A Study on a New Fractional Integral Inequality in Quantum Calculus

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Abstract

In this paper, we present a new fractional integral inequality in quantum calculus.

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1 Introduction and Preliminaries

In [2, 4, 5], for \(a, b \in \mathbb{C}\) and \(q \in (0, 1)\), we denote

\[(a; q)_\infty = \prod_{k=0}^\infty (1 - aq^k) \quad \text{and} \quad (a - qb)^{(\alpha)} = a^\alpha \frac{(\frac{aq}{a}; q)_\infty}{(q^{\alpha+1}\frac{b}{a}; q)_\infty}.

The q-Jackson integral [3] from 0 to \(a\) is defined by

\[
\int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^\infty f(aq^n)q^n.
\]

The fractional q-integral [5] of the Riemann-Liouville type is defined by
\[ J_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qx)^{(\alpha - 1)} f(x) dq x, \]

where \( \alpha > 0, \)

\[ \Gamma_q(\alpha) = \frac{1}{1 - q} \int_0^1 \left( \frac{u}{1 - q} \right)^{\alpha - 1} e_q(du) \quad \text{and} \quad e_q(x) = \prod_{k=0}^{\infty} (1 - qx). \]

In [1], Brahim and Taf presented a fractional \( q \)-integral inequality in quantum calculus as follows.

**Theorem 1.1.** [1] Let \( v \) be a positive function on \([0, \infty)\), and let \( f, g \) be functions on \([0, \infty)\) such that

\[ (f(x) - f(y))(g(x) - g(y)) \geq 0 \]

for all \( x, y \geq 0 \). Then

\[ J_{q_1}^\alpha v(t) J_{q_2}^\beta (fg)(t) + J_{q_2}^\alpha v(t) J_{q_1}^\beta (vf)(t) \]
\[ \geq J_{q_1}^\alpha f(t) J_{q_2}^\beta (vg)(t) + J_{q_2}^\alpha v(t) J_{q_1}^\beta (vf)(t) \]

where \( \alpha, \beta, t > 0 \) and \( q_1, q_2 \in (0, 1) \).

In this paper, we present a new fractional integral inequality in quantum calculus.

## 2 Results

**Theorem 2.1.** Let \( v \) be a positive function on \([0, \infty)\), and let \( f, g, h \) be functions on \([0, \infty)\) such that

\[ (f(x) - f(y))(g(x) - g(y))(h(x) + h(y)) \geq 0 \]

for all \( x, y \geq 0 \). Then

\[ J_{q_1}^\alpha (v fgh)(t) J_{q_2}^\beta v(t) + J_{q_1}^\alpha (v fg)(t) J_{q_2}^\beta (vh)(t) \]
\[ + J_{q_1}^\alpha (vh)(t) J_{q_2}^\beta (vfg)(t) + J_{q_1}^\alpha v(t) J_{q_2}^\beta (vfh)(t) \]
\[ \geq J_{q_1}^\alpha (vfh)(t) J_{q_2}^\beta (vg)(t) + J_{q_1}^\alpha (vgh)(t) J_{q_2}^\beta (vf)(t) \]
\[ + J_{q_1}^\alpha (vg)(t) J_{q_2}^\beta (vfh)(t) + J_{q_1}^\alpha (vf)(t) J_{q_2}^\beta (vgh)(t), \]

where \( \alpha, \beta, t > 0 \) and \( q_1, q_2 \in (0, 1) \).
Proof. By the assumption, for any $x, y$, we have

$$f(x)g(x)h(x) + f(x)g(x)h(y) + f(y)g(y)h(x) + f(y)g(y)h(y)$$

$$\geq f(x)g(y)h(x) + f(y)g(x)h(x) + f(y)g(x)h(y) + f(x)g(y)h(y).$$

Then

$$\int_0^t (t - q_1 x)^{(\alpha - 1)} (v f g h)(x)d_{q_1} x + \int_0^t (t - q_1 x)^{(\alpha - 1)} (v f g)(x)h(y)d_{q_1} x$$

$$+ \int_0^t (t - q_1 x)^{(\alpha - 1)} (v h)(x)(f g)(y)d_{q_1} x + \int_0^t (t - q_1 x)^{(\alpha - 1)} v(x)(f g h)(y)d_{q_1} x$$

$$\geq \int_0^t (t - q_1 x)^{(\alpha - 1)} (v f h)(x)g(y)d_{q_1} x + \int_0^t (t - q_1 x)^{(\alpha - 1)} (v g h)(x)f(y)d_{q_1} x$$

$$+ \int_0^t (t - q_1 x)^{(\alpha - 1)} (v g)(x)(f h)(y)d_{q_1} x + \int_0^t (t - q_1 x)^{(\alpha - 1)} (v f)(x)(g h)(y)d_{q_1} x$$

so

$$J_{q_1}^\alpha (v f g h)(t) + h(y)J_{q_1}^\alpha (v f g)(t)$$

$$+ (f g)(y)J_{q_1}^\alpha (v h)(t) + (f g h)(y)J_{q_1}^\alpha v(t)$$

$$\geq g(y)J_{q_1}^\alpha (v f h)(t) + f(y)J_{q_1}^\alpha (v g h)(t)$$

$$+ (f h)(y)J_{q_1}^\alpha (v g)(t) + (g h)(y)J_{q_1}^\alpha (v f)(t),$$

where $\alpha, t > 0$, $y \in (0, t)$ and $q_1 \in (0, 1)$.

Then

$$\int_0^t (t - q_1 x)^{(\alpha - 1)} v(y)J_{q_1}^\alpha (v f g h)(t)d_{q_1} y$$

$$+ \int_0^t (t - q_1 x)^{(\alpha - 1)} (v h)(y)J_{q_1}^\alpha (v f g)(t)d_{q_1} y$$

$$+ \int_0^t (t - q_1 x)^{(\alpha - 1)} (v f g)(y)J_{q_1}^\alpha (v h)(t)d_{q_1} y$$

$$+ \int_0^t (t - q_1 x)^{(\alpha - 1)} (v f g h)(y)d_{q_1} x$$

$$\geq \int_0^t (t - q_1 x)^{(\alpha - 1)} (v g)(y)J_{q_1}^\alpha (v f h)(t)d_{q_1} y$$

$$+ \int_0^t (t - q_1 x)^{(\alpha - 1)} (v f)(y)J_{q_1}^\alpha (v g h)(t)d_{q_1} y$$

$$+ \int_0^t (t - q_1 x)^{(\alpha - 1)} (v f h)(y)J_{q_1}^\alpha (v g)(t)d_{q_1} y$$

$$+ \int_0^t (t - q_1 x)^{(\alpha - 1)} (v g h)(y)J_{q_1}^\alpha (v f)(t)d_{q_1} y,$$
so

$$J_{q_1}^\alpha(v f g h)(t)J_{q_2}^\beta v(t) + J_{q_1}^\alpha(v f g)(t)J_{q_2}^\beta(v h)(t) + J_{q_1}^\alpha(v h)(t)J_{q_2}^\beta(v f g)(t) + J_{q_1}^\alpha(v)(t)J_{q_2}^\beta(v f g h)(t)$$

$$\geq J_{q_1}^\alpha(v f h)(t)J_{q_2}^\beta(v g)(t) + J_{q_1}^\alpha(v g h)(t)J_{q_2}^\beta(v f)(t) + J_{q_1}^\alpha(v g)(t)J_{q_2}^\beta(v f h)(t) + J_{q_1}^\alpha(v f)(t)J_{q_2}^\beta(v g h)(t),$$

where $\alpha, \beta, t > 0$ and $q_1, q_2 \in (0, 1)$.

\[ \square \]

References


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