Finite Photon Exchange Amplitude in 6D Space-Time with Quasi-classical Compact Dimensions

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Abstract

We consider the electron-muon scattering in six dimensions with two extra dimensions compactified on a noncommutative (NC) Orbifold $T_\theta/\mathbb{Z}_2$. Then we use the coherent state basis of the NC Orbifold to demonstrate the finiteness of the photon exchange amplitude in a typical electron-muon scattering process.

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1 Introduction

As a recently proposed approach to the NC spaces, coherent state formalism has attracted some interest because of its success in solving the unitarity and finiteness problems associated with the fields defined over a NC space-time [1-6]. In this approach the coordinates spanning the “quasi-classical” space-time manifold, emerge as the mean values of the coordinate operators sandwiched between the ladder operators constructed from the noncommutating coordinates. The finiteness is achieved by appearance of a damping factor in momentum space propagator which acts as a UV cut-off over the diverging integrals. In this letter we examine the coherent state formalism in a different set up by exploring the photon mediated electron-muon scattering in $\mathbb{R}^4 \times T^2_\theta$ space-time where two extra dimensions are compactified on a NC torus. We show how the noncommutativity parameter appears as a UV cut off making the photon exchange amplitude finite. We recover the usual divergent amplitude when the noncommutativity of the torus disappears. In this work we use the Latin indices for the six dimensional (Euclidean) space-time and Greek indices for the four dimensional case.
2 Quasi-classical NC Space

We consider a two dimensional noncommutative space, which means

\[ [\hat{y}^2, \hat{y}^1] = i\theta, \]

(1)

The associated ladder operators are [1, 2]

\[ A = \frac{1}{\sqrt{2}}(\hat{y}^2 + i\hat{y}^1), \]

(2)

\[ A^\dagger = \frac{1}{\sqrt{2}}(\hat{y}^2 - i\hat{y}^1). \]

Coherent state \(|\alpha\rangle\) associated with the above operators are defined as

\[ A|\alpha\rangle = \alpha|\alpha\rangle, \]

(3)

\[ \langle\alpha|A^\dagger = \langle\alpha|\alpha^* \]

Now a function like \(\hat{F}(\hat{y})\) defined over the noncommutative plane will be replaced by its mean value as \(F_\theta(x) = \langle\alpha|\hat{F}(\hat{y})|\alpha\rangle\) where \(x^i = \langle\alpha|\hat{y}^i|\alpha\rangle\) [1, 2]. In particular for the massive particle propagator in six dimensional Euclidean space-time with \([\hat{y}^6, \hat{y}^5] = i\theta\) one finds [3-5]

\[ g_\theta(p) = \frac{e^{-\frac{\theta}{2}(p_6^2 + p_5^2)}}{p^M p_M + m^2}. \]

(4)

3 Photon Exchange Amplitude in \(\mathbb{R}^4 \times T^2_\theta\) Space-Time

The six dimensional Lagrangian of the free gauge field plus the gauge fixing term is

\[ \mathcal{L}_{6D} = -\frac{1}{4} \mathcal{F}^{MN} \mathcal{F}_{MN} - \frac{1}{2\zeta} \left( \partial_\mu A^\mu + \frac{1}{\zeta} \partial_m A^m \right)^2, \quad m = 5, 6. \]

(5)

Now we assume toroidal compactification of the extra dimensions with a \(\mathbb{Z}_2\) orbifolding by identifying the points via \(x^m \rightarrow -x^m, \quad m = 5, 6\) [6]. In \(\zeta \rightarrow 0\) the effective interaction term in four dimension becomes

\[ \mathcal{L}_{4D,int} = e\bar{\psi}(x)\gamma_\mu A^{(0)}(x)\psi(x) + e\sqrt{2}\sum_{\vec{n}\neq 0} \bar{\psi}(x)\gamma_\mu A^{(\vec{n})}(x)\psi(x). \]

(6)

with \(\vec{n} = 0\) or \(\vec{n} = (n_1 = 0, n_2 > 0)\) or \(\vec{n} = (n_1 > 0, n_2 = 0, \pm 1, \ldots)\) [6, 7]. Compactifying the extra dimensions on a torus with radii \(R_1\) and \(R_2\) requires the
quantization of momentum along the extra dimensions as \((k_5, k_6) \rightarrow \left(\frac{n_1}{R_1}, \frac{n_2}{R_2}\right)\) which leads to KK modes with an infinite mass tower given by \(m_n^2 = \left(\frac{n_1}{R_1}, \frac{n_2}{R_2}\right)\).

So from (4) for the photon propagator we find [7]

\[
D_{\mu\nu} = g_{\mu\nu} k^2 + m_n^2 e^{-\frac{g}{2} m_n^2} \tag{7}
\]

Thus the \(t\)-channel electron-muon scattering amplitude for \(e + \mu \rightarrow e' + \mu'\) process becomes

\[
\mathcal{M} = e^2 \bar{u}(p_e')\gamma^\mu u(p_e) \left(\frac{g_{\mu\nu}}{t} + 2 \sum_n \frac{g_{\mu\nu}}{t + m_n^2} e^{-\frac{g}{2} m_n^2}\right) \bar{u}(p_m')\gamma^\nu u(p_m) \tag{8}
\]

where \(t = (p_e' - p_e)^2\). We rewrite (8) as \(\mathcal{M} = \mathcal{A} + \mathcal{B}_\theta\). The first term, \(\mathcal{A}\), is the familiar (finite) photon exchange amplitude, while the second term, \(\mathcal{B}_\theta\) is the divergent contribution made by the infinite tower of the KK modes. So with the help of proper time parameterization

\[
\frac{1}{X} = \int_0^\infty ds e^{-sX} \tag{9}
\]

and the Jacobi theta function defined as

\[
\Theta(\nu, \tau) = \sum_{n=-\infty}^{+\infty} e^{i\pi n^2 \tau + 2i\pi n\nu}, \tag{10}
\]

we rewrite \(\mathcal{B}_\theta\) in (8) as

\[
\mathcal{B}_\theta = g e^{i\theta} \int_0^\infty ds e^{-st} \Theta\left(0, \frac{i s}{\pi R_1^2}\right) \Theta\left(0, \frac{i s}{\pi R_2^2}\right) \tag{11}
\]

where \(g = 2e^2 \bar{u}(p_e')\gamma^\nu u(p_e) \bar{u}(p_m')\gamma^\nu u(p_m)\). Therefore we observe how the noncommutativity parameter \(\theta\) appears as a UV cut-off at the lower limit of integral. Let us compare the above result with the scattering amplitude of a commutative toroidal internal space

\[
\mathcal{B} = g \int_{\Lambda^{-2}}^\infty ds e^{-st} \Theta\left(0, \frac{i s}{\pi R_1^2}\right) \Theta\left(0, \frac{i s}{\pi R_2^2}\right) \tag{12}
\]

Here one must introduce the cut-off \(\Lambda^{-2}\) to make the expression (12) finite due to the divergent behavior of Jacobi function for small values of \(\tau\) [8]:

\[
\Theta(0, \tau) \sim \tau^{-\frac{1}{2}} \tag{13}
\]

So, assuming the internal space to be noncommutative leads to a finite result for the scattering amplitude. As is expected, one recovers the graviton exchange amplitude (12) by taking the limit \(\theta \rightarrow 0\) in (11), i.e.

\[
\lim_{\theta \rightarrow 0} \mathcal{B}_\theta(t) \big|_{\frac{1}{\ell^2} = \Lambda^{-2}} = \mathcal{B}(t) \tag{14}
\]
provided that we identify \( \frac{\theta}{2} = \Lambda^{-2} \). One may replace the sum in (14) with an integral [8]. Thus we write

\[
B_\theta = g \int d^2q \frac{e^{-\frac{\theta}{2}q^2}}{t + q^2}
\]  

(15)

which with the aid of [9]

\[
\int_x^\infty \frac{ds}{s}e^{-s} = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-x)^n}{nn!2^n}
\]  

(16)

results in

\[
B_\theta = \pi ge^{\frac{\theta}{2}t} \left( -\gamma - \ln \frac{\theta}{2}t - \sum_{n=1}^{\infty} \frac{(-t\theta)^n}{nn!2^n} \right)
\]  

(17)

The factor \( \gamma \) is the Euler-Mascheroni constant. Similarly, for the commutative case we find

\[
B = g \int d^2q \frac{1}{t + q^2} = -\pi g \ln \frac{t}{\Lambda^2}
\]  

(18)

which coincides with (17) in \( \theta \to 0 \) limit on setting \( \frac{\theta}{2} = \Lambda^{-2} \). Again equation (17) implies the finiteness of scattering amplitude in \( \mathbb{R}^4 \times T_\theta^2 \) space-time.

4 CONCLUSIONS

In this note we showed that how the finiteness of quantum field theory within the framework of coherent state formalism (at least at one loop level) could be observed also at a tree level scattering process where a typical matter-matter scattering process takes place via exchanging of the photon in space-time with noncommutative toroidal compact dimensions.

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References


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