Symmetry $p$-Adic $q$-Integrals on $\mathbb{Z}_p$
for $(h, q)$-Euler Numbers and Polynomials

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea
ryoocs@hnu.kr

Abstract
In [7], we studied the $(h, q)$-Euler numbers and polynomials. By using these numbers and polynomials, we investigate the alternating sums of powers of consecutive integers. By applying the symmetry of the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, we give recurrence identities the $(h, q)$-Euler polynomials.

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1 Introduction

In 80th and 90th there was demonstrated large interest to various $p$-adic physical models, see, for example, papers on $p$-adic models for spin glasses [1], $p$-adic string theory [11], $p$-adic quantum mechanics and field theory [10], $p$-adic differential equations [2]. Recently, $p$-adic analysis and non-Archimedean functional analysis has been developed with its applications in mathematical physics(see [3-9]). The Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. The Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics(see [1-11]).

As a consequence, many authors have studied the $q$-extension in various areas (see [3-9]). In this paper, by using the symmetry of $p$-adic $q$-integral on $\mathbb{Z}_p$, we obtain the recurrence identities the $(h, q)$-Euler polynomials.

Throughout this paper, we always make use of the following notations: $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. 
Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu_p(p)} = p^{-1} \). When one talks of \( q \)-extension, \( q \) is considered in many ways such as an indeterminate, a complex number \( q \in \mathbb{C} \), or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \) one normally assume that \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we normally assume that \( |q - 1|_p < p^{-\nu_p(p)} \) so that \( q^x = \exp(x \log q) \) for \( |x|_p \leq 1 \). Throughout this paper we use the notation:

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.
\]

Hence, \( \lim_{q \to 1} [x] = x \) for any \( x \) with \( |x|_p \leq 1 \) in the present \( p \)-adic case.

For \( g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\} \), the \( p \)-adic \( q \)-integral was defined by

\[
I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{[2]_q}{1 + q^{pN}} \sum_{x=0}^{p^{N-1}} g(x)(-q)^x, \text{ see } [3-9]. \tag{1.1}
\]

If we take \( g_1(x) = g(x+1) \) in (1.1), then we easily see that

\[
q I_{-q}(g_1) + I_{-q}(g) = [2]_q g(0). \tag{1.2}
\]

In [7], we defined the \((h, q)\)-Euler numbers and polynomials and investigate their properties. For \( h \in \mathbb{Z}, q \in \mathbb{C}_p \) with \( |1 - q|_p \leq 1 \), the \((h, q)\)-Euler polynomials \( \tilde{E}_{n,q}(x) \) are defined by

\[
\tilde{E}_{n,q}^{(h)}(x, t) = \sum_{n=0}^{\infty} \tilde{E}_{n,q}^{(h)}(x) \frac{t^n}{n!} = \frac{[2]_q}{q^h e^t + 1} e^t. \tag{1.3}
\]

The \((h, q)\)-Euler numbers \( \tilde{E}_{n,q}^{(h)} \) are defined by the generating function:

\[
\tilde{E}_{n,q}^{(h)}(t) = \sum_{n=0}^{\infty} \tilde{E}_{n,q}^{(h)} \frac{t^n}{n!} = \frac{[2]_q}{q^h e^t + 1}. \tag{1.4}
\]

The following elementary properties of the \((h, q)\)-Euler numbers \( \tilde{E}_{n,q}^{(h)} \) and polynomials \( \tilde{E}_{n,q}^{(h)}(x) \) are readily derived form (1.1), (1.2), (1.3) and (1.4) (see, for details, [7]). We, therefore, choose to omit details involved.

**Theorem 1.1** (Witt formula). For \( h \in \mathbb{Z}, q \in \mathbb{C}_p \) with \( |1 - q|_p < 1 \), we have

\[
\tilde{E}_{n,q}^{(h)} = \int_{\mathbb{Z}_p} q^{(h-1)x} x^n d\mu_{-q}(x),
\]

\[
\tilde{E}_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)x} (x + y)^n d\mu_{-q}(y).
\]
2 Symmetry \( p \)-adic \( q \)-integrals on \( \mathbb{Z}_p \) for \((h, q)\)-Euler numbers and polynomials

Let \( q \) be a complex number with \(|q| < 1\). By using (1.3), we give the alternating sums of powers of consecutive \((h, q)\)-integers as follows:

\[
\sum_{n=0}^{\infty} \tilde{E}_{n,q}^{(h)} \frac{t^n}{n!} = \frac{[2]_q}{q^h e^t + 1} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{nt}.
\]

From the above, we obtain

\[
- \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{(n+k)t} + \sum_{n=0}^{\infty} (-1)^n q^{h(n-k)} e^{nt} = \sum_{n=0}^{k-1} (-1)^n q^{h(n-k)} e^{nt}.
\]

Thus, we have

\[
- [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{(n+k)t} + [2]_q (-1)^{-k}q^{-hk} \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{nt} = [2]_q (-1)^{-k}q^{-hk} \sum_{n=0}^{k-1} (-1)^n q^{hn} e^{nt}.
\]

By using (1.3) and (1.4), and (2.1), we obtain

\[
- \sum_{j=0}^{\infty} \tilde{E}_{j,q}^{(h)}(k) \frac{t^j}{j!} + (-1)^{-k}q^{-hk} \sum_{j=0}^{\infty} \tilde{E}_{j,q}^{(h)} \frac{t^j}{j!} = [2]_q \sum_{j=0}^{\infty} \left((-1)^{-k}q^{-hk} \sum_{n=0}^{k-1} (-1)^n q^{hn} n^j\right) \frac{t^j}{j!}.
\]

By comparing coefficients of \( \frac{t^j}{j!} \) in the above equation, we obtain

\[
\sum_{n=0}^{k-1} (-1)^n q^{hn} n^j = \frac{(-1)^{k+1} q^{hk} \tilde{E}_{j,q}^{(h)}(k) + \tilde{E}_{j,q}^{(h)}}{[2]_q}.
\]

By using the above equation we arrive at the following theorem:

**Theorem 2.1** Let \( k \) be a positive integer and \( q \in \mathbb{C} \) with \(|q| < 1\). Then we obtain

\[
\tilde{T}_{j,q}^{(h)}(k-1) = \sum_{n=0}^{k-1} (-1)^n q^{hn} n^j = \frac{(-1)^{k+1} q^{hk} \tilde{E}_{j,q}^{(h)}(k) + \tilde{E}_{j,q}^{(h)}}{[2]_q}.
\]
Next, we assume that \( q \in \mathbb{C}_p \). We obtain recurrence identities the \((h, q)\)-Euler polynomials and the \( q \)-analogue of alternating sums of powers of consecutive integers. By using (1.1), we have

\[
q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l),
\]

where \( g_n(x) = g(x + n) \). If \( n \) is odd from the above, we obtain

\[
q^n I_{-q}(g_n) + I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l) \quad \text{(cf. [3-5])}. \tag{2.2}
\]

It will be more convenient to write (2.2) as the equivalent integral form

\[
q^n \int_{\mathbb{Z}_p} g(x + n) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = [2]_q \sum_{k=0}^{n-1} (-1)^k q^k g(k). \tag{2.3}
\]

Substituting \( g(x) = q^{(h-1)x} e^{xt} \) into the above, we obtain

\[
(q^{hn} e^{nt} + 1) \int_{\mathbb{Z}_p} q^{(h-1)x} e^{xt} d\mu_{-q}(x) = [2]_q \sum_{j=0}^{n-1} (-1)^j q^h e^{jt}. \tag{2.4}
\]

After some elementary calculations, we have

\[
\int_{\mathbb{Z}_p} q^{(h-1)x} e^{xt} d\mu_{-q}(x) = \frac{[2]_q}{q^h e^{t} + 1},
\]

\[
\int_{\mathbb{Z}_p} q^{(h-1)x} e^{(x+n)t} d\mu_{-q}(x) = e^{nt} \frac{[2]_q}{q^h e^{t} + 1}. \tag{2.5}
\]

By using (2.4) and (2.5), we have

\[
q^{hn} \int_{\mathbb{Z}_p} q^{(h-1)x} e^{(x+n)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} q^{(h-1)x} e^{xt} d\mu_{-q}(x) = \frac{[2]_q (1 + q^{hn} e^{nt})}{q^h e^{t} + 1}.
\]

From the above, we get

\[
\frac{[2]_q (1 + q^{hn} e^{nt})}{q^h e^{t} + 1} = \frac{[2]_q \int_{\mathbb{Z}_p} q^{(h-1)x} e^{xt} d\mu_{-q}(x)}{\int_{\mathbb{Z}_p} q^{(hn-1)x} e^{nt} d\mu_{-q}(x)}. \tag{2.6}
\]

By using (2.4) and (2.6), we arrive at the following theorem:

**Theorem 2.2** Let \( n \) be odd positive integer and \( h \in \mathbb{Z} \). Then we have

\[
\frac{\int_{\mathbb{Z}_p} q^{(h-1)x} e^{xt} d\mu_{-q}(x)}{\int_{\mathbb{Z}_p} q^{(hn-1)x} e^{nt} d\mu_{-q}(x)} = \sum_{m=0}^{\infty} \left( \tilde{T}^{(h)}_{m,q}(n-1) \right) \frac{t^m}{m!}.
\]
Let \( w_1 \) and \( w_2 \) be odd positive integers. By (2.4), Theorem 2.2, and after some elementary calculations, we obtain the following theorem.

**Theorem 2.3** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we have

\[
\frac{\int_{\mathbb{Z}_p} q^{(w_2-1)x_2} e^{w_2 x_2 t} d\mu_{-q}(x_2)}{\int_{\mathbb{Z}_p} q^{(w_1 w_2-1)x} e^{w_1 w_2 x t} d\mu_{-q}(x)} = \sum_{m=0}^{\infty} \left( \frac{\tilde{T}_{m,q}(w_2 - 1) w_2^m}{m!} \right) t^m.
\]

By (1.1), we obtain

\[
\frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} q^{(w_1-1)x_1} q^{(w_2-1)x_2} e^{(w_1 x_1 + w_2 x_2 + w_1 w_2 x) t} d\mu_{-q}(x_1) d\mu_{-q}(x_2)}{\int_{\mathbb{Z}_p} q^{(w_1 w_2-1)x} e^{w_1 w_2 x t} d\mu_{-q}(x)} = \frac{e^{w_1 w_2 x t} \int_{\mathbb{Z}_p} q^{(w_1-1)x_1} e^{w_1 x_1 t} d\mu_{-q}(x_1) \int_{\mathbb{Z}_p} q^{(w_2-1)x_2} e^{w_2 x_2 t} d\mu_{-q}(x_2)}{\int_{\mathbb{Z}_p} q^{(w_1 w_2-1)x} e^{w_1 w_2 x t} d\mu_{-q}(x)}.
\]

(2.7)

By using Theorem 2.3 and (2.7), after elementary calculations, we obtain

\[
a = \left( \int_{\mathbb{Z}_p} q^{(w_1-1)x_1} e^{(w_1 x_1 + w_1 w_2 x) t} d\mu_{-q}(x_1) \right) \left( \int_{\mathbb{Z}_p} q^{(w_2-1)x_2} e^{w_2 x_2 t} d\mu_{-q}(x_2) \right)^2 \left( \sum_{m=0}^{\infty} \frac{\tilde{E}_{m,q}^{(w_2)} (w_2 x) w_1^m}{m!} \right) \left( \sum_{m=0}^{\infty} \frac{\tilde{T}_{m,q}^{(w_2)} (w_2 - 1) w_2^m}{m!} \right).
\]

(2.8)

By using Cauchy product in the above, we have

\[
a = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} \tilde{E}_{j,q}^{(w_2)} (w_2 x) w_1^j \tilde{T}_{m-j,q}^{(w_2)} (w_2 - 1) w_2^{m-j} \frac{t^m}{m!}.
\]

(2.9)

By using the symmetry in (2.8), we obtain

\[
a = \left( \int_{\mathbb{Z}_p} q^{(w_2-1)x_1} e^{(w_2 x_2 + w_1 w_2 x) t} d\mu_{-q}(x_2) \right) \left( \int_{\mathbb{Z}_p} q^{(w_1-1)x_1} e^{x_1 w_1 t} d\mu_{-q}(x_1) \right) \left( \sum_{m=0}^{\infty} \frac{\tilde{E}_{m,q}^{(w_2)} (w_1 x) w_2^m}{m!} \right) \left( \sum_{m=0}^{\infty} \frac{\tilde{T}_{m,q}^{(w_1)} (w_2 - 1) w_1^m}{m!} \right).
\]

Thus we obtain

\[
a = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} \tilde{E}_{j,q}^{(w_2)} (w_1 x) w_2^j \tilde{T}_{m-j,q}^{(w_1)} (w_2 - 1) w_1^{m-j} \frac{t^m}{m!}.
\]

(2.10)

By comparing coefficients \( \frac{t^m}{m!} \) in the both sides of (2.9) and (2.10), we arrive at the following theorem.
Theorem 2.4 Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain
\[
\sum_{j=0}^{m} \binom{m}{j} \tilde{E}_{j,q}^{(w_1)}(w_2x)w_1^j \tilde{T}_{m-j,q}^{(w_2)}(w_1 - 1)w_2^{m-j} = \sum_{j=0}^{m} \binom{m}{j} \tilde{E}_{j,q}^{(w_2)}(w_1x)w_2^j \tilde{T}_{m-j,q}^{(w_1)}(w_2 - 1)w_1^{m-j},
\]
where \( \tilde{E}_{k,q}^{(h)}(x) \) and \( \tilde{T}_{m,q}^{(h)}(k) \) denote the \((h, q)\)-Euler polynomials and the \(q\)-analogue of alternating sums of powers of consecutive integers, respectively.

By using (2.9), we have
\[
a = \sum_{j=0}^{w_1-1} (-1)^j q^{w_2j} \int_{\mathbb{Z}_p} q^{(w_1-1)x_1} e^{(x_1 + w_2x + j \frac{w_2}{w_1})^{(w_1)}} d\mu_q(x_1)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{w_1-1} (-1)^j q^{w_2j} \tilde{E}_{n,q}^{(w_1)} \left( w_2x + j \frac{w_2}{w_1} \right) w_1^n \right) \frac{t^n}{n!}.
\]
By using the symmetry property in (2.11), we also have
\[
a = \sum_{j=0}^{w_2-1} (-1)^j q^{w_1j} \int_{\mathbb{Z}_p} q^{(w_2-1)x_2} e^{(x_2 + w_1x + j \frac{w_1}{w_2})^{(w_2)}} d\mu_q(x_2)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{w_2-1} (-1)^j q^{w_1j} \tilde{E}_{n,q}^{(w_2)} \left( w_1x + j \frac{w_1}{w_2} \right) w_2^n \right) \frac{t^n}{n!}.
\]
By comparing coefficients \( \frac{t^n}{n!} \) in the both sides of (2.11) and (2.12), we have the following theorem.

Theorem 2.5 Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we have
\[
\sum_{j=0}^{w_1-1} (-1)^j q^{w_2j} \tilde{E}_{n,q}^{(w_1)} \left( w_2x + j \frac{w_2}{w_1} \right) w_1^n = \sum_{j=0}^{w_2-1} (-1)^j q^{w_1j} \tilde{E}_{n,q}^{(w_2)} \left( w_1x + j \frac{w_1}{w_2} \right) w_2^n.
\]

Remark 2.6 Let \( w_1 \) and \( w_2 \) be odd positive integers. If \( q \to 1 \), we have
\[
\sum_{j=0}^{w_1-1} (-1)^j E_n \left( w_2x + j \frac{w_2}{w_1} \right) w_1^n = \sum_{j=0}^{w_2-1} (-1)^j E_n \left( w_1x + j \frac{w_1}{w_2} \right) w_2^n,
\]
where \( E_n(x) \) denotes the Euler polynomials (see [4, 8, 9]).
Substituting $w_1 = 1$ into (2.13), we arrive at the following corollary.

**Corollary 2.7** Let $w_2$ be odd positive integer. Then we obtain

\[
\tilde{E}_{n,q}(x) = \sum_{j=0}^{w_2-1} (-1)^j q^j \tilde{E}_{n/q}^{(w_2)} \left( \frac{x+j}{w_2} \right) w_2^n,
\]

where $\tilde{E}_{n,q}(x)$ denotes the $q$-Euler polynomials (see [8, 9]).

**References**


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