Regularized Inverse Problem of Magnetometric Resistivity Response over a Layered Earth

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Abstract

A regularization technique can be applied to inverse problems arising in geoelectrical resistivity sounding. We propose here to use a regularized inversion scheme for interpretation of magnetometric resistivity data gathered from a horizontally stratified layered earth. An iterative scheme using the regularized conjugate gradient method is applied to estimate the conductivity parameters of the ground. The L-curve criterion is used to determine a suitable value of the regularization parameter. The final inverted model obtained is qualitatively in good agreement with the real model from synthetic data. A comparison of inversion results obtained from our scheme, conventional conjugate gradient and Levenberg-Marquardt methods on a test data set clearly demonstrates an edge over the other two stated schemes as far as the robustness is concerned. The scheme described here has been successfully used for geophysical inversion of magnetometric resistivity data.

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1 Introduction

The electrical resistivity method was first applied by Conrad Schlumberger in 1912. The purpose of resistivity sounding is to investigate the change of formation resistivity with depth. The inverse problem in resistivity interpretation was reported as early as the 1930s. Many methods of inverse theory for geophysical problems were conducted, such as in Backus and Gilbert [1, 2], Marquardt [7]. One of the central problems in inversion theory was a solution of minimization problem for different functionals. This problem can be solved
directly in the linear case of a forward operator. However, in the general case of a nonlinear operator, the solution can only be found iteratively. There are many different approaches to the construction of iterative processes for functional minimization. One of the most widely used techniques for optimization is based on gradient-type methods. The formal solution of ill-posed inverse problem could result in unstable, unrealistic models. Regularization theory provides guidance for overcoming this difficulty (Zhdanov [10]).

In this study, we adapt and apply a regularization technique to inverse problems arising in geoelectrical resistivity sounding based on the measurement of static magnetic fields. The proposed method is the regularized conjugate gradient (RCG) method which is used to interpret magnetometric resistivity data gathered from a horizontally stratified layered earth. The L-curve criterion is applied to determine a suitable value of the regularization parameter. A comparison of this inversion scheme with conventional conjugate gradient (CG) and Levenberg-Marquardt (LM) methods on a test model is also presented.

2 Forward Problem

The forward problem expressed in terms of integral expression for magnetic field due to a semi-infinite source in a horizontally stratified layered earth with all layers possessing constant conductivities was described and discussed by Chen and Oldenburg [4]. The azimuthal component of the magnetic field, denoted by $H_k$, in source region layer $k$ is given as

$$H_k(r, z) = \int_0^\infty \lambda \left( \frac{I}{2\pi \lambda} + A_k e^{-\lambda(z-h_{k-1})} + B_k e^{\lambda(z-h_{k-1})} \right) J_1(\lambda r) d\lambda,$$

(1)

and for a source-free region,

$$H_k(r, z) = \int_0^\infty \lambda \left( A_k e^{-\lambda(z-h_{k-1})} + B_k e^{\lambda(z-h_{k-1})} \right) J_1(\lambda r) d\lambda,$$

(2)

where $A_k$ and $B_k$ are undetermined coefficients that can be efficiently calculated through a recursive scheme (see Sripanya and Yooyuanyong [9]), and $J_1$ is the Bessel function of the first kind of order one. Both of the above equations can be integrated by a quadrature and continued fraction expansion technique (e.g., Chave [3]).

3 Regularized Inversion Scheme

The main objective in our inversion method is to obtain a geologically interpretable model that can adequately reproduce observations. This is accomplished by posing the inverse problem as an optimization problem in which an objective function of the earth model is minimized.
3.1 Regularized Conjugate Gradient Method

Let us consider a geophysical inverse problem described by the following operator equation
\[ d = A m, \]
where \( m \) is a model vector, \( d \) is an observed data vector and \( A \) is a nonlinear operator. In the framework of general regularization theory, a solution of inverse problem can be reduced to minimization of the Tikhonov parametric functional, namely
\[ \min \mathcal{F}_\alpha(m), \]
where
\[ \mathcal{F}_\alpha(m) = \| A m - d \|^2 + \alpha \| W m \|^2, \]
\( \alpha \) is a regularization parameter, \( W \) is a stabilization operator and \( \| \cdot \| \) denotes the Euclidean norm. The regularized least squares problem (4) can be solved by applying a nonlinear conjugate gradient method. This technique generates a sequence \( m_k \) for \( k \geq 1 \), starting from an initial guess \( m_0 \) and using the recurrence
\[ m_{k+1} = m_k + \lambda_k h_k, \]
where \( \lambda_k \) is a positive step size obtained by a line search, and \( h_k \) is a search direction generated by the rule
\[ h_0 = -g_0, \quad h_{k+1} = -g_{k+1} + \gamma_k h_k, \]
where \( g_k = \nabla \mathcal{F}_\alpha(m_k) \) and \( \gamma_k \) is an update parameter which can be determined by using the Fletcher-Reeves formula
\[ \gamma_k = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k}, \]
or the Polak-Ribiére formula
\[ \gamma_k = \frac{(g_{k+1} - g_k)^T g_{k+1}}{g_k^T g_k}. \]

3.2 Stabilization Operator

Stabilization operator \( W \) can be shaped to meet a smooth solution requirement, and its preferable implementations are the derivative of the first or second order of a solution with respect to space variables, in the case of a layered earth, \( z \) variable (Constable et al. [5]). For the first-order derivative, \( W \) is called the flattest operator, and the corresponding discrete form is
\[ W = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}_{(M-1) \times M}. \]
For the second-order derivative, \( W \) can be written as
\[
W = \begin{bmatrix}
1 & -2 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & -2 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & -2 & 1 \\
\end{bmatrix}_{(M-2) \times M},
\]
which will yield a smoothest model in inversions.

### 3.3 Choice of a Regularization Parameter

There are several heuristic ways to proceed in order to select a regularization parameter, but the criterion based on the L-curve construction (Hansen and O’Leary [6]) is certainly the most used. The method offers a convenient way to display regularized information as a function of the regularization parameter. The L-curve is a parametric plot of the norm of a regularized solution versus the norm of the corresponding residual. The idea of the L-curve criterion is to choose a regularization parameter related to the characteristic L-shaped corner of the L-curve which corresponds to a good balance between minimization of these two quantities. The L-curve’s corner is defined as a point on the curve
\[
(R(\alpha), S(\alpha)) = (\log \|Am_\alpha - d\|, \log \|m_\alpha\|),
\]
which has a maximum curvature. The curvature \( \kappa \) is usually given by the formula
\[
\kappa(\alpha) = \frac{R'S'' - R''S'}{(R')^2 + (S')^2}^{3/2},
\]
where the differentiation is with respect to \( \alpha \).

### 4 Numerical Experiments

In our inverse model example, we simulate the reflection of magnetic radiation data from a forward model of practical interest. The model of a simple case for the ground structure is used to investigate the electrical conductivity profile. The algorithm for regularized inversion is applied to find the model parameters of conductivity variation.

#### 4.1 RCG Solution

As a sample test, we consider the synthesis model of a 2-layered earth. The overburden for our model has a constant conductivity \( \sigma_1 \) with thickness \( h \) over the host medium having constant conductivity \( \sigma_2 \) with infinite depth. The
values of the model parameters are given in Table 1. The magnetometric resistivity data are generated by the forward problem of the example model for our sample test. The buried depth of the current source for our model is 5 metres. The electric current of 1 ampere is used in our computations. Random errors up to 3% are superimposed on the scaled magnetic fields to simulate the set of real data. The parameter $\sigma_1$ is a conductivity of the earth’s surface, which can be assumed to be known from the measurement. The iterative procedure using the regularized conjugate gradient method is applied to estimate the unknown parameters $h$ and $\sigma_2$. We start the iterative process to find the values of the model parameters with initial guess values $h = 5$ m and $\sigma_2 = 1$ S/m. The suite of RCG solutions is obtained for a range of 15 values of $\alpha$, ranging between approximately $3.9 \times 10^{-14}$ and $2.4 \times 10^{-16}$, after using 5 iterations for each $\alpha$. The L-curve of the model norm versus the data misfit is plotted as shown in Figure 1. When the regularization parameter $\alpha$ decreases, the curve moves from the lower right to the upper left. Figure 2 shows the curvature $\kappa$ of the L-curve as a function of the regularization parameter $\alpha$. The optimal value for $\alpha$ is located at the point with maximum curvature of the L-curve. The solution best satisfying the L-curve criterion corresponds to $\alpha = 2.1248908 \times 10^{-15}$ in which the misfit norm is roughly $1.5502539 \times 10^{-6}$. Figure 3 shows the RCG solution for our sample test in comparison with the true model.

### 4.2 Comparison of Inversion Schemes

The present scheme for regularized inversion of magnetometric resistivity data is compared with conventional conjugate gradient and Levenberg-Marquardt methods (Press et al. [8]) in terms of the final model obtained from the same starting model, and the number of iterations taken to attain the final model parameters. The usefulness of the scheme proposed here is demonstrated by Table 2 which compares the results obtained from three inversion schemes tested on the example model in Section 4.1.
Figure 1: L-curve of solution norm versus corresponding residual norm. Characteristic L-shaped corner as a point on curve with maximum curvature.

Figure 2: Curvature $\kappa$ of L-curve as a function of regularization parameter $\alpha$.

Figure 3: RCG solution shown in comparison with true model.
### Table 2: Comparison table of results obtained from three inversion schemes.

<table>
<thead>
<tr>
<th>Inversion Schemes</th>
<th>RCG</th>
<th>CG</th>
<th>LM</th>
</tr>
</thead>
<tbody>
<tr>
<td>no. of iterations</td>
<td>5</td>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>$h$</td>
<td>7.8876023</td>
<td>7.8420849</td>
<td>7.8763562</td>
</tr>
<tr>
<td>% error in $h$</td>
<td>1.4049709</td>
<td>1.9739394</td>
<td>1.5455477</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>$2.4352617 \times 10^{-1}$</td>
<td>$2.3798302 \times 10^{-1}$</td>
<td>$2.2895288 \times 10^{-1}$</td>
</tr>
<tr>
<td>% error in $\sigma^2$</td>
<td>2.5895301</td>
<td>4.8067918</td>
<td>8.4188462</td>
</tr>
<tr>
<td>residual norm</td>
<td>$1.5502539 \times 10^{-6}$</td>
<td>$1.7030973 \times 10^{-6}$</td>
<td>$4.8460740 \times 10^{-6}$</td>
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<tr>
<td>solution norm</td>
<td>7.8913608</td>
<td>7.8456950</td>
<td>7.8796831</td>
</tr>
<tr>
<td>error norm</td>
<td>$1.1258395 \times 10^{-1}$</td>
<td>$1.5837172 \times 10^{-1}$</td>
<td>$1.2542239 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

### 5 Discussions and Conclusions

An inversion scheme for nonlinear ill-posed problems arising in geoelectrical resistivity sounding is presented in this study. The proposed method is the regularized conjugate gradient method which is used to interpret magnetometric resistivity data gathered from a horizontally stratified layered earth. The example model of a simple case for the ground structure is used to investigate the electrical conductivity profile. An iterative scheme using the regularized conjugate gradient method is applied to estimate the model parameters of conductivity variation. The L-curve criterion is used to select an optimal value of the regularization parameter. The scheme has taken only 5 iterations to attain the true model parameters. The graphs of the true and estimated conductivity models are plotted as shown in Figure 3. We see that the graph of the estimated model is close to the true model. The final inverted model obtained is qualitatively in good agreement with the real model from synthetic data. The inversion scheme leads to very good result and has high speed of convergence. A comparison of inversion results obtained from our scheme, conventional conjugate gradient and Levenberg-Marquardt methods on a test data set clearly demonstrates an edge over the other two stated schemes as far as the robustness is concerned (see Table 2). This illustrates the advantage in using the RCG method which gives the result much better than using another method of inversion. The example of using the RCG scheme described here has been successfully applied for geophysical inversion of magnetometric resistivity data.
References


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