Quantum Mechanics of Photons

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Abstract

We present a short review of the concept of photon wave function and the photon wave equation which represents the Schrödinger equation for the photon. Photon wave function puts the photon on the same footing as particles with mass and presents the first quantization formulation for electromagnetic fields. Because the photon has zero rest mass and it always moves with speed of light, the quantum mechanical formulation is somewhat different from other particles. We present how the concept of photon wave function is developed from the works of Bialynicki-Birula and Sipe and the recent applications of this concept.

Keywords: Photon, Electromagnetic field, Light quanta, Photon wave function, Photon wave equation, Photonic crystal, Quantum coherence.

1 Introduction

Photon as a quantum of light energy was first introduced by Max Planck in the year 1900 in order to explain the spectral distribution of electromagnetic waves emitted by a blackbody. In 1905 Einstein suggested particle nature of light both for free fields and for light interacting with matter to explain the photoelectric effect. But, it is not until 1926 the name “photon” was suggested by the chemist Gilbert Lewis for the quantum of light energy. For massive particles the classical description is given by Hamilton’s variational principle and the nonrelativistic quantum mechanical description, by the Schrödinger equation. This equation represents the first quantization procedure wherein the particle shows the wave nature and the physical quantities like position, momentum etc. are elevated to the status of operators. To describe the relativistic quantum mechanical nature of particles, second quantization procedure is required wherein the wavefunctions are elevated to the status of operators. For electromagnetic fields, Maxwell equations are considered to be the classical description and the quantization of
these fields gives the relativistic quantum description. This is the usual second quantization procedure for the electromagnetic fields. There seems to be no first quantization procedure for electromagnetic fields. Bialynicki-Birula [1,2,3,5] and Sipe [13] showed that Maxwell equations are the quantum description of electromagnetic fields at the first quantum level and the Fermat principle for light forms the classical description of electromagnetic fields similar to the Hamilton’s variational principle for particles with mass. The Maxwell equations can be recast in the functional form of Schrödinger equation and the photon wave function with corresponding photon wave equation can be defined from it. The photon wave equation which is the Schrödinger equation of photon describes the quantum mechanical properties of a photon. This equation can be used to study the quantum mechanical properties of the electromagnetic fields without resorting to the elaborate quantum electrodynamic methods.

2 Photon Wave Function

Photon wave function is still not fully accepted among the scientific community, as it doesn’t have all the properties of the wave function of particles with mass in quantum mechanics. This arises mainly because of nonlocalizable nature of the photon and the consequent difficulty in defining the position operator for photon. This means photon wave function cannot exist in position eigenstate. However photon wave function in momentum representation is well established and its momentum operator is well defined. However, position operator for photon consistent with all the usual rules of quantum mechanics is constructed by Hawton [6]. If we adopt a less stringent view on position representation of wave function, it is possible to define photon wave function in position representation. Bialynicki-Birula [1,2,3] and Sipe [13] defined a photon wave function, whose modulus squared gives the photon’s mean energy density at given position. The validity of this definition stems from the fact that it satisfies the rules of non-relativistic quantum mechanics, except for position operator, of course, and that second quantization of this photon wave function reproduces the usual quantum electrodynamics of photons. It is also possible to arrive at photon wave mechanics as a special case from quantum electrodynamics.

The origin of photon wave function is the complexified form of Maxwell equations that were used by Riemann, Silberstein and Bateman [3]. The complex form of Maxwell equations are given as

\[
\frac{\partial \mathbf{F}(\mathbf{r},t)}{\partial t} = \epsilon \nabla \times \mathbf{F}(\mathbf{r},t) \quad (1)
\]

\[
\nabla \cdot \mathbf{F}(\mathbf{r},t) = 0 \quad (2)
\]

where
Quantum mechanics of photons

\[ F(r, t) = \frac{D(r, t)}{\sqrt{2\varepsilon_0}} + i \frac{B(r, t)}{\sqrt{2\mu_0}} \]  

Substitution of Eq. (3) in Eqs. (1) and (2) reproduce the usual form of Maxwell equations. The complex function \( F(r, t) \) is known as Riemann-Silberstein vector and it is defined as the wave function for single photon. The rational for the above definition lays in the fact that the dynamical quantities like the energy density and the Poynting vectors can be expressed as bilinear expressions of \( F(r, t) \).

\[ E = \int d^3r F^* \cdot F \]  
\[ P = \frac{1}{2ic} \int d^3r F^* \times F \]

These expressions look similar to the expressions for expectation values of physical quantities in quantum mechanics.

An arbitrary polarization state of electromagnetic wave can be expressed as combination of left and right circularly polarized states. To represent a photon with arbitrary polarization state, we can use two state functions of the photon written as \( F^+(r, t) = \frac{D(r, t)}{\sqrt{2\varepsilon_0}} + i \frac{B(r, t)}{\sqrt{2\mu_0}} \) and \( F^-(r, t) = \frac{D(r, t)}{\sqrt{2\varepsilon_0}} - i \frac{B(r, t)}{\sqrt{2\mu_0}} \) which form the left and the right circularly polarized states respectively. In general the photon wave function is given as six component column vector as

\[ \psi(r, t) = \begin{pmatrix} F^+(r, t) \\ F^-(r, t) \end{pmatrix} \]

Even though it is a six component quantity, all the information of photon can be obtained from either \( F^+ \) or \( F^- \) alone. The physical interpretation of the photon wave function \( \psi(r, t) \) is that the modulus square of this wave function gives the electromagnetic energy density at given position and time.

\[ |\psi|^2 = \frac{\varepsilon_0}{2} |E|^2 + \frac{|B|^2}{2\mu_0} \]

A different approach in defining the photon wave function which describes a single- or many photon system is given by D. Dragoman [4]. In this approach a photon is associated with spectral and polarization component of the classical electromagnetic field and the coefficients of electric and magnetic fields in the photon wave function are such that its modulus square gives the probability.
density amplitude of photons. This is in contrast to the meaning of the
Riemann-Silberstein photon wave function interpretation which is in terms of
electromagnetic energy density. The wave function as defined by D. Dragoman
[4] is given as

$$\psi(r,t) = \sqrt{\frac{\varepsilon_0 E}{\mu_0 H}}$$

(8)

where electric and magnetic fields can take complex values. This is to account for
the phase-related phenomena like interference in which the Hannay angle [4] is
measured.

3 Photon Wave Equation

The quantum mechanical properties of photons can be described by the photon
wave equation which is in the same functional form as the Schrödinger equation.
The photon wave equation can be deduced from the Maxwell equations

$$\frac{\partial E}{\partial t} = c \nabla \times H, \quad \frac{\partial H}{\partial t} = -c \nabla \times E, \quad D = E, \quad B = H,$$

by rewriting them as

$$i\hbar \frac{\partial \psi}{\partial t} = H_f \psi$$

(9)

with following definitions

$$\psi = \begin{pmatrix} E(r,t) + iH(r,t) \\ E(r,t) - iH(r,t) \end{pmatrix}, \quad H_f = c \begin{pmatrix} p \cdot S & 0 \\ 0 & -p \cdot S \end{pmatrix} \quad \text{and} \quad p = -i\hbar \nabla$$

The operators $H_f$ and $p$ are respectively the Hamiltonian and momentum
operators of photon and $S$ is the photon spin matrix (spin 1 matrix) which is
defined as $S_{\mu\nu} = -i\varepsilon_{\mu\nu}$ (antisymmetric Levi-Civita symbol). In explicit
form $S_{\mu\nu}$ are given as
Quantum mechanics of photons

\[
S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
and \( S = (S_x, S_y, S_z) \).

It is also possible to derive the photon wave equation from the Einstein energy-momentum-mass relation \([14,15]\) \( E = (c^2|p|^2 + m^2 c^4)^{1/2} \). For photon \( m = 0 \) and \( E = c\sqrt{p \cdot p} \). Since momentum-space photon wave function \( \psi(p) \) is well defined we can get the following equation

\[
E\psi(p) = c\sqrt{p \cdot p}\psi(p)
\]

Any vector field can be resolved into transverse and longitudinal parts. So with the superscripts \( T \) and \( L \) representing transverse and longitudinal parts respectively we can write \( \psi(p) = \psi^T(p) + \psi^L(p) \) and using the vector identity

\[
p \times p \times \psi^T(p) = -p \cdot p \psi^T + p(p \cdot \psi^T(p)) = -p \cdot p \psi^T(p)
\]
we arrive at the equation for photon \( E\psi^T(p) = \pm ic p \times \psi^T(p) \). Dropping the superscript and writing in the form of energy-eigenvalue equation

\[
H_j\psi(p) = E\psi(p)
\]
we arrive at the expression for Hamiltonian \( H_j = ic p \times \) and the time-dependent photon wave equation can be written as given in Eq. (9).

The photon wave equation for the alternative form of photon wavefunction defined in Eq.(8) is similar to Eq.(9) and is written as [4]

\[
i\hbar \frac{\partial \psi}{\partial t} = ic(S \cdot p)(J \psi) = H_j \psi
\]
where \( S \) is spin-1 matrix as before and \( J = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} \) with \( \mathbf{I} \) denoting 3×3 identity or unit matrix. The parallel between the photon and the particle with mass is well summarized by M.O.Scully et al., [12] and A. Muthukrishnan et al., [9] in the following tabular form. Here \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli matrices, \( \varphi \) is the four component wavefunction for neutrino, \( \psi \), the six component photon wavefunction, \( n \), the refractive index of medium, and \( L \), the Lagrangian for the
neutrino. Eikonal physics gives classical description of light and neutrino in form of Fermat and Hamilton’s variational principles. “Wave” or quantum mechanics of light and neutrino are described by Maxwell equations and Dirac equations respectively. Quantum field description of photon and neutrino are given by the dynamics of corresponding quantum field operators $\hat{a}_k$ and $\hat{c}_k$.

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<thead>
<tr>
<th>Eikonal Physics</th>
<th>Photon</th>
<th>Neutrino</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ray Optics (Fermat Principle):</td>
<td>$\delta \int nds = 0$</td>
<td>Classical Mechanics (Hamiltonian Principle):</td>
</tr>
<tr>
<td>“Wave” Mechanics</td>
<td>Maxwell Equations:</td>
<td>Dirac Equations:</td>
</tr>
<tr>
<td>Quantum Field Theory</td>
<td>$\hat{E}^+ (\mathbf{r}, t) = \sum_k \hat{a}_k (t) E_k (\mathbf{r})$</td>
<td></td>
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<td>\psi\rangle = -\frac{i}{\hbar} H \psi$</td>
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</tbody>
</table>

## 4 Applications of Photon Wave Equation

The concept of photon wave function was introduced to bring about harmony between photons and particles with mass. It doesn’t remain as mere theoretical exercise and there are various applications in recent time. Photon wave equation is used to study the propagation of photons in medium [8,10,16], the quantum properties of electromagnetic waves in structured media [14,15] and the scattering of electromagnetic waves in both isotropic and anisotropic inhomogeneous media [7]. J.Zaleśny [16] showed that interaction between photon and medium is described by some scalar and vector potentials. The scalar potential is related to permittivity and permeability of stationary medium and vector potential to the velocity of the medium. For example, the photon passing through a photonic crystal is equivalent to the photon passing through a periodic potential. Recently another approach for the interaction between photon and medium is given by P.L.Saldanha and C.H.Monken [11] in which interaction is given by a term proportional to the current density induced in the media due the presence of photon. The quantum coherence properties and the decoherence of photons in medium, two-photon and multi-photon wave mechanics are described in detail in [14]. Quantum mechanical theory of photons will be useful in studying the interference and entanglement of photons [9,12].

## 5 Conclusions

The equivalence between Maxwell equations and Schrödinger equation shows that the Maxwell equations actually describe the quantum properties of
Quantum mechanics of photons

Electromagnetic radiation at first quantization level. The concept of photon wave function is also useful in studying the quantum properties of radiation in homogenous and inhomogeneous medium in relatively simpler form than the full quantum electrodynamical treatment. There are still open questions in covariant Lagrangian formalism and consequently in the Feynman path integral formalism for photon wave function.

References