On a Subclass of k-Uniformly Starlike Functions Associated with the Generalized Hypergeometric Functions

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Abstract

In the present investigation, we obtain coefficient bounds and extreme points of the subclass of starlike functions associated with Wright’s generalized hypergeometric functions. Furthermore, we discuss radius of convexity and closure properties for functions in this generalized class.

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1 Introduction

Let $S$ be the class of functions $f$ that are analytic in the unit disc $U = \{z : |z| < 1\}$ with $f(0) = 0$.

Denote by $T$, the subclass of $S$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad (z \in U), \quad (1.1.1)$$

and also denote $T_1$, the subclass of $S$ consisting of functions of the form

$$f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k, \quad (a_1 > 0, a_k \geq 0), \quad (1.1.2)$$
For complex parameters $f$ and, the function where

$F$ Following Goodman [7, 8], Rønning [6] defined two subclasses of $S$ in terms of the $\Gamma$ function, by

Let $T_\mu$ be the subclass of $T_1$ satisfying

where, $(-1 < z_0 < 1; z_0 \neq 0)$ or, $f'(z_0) = 1 \ (-1 < z_0 < 1)$. Let the generalized hypergeometric function $qF_s(\alpha_1, \alpha_2, ...; \beta_1, \beta_2, ..., \beta_s; z)$ defined by

$qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) = \sum_{k=0}^{\infty} (\alpha_1)_k, (\alpha_2)_k, ..., (\alpha_q)_k \frac{z^k}{(\beta_1)_k, (\beta_2)_k, ..., (\beta_s)_k k!}$

where

$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U})$,

for $\mathbb{N}$ denotes the set of all positive integers and $(x)_k$ is the Pochhammer symbol defined, in terms of the $\Gamma$ function, by

$(x)_k = \begin{cases} 1 & \text{for } k = 0, \\ x(x+1)(x+2)\ldots(x+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, \ldots\}. \end{cases}$

Corresponding to a function $h_{q,s}(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z)$ defined by

$h_{q,s}(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) = zqF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z)$,

Dziok-Srivastava [2] introduced a convolution operator on $A$ such that

$H_{q,s}(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) : A \rightarrow A,$
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is defined by

$$H(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) = h(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) \ast f(z)$$

$$= z + \sum_{k=2}^{\infty} \Gamma_k a_k z^k,$$

where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1}, (\alpha_2)_{k-1}, ..., (\alpha_q)_{k-1}}{(\beta_1)_{k-1}, (\beta_2)_{k-1}, ..., (\beta_s)_{k-1}(1)_{k-1}}.$$

Definition 1.1.1

Let

$$\varphi^m_{l, \lambda_1, \lambda_2}(z)(\alpha_i, \beta_j; z) = \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) \delta_k z^k,$$

where

$$\delta_k = \frac{(1 + \lambda_1(k - 1) + l)^{m-1}}{(1 + l)^{m-1}(1 + \lambda_2(k - 1))^m},$$

$$i = \{1, 2, ..., q\}, j = \{1, 2, ..., s\}, z \in U,$$

and ($z \in U, b \neq 0, -1, -2, -3, ...), \lambda_2 \geq \lambda_1 \geq 0, l \geq 0, m \in \mathbb{Z}$, also $(x)_k$ is the Pochhammer symbol.

We define a linear operator $D^m_{l, \lambda_1, \lambda_2}(\alpha_i, \beta_j) : A \rightarrow A$ by the following Hadamard product:

$$D^m_{l, \lambda_1, \lambda_2}(\alpha_i, \beta_j)f(z) := \varphi^m_{l, \lambda_1, \lambda_2}(z)(\alpha_i, \beta_j; z) \ast f(z)$$

$$D^m_{l, \lambda_1, \lambda_2}(\alpha_i, \beta_j)f(z) = z + \sum_{k=2}^{\infty} \delta_k \Gamma_k(\alpha_1)a_k z^k. \quad (1.1.4)$$

Note that:

$$D^1_{0, 0, 0}(\alpha_i, \beta_j)f(z) = H(\alpha_i, \beta_j)f(z),$$

$$(1 + l)D^{m+1, \lambda_1, \lambda_2}_{l, \lambda_1, \lambda_2}(\alpha_i, \beta_j)f(z) = (1 - \lambda_1 + l)[D^m_{l, \lambda_1, \lambda_2}(\alpha_i, \beta_j) \ast \varphi^1(\lambda_1, \lambda_2, l)(z)]H(\alpha_i, \beta_j)f(z) +$$
\[ \lambda_1 z[D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j) * \varphi^1(\lambda_1, \lambda_2, l)(z)]' H(\alpha_i, \beta_j) f(z) \]

\[ = D_\lambda[D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j) * \varphi^1(\lambda_1, \lambda_2, l)(z)] H(\alpha_i, \beta_j) f(z), \lambda_1, \lambda_2 \geq 0, \]

\[ D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j) = D_\lambda[D_l^{m-1,\lambda_1,\lambda_2}(\alpha_i, \beta_j) H(\alpha_i, \beta_j) f(z)), \]

where \( m \in \mathbb{N} = \{1, 2, 3, \ldots, \} \), and \( \varphi^1(\lambda_1, \lambda_2, l)(z) \) analytic function given by

\[ \varphi^1(\lambda_1, \lambda_2, l)(z) = z + \sum_{k=2}^{\infty} \frac{1}{(1 + \lambda_2(k - 1)) z^k}. \]

Let \( f \in A \) satisfies (1.1.3). Then we have the following:

\[ z[D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j) f(z)]' = \alpha_1 D_l^{m,\lambda_1,\lambda_2}(\alpha_1 + 1, \beta_j) f(z) - (\alpha_1 - 1) D_l^{m,\lambda_1,\lambda_2}(\alpha_1, \beta_j) f(z). \]

(1.1.5)

For \( (\beta_1 \neq 1, 0, -1, \ldots) \)

\[ z[D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_1) f(z)]' = (\beta_1 - 1) D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_1 - 1) f(z) - (\beta_1 - 2) D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_1) f(z). \]

(1.1.6)

For convenience, we write

\[ D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j) f(z) = D_l^{m,\lambda}[\alpha_1] f(z). \]

This operator \( D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j) f(z) \) includes various other linear operators which were considered earlier in the literature. Let us see some of the examples:

For \( m = 1 \) and \( \lambda_2 = 0 \), we obtain

\[ D_0^{1,0,0}(\alpha_i, \beta_j) f(z) = H_{q,s}(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z), \]

which was given by Dziojk-Srivastava [2].

For \( \alpha_i = 1 \) and \( \beta_j = 1 \), we obtain

\[ D_l^{m,\lambda_1,\lambda_2}(1, 1) f(z) = I^m(\lambda_1, \lambda_2, l, n) f(z), \]

as given in [1].

For \( s = 1 \) and \( q = 2 \), we obtain the linear operator:

\[ D_0^{1,0,0}(\alpha_1, \alpha_2, \beta_1) f(z) = F(\alpha_1, \alpha_2, \beta_1) f(z), \]
which was introduced by Hohlov [4]. Moreover, putting $\alpha_2 = 1$, we obtain the Carlson-Shaffer operator [5]:

$$D_0^{1,0,0}(\alpha_1, 1, \beta_1)f(z) = L(\alpha_1, \beta_1)f(z).$$

Ruscheweyh [3] introduced an operator

$$D_0^{1,0,0}(\lambda + 1, 1, 1)f(z) = D^\lambda f(z).$$

For $-1 \leq \gamma < 1$, we let $D_{I}^{m,\lambda}([\alpha_1], \gamma, \beta, z_0)$ denote the subclass of starlike functions corresponding to the family $UCV$ for functions $f$ of the form (1.1.1) such that

\[ \Re \left\{ \frac{z(D_{I}^{m,\lambda}([\alpha_1])f(z))'}{(1 - \lambda)D_{I}^{m,\lambda}([\alpha_1])f(z) + \lambda z(D_{I}^{m,\lambda}([\alpha_1])f(z))'} - \gamma \right\} > 0, \]

\[ \beta \left| \frac{z(D_{I}^{m,\lambda}([\alpha_1])f(z))'}{(1 - \lambda)D_{I}^{m,\lambda}([\alpha_1])f(z) + \lambda z(D_{I}^{m,\lambda}([\alpha_1])f(z))'} - 1 \right|, \quad \forall (z \in \mathbb{U}, -1 \leq \gamma < 1), \beta \geq 0. \]  

(1.1.7)

We let $TD_{I}^{m,\lambda}([\alpha_1], \gamma, \beta, z_0) = D_{I}^{m,\lambda}([\alpha_1], \gamma, \beta, z_0) \cap T^\mu$, be the subclass of $T^\mu$ consisting of functions of the form (1.1.3) and satisfying the analytic criterion (1.1.7).

Using the techniques of Silverman [9] and motivated by the earlier works [10, 11, 12, 13] and [14], in this paper we obtain the coefficient bounds, distortion bounds, extreme points, radius of starlikeness and closure theorem for the functions belonging to the class $TD_{I}^{m,\lambda}([\alpha_1], \gamma, \beta, z_0)$.

## 2 Main Results

**Theorem 2.2.1** A function $f$ of the form (1.1.2) is in the class $D_{I}^{m,\lambda}([\alpha_1], \gamma, \beta, z_0)$ if, and only if,

$$\sum_{k=2}^{\infty} [k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)]\delta_k \Gamma_k(\alpha_1) a_k \leq a_1(1 - \gamma),$$  

(2.2.1)

$-1 \leq \gamma < 1, \beta \geq 0.$
The proof of the Theorem 2.2.1 is similar to that of Theorem 2.2, in [15], hence we omit the details.

**Theorem 2.2.2** A function \( f \) of the form (1.1.2) is in the class \( TD^m_{i}(\alpha_1, \gamma, \beta, z_0) \) if, and only if,
\[
\sum_{k=2}^{\infty} \left\{ \frac{[k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)]}{1 - \gamma} \delta_k \Gamma_k(\alpha_1) - [(1 - \mu) + k\mu]z_0^{k-1} \right\} a_k \leq 1, \tag{2.2.2}
\]
\(-1 \leq \gamma < 1, \beta \geq 0\).

**Proof:** Since \((1 - \mu)\frac{f(z_0)}{z_0} + \mu f'(z_0) = 1\), where \((-1 < z_0 < 1; z_0 \neq 0)\) \quad 0 \leq \mu \leq 1,
we have
\[
a_1 = 1 + \sum_{k=2}^{\infty} [(1 - \mu) + k\mu]a_k z_0^{k-1}.
\]
Substituting for \(a_1\) in (2.2.1), we get (2.2.2).

**Corollary 2.1** Let the function \( f \) defined by (1.1.2) be in the class \( TD^m_{i}(\alpha_1, \gamma, \beta, z_0) \).
Then
\[
a_k \leq \left\{ \frac{[k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)]}{1 - \gamma} \delta_k \Gamma_k(\alpha_1) - [(1 - \mu) + k\mu]z_0^{k-1} \right\}^{-1},
\]
k \geq 2, -1 \leq \alpha < 1, \quad \text{with equality for}
\[
f(z) = \frac{[k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)]\delta_k \Gamma_k(\alpha_1)z - (1 - \gamma)z^k}{[k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)]\delta_k \Gamma_k(\alpha_1) - (1 - \gamma)[(1 - \mu) + k\mu]z_0^{k-1}}.
\]

**Theorem 2.2.3** Let
\[
f_1(z) = z \tag{2.2.3}
\]
and
\[
f_k(z) = \frac{[k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)]\delta_k \Gamma_k(\alpha_1)z - (1 - \gamma)z^k}{[k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)]\delta_k \Gamma_k(\alpha_1) - (1 - \gamma)[(1 - \mu) + k\mu]z_0^{k-1}},
\]
where \( k \geq 2 \). Then \( f \in TD^m_{i}(\alpha_1, \gamma, \beta, z_0) \) if, and only if, it can be expressed in the form
\[
f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z), \quad \text{where} \quad \lambda_k \geq 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \lambda_k = 1.
\]

**Proof:** The proof of Theorem 2.2.3 follows similarly to the proof of the theorem on extreme points given by Silverman [9].
3 Closure Theorems

Let the functions $f_j(z)$ be defined for $j = 1, 2, 3, ..., n$ by

$$f_j(z) = a_{1,j}z - \sum_{k=2}^{\infty} a_{k,j}z^k,$$  \hfill (3.3.1)

where $a_{1,j} > 0$, $a_{k,j} \geq 0$, $z \in \mathbb{U}$.

**Theorem 3.3.1** Let the function $f_j(z)$ of the form (3.3.1) be in the class $TD_i^{m,\lambda}([\alpha_1], \gamma, \beta, z_0)$. Then the function $h$ defined by

$$h(z) = \sum_{j=1}^{n} d_j f_j(z), \quad d_j \geq 0,$$  \hfill (3.3.2)

is also in the same class $TD_i^{m,\lambda}([\alpha_1], \gamma, \beta, z_0)$, where

$$\sum_{j=1}^{n} d_j = 1.$$  \hfill (3.3.3)

**Proof:** In the view of (3.3.2), we have

$$h(z) = b_1 z + \sum_{k=2}^{\infty} b_k z^k,$$  \hfill (3.3.4)

where $b_1 = \sum_{j=1}^{n} d_j a_{1,j}$ and $b_k = \sum_{j=1}^{n} d_j a_{k,j}$, $k = 2, 3, ...$.

Since $f_j(z) \in TD_i^{m,\lambda}([\alpha_1], \gamma, \beta, z_0)$, $(j = 1, 2, ..., n)$, and by applying Theorem 2.2.2, we obtain

$$\sum_{k=2}^{\infty} \left\{ \frac{[k(1+\beta) - (\gamma+\beta)(1+k\lambda-\lambda)]}{1-\gamma} \delta_k \Gamma_k(\alpha_1) - [(1-\mu) + k\mu]z_0^{k-1} \right\} a_{k,j} \leq 1,$$

$(j = 1, 2, ..., n)$.

Therefore, we have

$$\sum_{k=2}^{\infty} \left\{ \frac{[k(1+\beta) - (\gamma+\beta)(1+k\lambda-\lambda)]}{1-\gamma} \delta_k \Gamma_k(\alpha_1) - [(1-\mu) + k\mu]z_0^{k-1} \right\} \left( \sum_{j=1}^{n} d_j a_{k,j} \right)$$

$$= \sum_{j=1}^{n} d_j \left( \sum_{k=2}^{\infty} \left\{ \frac{[k(1+\beta) - (\gamma+\beta)(1+k\lambda-\lambda)]}{1-\gamma} \delta_k \Gamma_k(\alpha_1) - [(1-\mu) + k\mu]z_0^{k-1} \right\} a_{k,j} \right)$$

$$\leq \sum_{j=1}^{n} d_j = 1,$$ (by Theorem 2.2.2 and by (3.3.3)).

This implies that $h(z) \in TD_i^{m,\lambda}([\alpha_1], \gamma, \beta, z_0)$, and the proof is complete.
4 Radius of Convexity and Starlikeness

In this section, we obtain the radius of starlikeness of order \( \eta (0 \leq \eta < 1) \), radius of convexity of order \( \eta (0 \leq \eta < 1) \), for the class \( TD^m_\lambda ([\alpha_1], \gamma, \beta, z_0) \).

**Theorem 4.4.1** Let \( f \in TD^m_\lambda ([\alpha_1], \gamma, \beta, z_0) \). Then

(1) \( f \) is starlike of order \( \eta (0 \leq \eta < 1) \), in the disk \( |z| < r_1 \); that is, \( \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \eta \), where

\[
  r_1 = \inf_{k \geq 2} \left\{ \delta_k \Gamma_k(a_1) \frac{1 - \eta [k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)]}{k - \eta} \right\}^{1/k-1}.
\]

(2) \( f \) is convex of order \( \eta (0 \leq \eta < 1) \), in the disk \( |z| < r_2 \); that is, \( \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \eta \), where

\[
  r_2 = \inf_{k \geq 2} \left\{ \delta_k \Gamma_k(a_1) \frac{1 - \eta [k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)]}{k(1 - \gamma)} \right\}^{1/k-1}.
\]

Each of these results are sharp for the extremal function \( f \) given by (2.2.3).

**Proof:** Given \( f \in T_1 \), and \( f \) is starlike of order \( \eta \), we have

\[
  \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \eta. \tag{4.4.1}
\]

For the left hand side of (4.4.1), we have

\[
  \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \sum_{k=2}^{\infty} \frac{(k-1)a_k|z|^{k-1}}{a_1 - \sum_{k=2}^{\infty} a_k|z|^{k-1}}.
\]

The last expression is less than \( 1 - \eta \), if

\[
  \sum_{k=2}^{\infty} \frac{(k-\eta)}{1-\eta} a_k |z|^{k-1} < a_1. \tag{4.4.2}
\]

Substituting \( a_1 = 1 + \sum_{k=2}^{\infty} [(1 - \mu) + k\mu] a_k z_0^{k-1} \) in (4.4.2), we get

\[
  \sum_{k=2}^{\infty} \left\{ \frac{(k-\eta)}{1-\eta} |z|^{k-1} - [(1 - \mu) + k\mu] z_0^{k-1} \right\} a_k \leq 1. \tag{4.4.3}
\]
Using the fact that \( f \in TD_{l}^{m,\lambda}([\alpha_1], \gamma, \beta, z_0) \) if, and only if,
\[
\sum_{k=2}^{\infty} \left\{ \frac{[k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)]}{(1 - \gamma)} \delta_k \Gamma_k(\alpha_1) - [(1 - \mu) + k\mu]z_0^{k-1} \right\} a_k \leq 1,
\]
we can say (4.4.1) is true if
\[
(1 - \mu) + k\mu \leq \sum_{k=2}^{\infty} \frac{(k - \eta)}{1 - \eta} |z|^{k-1} \left( \frac{k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)}{(1 - \gamma)} \right) \delta_k \Gamma_k(\alpha_1) - [(1 - \mu) + k\mu]z_0^{k-1}.
\]
Or, equivalently,
\[
|z|^{k-1} < \frac{1 - \eta}{k - \eta} \frac{k(1 + \beta) - (\gamma + \beta)(1 + k\lambda - \lambda)}{(1 - \gamma)} \delta_k \Gamma_k(\alpha_1),
\]
which yields the starlikeness of the family.

(2) Using the fact that \( f \) is convex if, and only if, \( zf' \) is starlike, we can prove (2) similar to the proof of (1).

**Remark 4.1** We note that the radius of starlikeness and convexity are independent of the fixed point \( z_0 \).

## 5 Convex families

Suppose \( B \) is nonempty subset of the real interval \((0, 1)\). Define \( TD_{l}^{m,\lambda}([\alpha_1], \gamma, \beta, B) \) by
\[
TD_{l}^{m,\lambda}([\alpha_1], \gamma, \beta, B) = \bigcup_{z_i \in B} TD_{l}^{m,\lambda}([\alpha_1], \gamma, \beta, z_i).
\]

If \( B \) consists of a single element, say \( z_0 \), then \( TD_{l}^{m,\lambda}([\alpha_1], \gamma, \beta, z_0) \) is a convex family. Because, if \( f_1(z) \) and \( f_2(z) \) are in \( TD_{l}^{m,\lambda}([\alpha_1], \gamma, \beta, z_0) \), then it can be seen that for \( 0 \leq \xi \leq 1, \xi f_1(z) + (1 - \xi)f_2(z) \) is in \( TD_{l}^{m,\lambda}([\alpha_1], \gamma, \beta, z_0) \). To examine this class for other subsets of \( B \), we prove the following lemma.

**Lemma 5.5.1** If \( f \in TD_{l}^{m,\lambda}([\alpha_1], \gamma, \beta, z_0) \cap TD_{l}^{m,\lambda}([\alpha_1], \gamma, \beta, z_1) \), where \( z_0 \) and \( z_1 \) are distinct positive numbers, then \( f(z) = z \).
**Proof:** For a function $f$ of the form (1.1.2), we have

\[ a_1 = 1 + \sum_{k=2}^{\infty} [(1 - \mu) + k\mu]a_kz_0^{k-1}, \]

and

\[ a_1 = 1 + \sum_{k=2}^{\infty} [(1 - \mu) + k\mu]a_kz_1^{k-1}. \]

That is,

\[ [(1 - \mu) + k\mu]a_k[z_1^{k-1} - z_0^{k-1}] = 0. \]

Hence $a_k \equiv 0$ for $k \geq 2$, and so the result follows.

**Theorem 5.5.2** If $B$ is contained in the interval $(0, 1)$, then $TD_t^{m,\lambda}([\alpha_1], \gamma, \beta, B)$ is a convex family if, and only if, $B$ is connected.

**Proof:** Let $B$ be connected. Suppose $z_0, z_1 \in B$ with $z_0 \leq z_1$. If $f$ of the form (1.1.2) is in $TD_t^{m,\lambda}([\alpha_1], \gamma, \beta, z_0)$ and $g(z) = b_1z - \sum_{k=2}^{\infty} b_kz^k$ is in $TD_t^{m,\lambda}([\alpha_1], \gamma, \beta, z_1)$, then for $0 \leq \xi \leq 1$, we shall prove that there exists $z_2(0 \leq z_2 \leq z_1)$ such that $h(z) = \xi f(z) + (1 - \xi)g(z)$ is in $TD_t^{m,\lambda}([\alpha_1], \gamma, \beta, z_2)$. Set

\[ t(z) = \left[ (1 - \mu)\frac{h(z)}{z} + \mu h'(z) \right] \]

\[ = \xi \left[ a_1 - \sum_{k=2}^{\infty} [(1 - \mu) + k\mu]a_kz_0^{k-1} \right] + (1 - \xi) \left[ b_1 - \sum_{k=2}^{\infty} [(1 - \mu) + k\mu]b_kz_1^{k-1} \right] \]

and

\[ t(z) = 1 + \xi \sum_{k=2}^{\infty} [(1 - \mu) + k\mu]a_k(z_0^{k-1} - z_1^{k-1})] + (1 - \xi) \sum_{k=2}^{\infty} [(1 - \mu) + k\mu]b_k(z_1^{k-1} - z_1^{k-1})]. \]

(5.5.1)

We observe that $f$ is real when $z$ is real with $t(z_0) \geq 1$ and $t(z_1) \leq 1$. Hence for some $z_1, z_0 \leq z_2 \leq z_1$, we have $t(z_2) = 1$. Since $z_1, z_2$ and $\xi$ are arbitrary, the family $TD_t^{m,\lambda}([\alpha_1], \gamma, \beta, B)$ is convex. Conversely, suppose $B$ is not connected. Then we can take $z_0, z_1 \in B, z_2 \notin B$ such that $z_0 < z_2 < z_1$ (remember $z$ is real here). Let us assume
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\( f \) and \( g \) are not both identity function. Then using (5.5.1), letting \( z = z_2 \) and allowing \( \xi \) to vary, we have

\[
t(z) = t(z_2, \xi) = 1 + \xi \sum_{k=2}^{\infty} \left[ ((1 - \mu) + k\mu) a_k (z_0^{k-1} - z^{k-1}) \right] + (1 - \xi) \sum_{k=2}^{\infty} \left[ ((1 - \mu) + k\mu) b_k (z_1^{k-1} - z^{k-1}) \right].
\]

Since \( t(z_2, 0) > 1 \) and \( t(z_2, 1) < 1 \), there must exist \( \xi_0 (0 < \xi_0 < 1) \), for which \( t(z_2, \xi_0) = 1 \). Hence \( h \in TD^{m, \lambda}(\alpha_1, \gamma, \beta, z_2) \) for \( \xi = \xi_0 \). Since \( z_2 \notin B \), according to Lemma 5.5.1, it follows that \( h \in TD^{n, \lambda}(\alpha_1, \gamma, \beta, B) \). Therefore \( TD^{n, \lambda}(\alpha_1, \gamma, \beta, B) \) is not a convex family.

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References


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