of universe. If $E > 0$ than E and m remain independent functions. The quantization scheme is extended to the cosmological LTB model with cosmological constant term where similar results are shown to hold.

Keywords: LTB cosmology; Cosmological Kepler-like equation; Quantization; Solution; Universe wave function

1 Introduction

It is well known that the Lemaitre-Tolman-Bondi (LTB) cosmological model [3, 13, 18] describes an isotropic inhomogeneous universe filled with dust matter (without pressure) in comoving coordinate system of line element:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - e^{\gamma(r,t)}dr^2 - Y^2(r,t)(d\theta^2 + \sin^2\theta d\varphi^2) \quad (1)$$

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An interesting aspect of the model is that the Einstein field equation can be equivalently described, after a partial integration, by the equation (e.g., [12, 23]):

$$\frac{\dot{Y}^2}{2} - G \frac{M(r)}{Y} = E(r) \quad (2)$$

($\dot{Y} = \partial Y / \partial t$) where E and m are arbitrary time independent integration functions derived from the (t, r) Einstein equation and from the conservation of the energy momentum tensor respectively, such that

$$\exp \gamma = \frac{Y'^2(r, t)}{1 + 2E(r)}, \quad M(r) = 4\pi \int_0^r dr Y' Y^2 \eta \quad (3)$$

($Y' = \partial Y / \partial r$) and $\eta(r, t)$ is the proper energy density. The equation (2) can be integrated both directly and in parametric form (e. g., [5, 12]). In case a cosmological term Λ is considered, eq. (2) is modified by the addition of a left hand term $\Lambda Y^2/3$ and the equation can be as well integrated exactly [23, 26, 27].

In recent years a great effort has been done to properly formulate a Quantum General Relativity or a Quantum Gravity. Such quantization would be of interest not only on theoretical ground but also for application to the physics of the early Universe (e., g. [11]). Important directions of that study are the well developed semi-classical approach (see e., g., [2, 7]); the canonical quantization procedure that leads to the Wheeler-DeWitt equation (e. g., [6, 22, 17]); the loop quantum gravity and string theory (e.g. [8, 16]). The mentioned studies have been developed both in general and specific space-time model. In particular the LTB model has been considered an useful framework where to apply the mentioned quantization procedures. For the application of the semi classical quantization see e.g. [24, 25]. For the formulation of the Wheeler-DeWitt equation and consequences such as dust collapse, quantum correction, see e. g., [10, 20, 21]

In the present paper we propose a quantization of the LTB scheme that is different and that it has not the generality of the previously recalled ones. The quantization is performed in an elementary way. Here we consider as central the equation (2) and the associated Hamiltonian H . The canonical Schrödinger quantization procedure is then applied to H thus obtaining the Schrödinger operator $\hat{H}$. The wave function ψ satisfying the corresponding Schrödinger equation is interpreted as the "wave function of the Universe". The possible values of E are interpreted to be given by the eigenvalues of $\hat{H}$. The problem can be solved exactly by standard result on Schrödinger equation of a particle on the half line in Coulomb potential. In case $E < 0$ there results a discrete set of values of E such that one can write $m(r) \propto n\sqrt{-E(r)}$, $n = 1, 2, 3, ..$ and E arbitrarily given. In this case the wave function ψ can be expressed,

in general, as superposition of universe pure states. For $E > 0$ both E and m remain independent arbitrary functions and an interpretation of ψ can be given like the case $E < 0$. Finally the "quantization" procedure is extended to the case of LTB model with cosmological constant Λ . Results similar to the case $\Lambda = 0$ are obtained by applying recent results on the solution of 1D Schrödinger equation for particle in Coulomb plus harmonic potential.

2 Quantization scheme

After multiplying by μ , the equation (2) reads

$$\frac{1}{2}\mu\dot{Y}^2 - G\frac{\mu M(r)}{Y} = \mu E(r) \quad (4)$$

that can be interpreted as the energy equation of a one-dimensional particle of mass μ , coordinate Y , energy $\mu E(r)$ and potential energy $-G\mu m(r)/Y$ where r is considered as a parameter. $E(r)$ is energy per unit mass (here it has been set $c = 1$). The Lagrangian and the Hamiltonian of the particle system are then

$$L = \frac{1}{2}\mu\dot{Y}^2 + G\frac{\mu m(r)}{Y} \quad (5)$$

$$H = \frac{p^2}{2\mu} - G\frac{\mu m(r)}{Y}, \quad p = \frac{\partial L}{\partial \dot{Y}} \quad (6)$$

We now consider the Schrödinger operator $\hat{H}$ obtained from H by canonical quantization $p \rightarrow -i\hbar\partial/\partial Y$, $Y \rightarrow \cdot Y$ and Y as independent variable. The associated Schrödinger equation for the wave function ψ is then

$$\hat{H}\psi \equiv -\frac{\hbar^2}{2\mu}\frac{\partial^2\psi}{\partial Y^2} - G\frac{\mu m(r)}{Y}\psi = i\hbar\frac{\partial\psi}{\partial\tau} \quad (7)$$

We interpret $\psi(Y, \tau)$ as the wave function of the Universe and the eigenvalues of $\hat{H}$ as the possible values of μE . By setting $\psi(Y, \tau) = T(\tau)u(Y)$, eq. (7) can be separated as usual to obtain

$$T = \exp\left(-\frac{i}{\hbar}W\tau\right) \quad (8)$$

$$\frac{d^2u}{dY^2} + \frac{2\mu}{\hbar^2}\left(W + G\frac{\mu m(r)}{Y}\right)u = 0 \quad (9)$$

with W the separation constant and $u \in L^2((0, +\infty), dY)$. Formally, (9) is the radial equation of the 1D Hydrogen atom problem. It coincides with the equation for $u \in L^2((0, +\infty), dr)$ where $u(r) = rR(r)$, $R(r)$ being the radial

solution of the 3D Schrödinger equation for the Hydrogen atom in a state of zero centrifugal potential ($l = 0$). Therefore, for $E < 0$, one has (e.g. [14])

$$\mu E(r) \equiv W_n(r) = -\frac{(G\mu m(r))^2}{2\hbar^2} \frac{\mu}{n^2}, \quad n = 1, 2, 3, \dots \quad (10)$$

$$u \equiv u_n = c_n L_n^1(2k_n Y) \exp(-k_n Y), \quad n = 1, 2, 3, \dots \quad (11)$$

where c_n is a normalization coefficient [4] $k_n = \sqrt{2\mu|W_n|}/\hbar = G\mu^2/(n\hbar)$ and the L_n^1 's the Laguerre polynomials. The result (10) can be written in the form

$$m(r) = n \cdot \frac{\hbar}{G\mu} \sqrt{-2E(r)}, \quad n = 1, 2, 3, \dots \quad (12)$$

that represents, once E is given, a sort of "cosmological quantization" of the mass function $m(r)$.

By combining (10) and the condition $1 + 2E > 0$ (that follows from (3)) one has the constraint

$$m(r) \leq \frac{\hbar}{G\mu^{\frac{3}{2}}} \cdot n \quad (13)$$

that gives upper bounds to $m(r)$, independent of r , and that could always be satisfied, for given bounded m , with sufficient large n . Note that (13) implies $m(r) < \infty$ for every r . As it appears from the very definition, m is the mass contained in a sphere of radius Y . It is useful to compare with the mass $M(r)$ contained in a sphere of radius r whose definition and relation to m are

$$M(r) = 4\pi \int_0^r dr \eta(r, t) |g|^{\frac{1}{2}} = 4\pi \int_0^r dr \frac{Y^2 Y' \eta(r, t)}{\sqrt{1 + 2E}} \quad (14)$$

$$M(r) \geq m(r) \quad \text{if} \quad E(r) < 0 \quad (15)$$

$$M(r) \leq m(r) \quad \text{if} \quad E(r) > 0 \quad (16)$$

g the determinant of the metric tensor. If $M(r)$ is a non decreasing function of r , $M(\infty)$ can be interpreted as the total mass contained in the Universe. Therefore if the above scheme applies to our Universe for $E < 0$, $M(\infty)$ can be identified, "missing mass" included, to the value of about $6 \cdot 10^{52}$ Kg [19]. From (15) then also $m(r)$ is finite and we have consistency of the quantization scheme with at least one experimental result.

The eigenvalue problem (9) admits of solution also for arbitrary $W = \mu E > 0$. The corresponding eigenstates $u_W(Y, E, m)$ are improper solutions whose expression can be again found in text books of quantum mechanics [4, 14]. In this case by the identification $W \equiv \mu E$, both m and E remain independent arbitrary functions because also the condition $1 + 2E > 0$ is automatically satisfied. For what concerns the interpretation of the wave function $\psi(Y, \tau)$ consider for $E < 0$, a solution of (7) of the form

$$\psi(Y, \tau) = \sum_n c_n e^{-\frac{i}{\hbar} W_n \tau} u_n(Y) \quad (17)$$

ψ is interpreted in this case as the wave function of the universe that is superposition of universe pure states u_n , $|c_n|^2$ being the probability of finding the universe in the pure state u_n with energy W_n given by (10). Going back to the cosmological model the state u_n corresponds to the solution of the classical cosmological equation (2) with fixed $E < 0$ and $m \equiv m_{n,E}(r)$ given in (12). Such cosmological solutions of (2) can be explicitly given, as it well known [12, 5, 23]. Analogously if $E > 0$, ψ can be given as a continuous superposition in W of the universe states $u_W(Y, E, m)$'s. Also in this case an interpretation similar to the one just mentioned relative to (17) can be given according to the canonical interpretation of improper states in quantum mechanics.

3 Developments and comments

In the previous Section a quantization of the LTB model has been proposed that is based on the quantization procedure of the Kepler-like cosmological equation. The procedure has not the generality of the canonical quantizations of General Relativity mentioned in the introduction and it is specific of the LTB model. It has however some positive aspects.

Firstly it results automatically covariant being based on equation (2).

Secondly the procedure can be extended to the quantization of the LTB model with cosmological term Λ . Indeed if Λ is taken into account one obtains, from the Einstein field equation, instead of (2), the equation (e.g., [23]).

$$\frac{\dot{Y}^2}{2} - G \frac{m(r)}{Y} + \frac{\Lambda}{3} Y^2 = E(r) \quad (18)$$

with E, m , that are again as in (3), arbitrary independent functions r . By applying the previous Schrödinger quantization, one now obtains the equations

$$\hat{H}_\Lambda \psi \equiv -\frac{\hbar^2}{2\mu} \frac{\partial^2 \psi}{\partial Y^2} + V(Y) \psi = i\hbar \frac{\partial \psi}{\partial \tau} \quad (19)$$

$$\frac{d^2 u}{dY^2} + \frac{2\mu}{\hbar^2} (W - V(Y)) u = 0 \quad (20)$$

$$V(Y) = -G \frac{\mu m(r)}{Y} + \frac{\Lambda}{3} Y^2 \quad (21)$$

where $\psi(Y, \tau) = \exp(-iW\tau/\hbar)u(Y)$ that is interpreted again as the wave function of the Universe. The equation (20) is formally analogous to the radial Schrödinger equation with Coulomb plus harmonic potential in case of zero orbital angular momentum. Therefore by assuming an infinite positive barrier in $Y = 0$, if $Gm(r)\mu > 0$, $\Lambda > 0$, by applying qualitative considerations concerning the one dimensional Schrödinger operator (e. g., [15]) one obtains that $\hat{H}_\Lambda$ has a discrete spectrum. The corresponding eigenvalues can be obtained as

in [1]. They are of the form $W_n \propto (n + 3/2)c$, $n = 1, 2, \dots$ with eigenfunctions $u_n \cong \left(\sum_o^n a_k Y^{k+1} \right) \exp(-cY^2/2)$. The a_k 's and c result determined by recurrence relation. Moreover c expresses a relation between Λ and $m(r)$ [1]. If one identifies μE with W one has the result $E(r) \propto (n + 3/2)c$ that expresses also a relation between E and m . This relation selects possible solutions of (18) that in turns can be explicitly given (see e. g., [23, 27]).

Note that if $Gm\mu > 0$ but $\Lambda < 0$, one has $V(Y) \rightarrow -\infty$ for both $Y \rightarrow 0^+$ and $Y \rightarrow +\infty$ so that $V(Y)$ has a (negative) maximum in $\mathbf{R}^+$ and $\hat{H}_\Lambda$ is not bounded from below [15]. There is therefore, according to general result in Schrödinger one dimensional operator theory the possibility of a tunneling effect for wave packets solutions of (19) when $\Lambda < 0$. Finally it is clear that one can give again an interpretation of the wave function of the Universe, similar to that of the previous Section, for both $\Lambda > 0$ and $\Lambda < 0$.

A further step in the formulation of the quantization procedure previously proposed, is of considering also fields interacting with the gravitational one.

An elementary procedure could be of quantizing the interacting field in the background of the (quantized) LTB model. This resembles a semi classical attitude for which however the time evolution of the wave function of the Universe is not affected by the interacting field.

A more consistent point of view is of coupling the spin field to gravity. This can be done as usual by adding the energy momentum tensor of the field to the source term of the Einstein field equation. One can then proceed in decoupling the equations by first expressing the energy momentum tensor of the field in terms of the solution of its own equation. Then by a first integration of the Einstein field equation one should be left with a generalization of (18). The procedure should be completed by quantizing the resulting cosmological equation as in the previous Sections. In this way the wave function of the Universe results affected both by gravitation and by interacting field. Even if conceptually consistent, the procedure may be very difficult and cumbersome. The reason is the involved dependence of the coefficients of the field equation on the metric coefficients and of the structure of the energy momentum tensor that becomes rapidly very complex [9] by increasing the spin. A tentative in this direction is currently under investigation for fields with the lowest value of the spin.

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Received: June 7, 2013