On the Nature of Fluctuations Associated with Fractional Poisson Process

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Abstract

The paper studies the nature of fluctuations associated with the fractional Poisson process (FPP): (i) The analytic expressions of the exact solutions for various initial conditions are displayed with the method of the inverse Lévy transform; (ii) The expression of the logarithmic cumulants of the waiting time distribution are also obtained; (iii) The Fano and the Allan factor of the event number fluctuation are obtained; (iv) The fractal and clustering nature of fluctuations of the FPP is discussed in conjunction with (a) experimental observations of spike counting in various neural systems, (b) doubly stochastic processes for the Poisson process, and (c) the Fokker-Planck approximation.

Keywords: fractal fluctuations, fractional poisson process, Fano and Allan factor, logarithmic-cumulants for the waiting-time distribution

1 Introduction

Analyses of neural spike train in brain and/or heart beat interval is regarded as an important fundamental information on inherent dynamics in biological systems. Various experimental and theoretical studies have been performed in vivo and in vitro [1-23] in various biological subsystems.

In the field of neural systems, they discuss two types of codes: “time code” associated with the inter spike interval (ISI), and “rate code” associated with the counting number distribution [1-23]. If the appearance of neural spike train is subjected to the homogeneous Poisson process, the ISI distribution has the exponential form, and the counting number distribution has the Poissonian
form. This is not the case in actual systems. Many experimental results in neural systems establish the fact that the ISI distribution in vitro is subjected to the Gamma distribution, not the exponential distribution. However, in vivo, its variability is large enough and it seems that the parameters of the Gamma process may be subjected to a kind of doubly stochastic processes. Up to now, various processes like inhomogeneous Poisson process [7], doubly Poisson process [1], doubly Gamma process semi-parametric model, mixture distribution ([14-15]), fractal point process [17-22] have been examined. However, the origin of high variability of spike activities, biological and physiological meaning of the activity are not clarified completely. Also, there is no successful unified theory to give consistent explanations of the various features of experimental findings in neural systems.

In the field of heart beat analysis, it is shown that there exist a long-memory in the fluctuations of the RR interval. Kiyono et al [24-26] tried to describe the nature of their fluctuations with long-memory based on the probability distribution function (pdf) of the RR interval and the scaling property of the pdf. In spite of extensive studies, there are only a few successful phenomenological stochastic dynamical models due to the mathematical difficulty for describing non-Poissonian processes with a long-memory and a large-variations.

In our previous paper [27-28], we have studied a generalized birth process which take into account the effect of memory and the number of state $n$ as

$$\frac{d}{dt}p(n, t) = \lambda(n - 1, t)p(n - 1, t) - \lambda(n, t)p(n, t) ,$$

(1)

where the rate of birth is assumed to have

$$\lambda(n, t) = \kappa(t)(\alpha n + \beta) ,$$

(2)

with arbitrary time-function $\kappa(t)$, and $\alpha, \beta$ are constants (cf. [29]). Also, we have studied more sophisticated one with the rate of birth as

$$\lambda(n, t) = \alpha(t)n + \beta(t) .$$

(3)

The model in eq.(1) with the birth rate in eq.(3) is also exactly solvable [30]. These are classified into the non-stationary type Master equation. It is found that there appears the mono-fractal nature of fluctuations, i.e., $\langle n(t) \rangle, \sigma_n(t)^2$, the Fano and the Allan factor are proportional to $t^\nu$ (0 < $\nu$ < 2 being fractional power) only when $\kappa(t), \alpha(t)$ and/or $\beta(t)$ takes the form $b/(1 + at)$.

In the framework of memory function type Master equation [31-32], the continuous time random walk (CTRW) is one of the most popular model in taking into account the effect of a long-memory. It is worth to compare the results of the memory-function formalism to those of the non-stationary one.

In this paper, we will study a fractional Poisson process (FPP) in the unified way that both the event counting process (the counting number distribution
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$p(n, t)$ and the waiting time process of event (the inter-event distribution $f(\tau)$) are studied simultaneously. Namely, with the use of the obtained exact solution for the FPP, both the rate code and the time code will be discussed associated with the nature of its fluctuation.

Section 2 describes the fractional Poisson process model, the inverse Lévy transform, the exact solution and the expressions of statistical fluctuations of the moments. Section 3 compares the FPP with a doubly stochastic process. Section 4 discusses theories and experiments of spike counting and interspike interval. The final section is devoted to summary and remarks.

2 Fractional Poisson Process

2.1 Model

Let us consider the fractional Poisson process (FPP) with fractional derivative $(0 < \mu \leq 1)$

$$\frac{D_t^\mu}{D_t} p_\mu(n, t) = \beta p_\mu(n - 1, t) - \beta p_\mu(n, t), \quad (4)$$

where $D_t^\mu$ is the Caputo fractional derivative defined by

$$D_t^\mu f(t) \equiv \frac{1}{\Gamma(1 - \mu)} \int_0^t \frac{f'(\tau)}{(t - \tau)^\mu} d\tau. \quad (5)$$

The other definition of fractional derivative due to Riemann-Liouville is defined by

$$D_t^\mu f(t) \equiv \frac{1}{\Gamma(1 - \mu)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t - \tau)^\mu} d\tau. \quad (6)$$

The relation between the two derivatives is

$$D_t^\mu f(t) = D_t^1 f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\mu}}{\Gamma(k - \mu + 1)} f^{(k)}(0^+). \quad (7)$$

Rewriting eq. (4) in the form,

$$\frac{\partial}{\partial t} p_\mu(n, t) = D_t^{1-\mu} \left[ \beta p_\mu(n - 1, t) - \beta p_\mu(n, t) \right], \quad (8)$$

one can see that the model in eq. (4) can be regarded as a memory function type Master equation.

2.2 Inverse Lévy Transform

An inverse Lévy transform is defined [32,33] by

$$p_\mu(n, t) = \int_0^\infty K_\mu(\tau, t)p_1(n, \tau) d\tau, \quad (9)$$
where the integral kernel $K_\mu(\tau, t)$ is expressed in terms of the one-sided cumulative Lévy distribution $L_\mu(z)$ (cf. Appendix A):

$$K_\mu(\tau, t) = \frac{d}{d\tau} \left[ 1 - L_\mu(t/\tau^{1/\mu}) \right]. \quad (10)$$

When the analytic solution for $\mu = 1$ (i.e., $p_1(n, t)$ is given), the solution of the fractional Poisson process $p_\mu(n, t)$ can be obtained. The generating function in the FPP (4) can be derived from $g_1(z, t)$ as

$$g_\mu(z, t) = \int_0^\infty K_\mu(\tau, t) g_1(z, \tau) \, d\tau. \quad (11)$$

### 2.3 Probability mass

Now let us derive the exact solution of probability mass $p_\mu(n, t)$ via the generating function $g_1(z, t)$ for $\mu = 1$, which is subjected to

$$\frac{\partial}{\partial t} g_1(z, t) = \beta(z - 1) g_1(z, t). \quad (12)$$

The solution under the initial condition (IC1) $p_1(n, 0) = \delta_{n,n_0}$ (i.e., $g_1(z, 0) = z^{n_0}$) is given by

$$g_1(z, t) = z^{n_0} \exp \left\{ \beta(z - 1)t \right\}. \quad (13)$$

With the use of the inverse Lévy transform in eq.(11), one obtains $g_\mu(z, t)$ for the IC1 (i.e., $g_\mu(z, 0) = z^{n_0}$) as

$$g_\mu[z, s] = z^{n_0} E_\mu \left( \beta t^\mu (z - 1) \right), \quad (14)$$

where $E_\mu(z)$ is the Mittag-Leffler function defined by

$$E_\mu(z) = \sum_{m=0}^\infty \frac{z^m}{\Gamma(\mu m + 1)}. \quad (15)$$

The probability mass function for $p(n, 0) = \delta_{n,n_0}$ is obtained from $g_\mu(z, t)$ as

$$p_\mu(n, t) = \frac{1}{n!} \frac{\partial^n}{\partial z^n} g_\mu(z, t) \bigg|_{z=0} = \left( \beta t^\mu \right)^{(n-n_0)} \frac{\infty}{(n-n_0)!} \sum_{k=0}^\infty \frac{(k + n - n_0)!}{k!} \frac{(-\beta t^\mu)^k}{\Gamma(\mu(k + n - n_0) + 1)} \cdot (n \geq n_0). \quad (16)$$

When the initial condition cannot be controllable, one must take the average over the initial values. For the initial condition (IC2) $p(n, 0) = \delta_{n,n_0}$
\( \delta_{n,n_0}(\theta)^n \exp(-\theta)/n! \) (i.e., the Poisson distribution with the parameter \( \theta \)), one obtains the generating function which is averaged over the initial condition as

\[
g_\mu(z,t) = \exp(\theta(z-1))E_\mu\left(\beta(z-1)t^\mu\right).
\]

(17)

The pmf for the IC2 is obtained after the summation over the initial value \( n_0 \) as

\[
p_\mu(n,t) = \frac{1}{n!} \frac{\partial^n}{\partial z^n} g_\mu(z,t) \bigg|_{z=0} = \sum_{k=0}^{n} \frac{(-x)^{n-k}\theta^k}{(n-k)!k!} E^{(n-k)}_\mu(x) \bigg|_{x=-\beta t^\mu},
\]

(18)

where \( E^{(k)}_\mu(x) \) is defined by

\[
E^{(k)}_\mu(x) = \frac{d^k}{dx^k} E_\mu(x).
\]

(19)

In any case, the generating function and the pmf are expressed in terms of the Mittag-Leffler function \( E_\mu(x) \) and its derivatives \( E^{(k)}_\mu(x) \). Figure 1 shows the profile of the Mittag-Leffler function \( E_\mu(-t^\mu) \) [(a) \( \mu = 1 \) (solid line); (b) \( \mu = 0.75 \) (dotted line); (c) \( \mu = 0.50 \) (dashed line); (d) \( \mu = 0.25 \) (dash-dot line)] as a function of time \( t \).

![Mittag-Leffler function profile](image)

Figure 1: Behavior of the Mittag-Leffler function \( E_\mu(-t^\mu) \) [(a) \( \mu = 1 \) (solid line); (b) \( \mu = 0.75 \) (dotted line); (c) \( \mu = 0.50 \) (dashed line); (d) \( \mu = 0.25 \) (dash-dot line)]

### 2.4 Moments

The moments of the counting number during the region \( T \) in the fractional Poisson process under the IC1 are obtained with the generating function in eq.(14) as

\[
\langle n(T) \rangle = n_0 + \frac{\beta T^\mu}{\Gamma(\mu + 1)},
\]

(20)
\[ \langle n(T)^2 \rangle = n_0^2 + (2n_0 + 1) \frac{\beta T^\mu}{\Gamma(\mu + 1)} + 2(\beta T^\mu)^2 \frac{2}{\Gamma(2\mu + 1)}, \tag{21} \]

and

\[ \langle n(T)^3 \rangle = n_0^3 + (3n_0^2 + 3n_0 + 1) \frac{\beta T^\mu}{\Gamma(\mu + 1)} + 6(n_0 + 1) \frac{(\beta T^\mu)^2}{\Gamma(2\mu + 1)} + 6 \frac{(\beta T^\mu)^3}{\Gamma(3\mu + 1)}, \tag{22} \]

where \( \langle X(n) \rangle \) denotes the statistical average of a variable \( X(n) \) as a function of \( n \): \( \langle X(n) \rangle = \sum_{n=0}^{\infty} X(n) P(n,t) \). It is easy to see that the variance is independent of \( n_0 \) as

\[ \sigma_n^2(T) = \langle n(T)^2 \rangle - \langle n(T) \rangle^2 = \frac{2(\beta T^\mu)^2}{\Gamma(2\mu + 1)} + \frac{\beta T^\mu}{\Gamma(\mu + 1)} - \left( \frac{\beta T^\mu}{\Gamma(\mu + 1)} \right)^2. \tag{23} \]

When the IC2 is adopted, one obtains with the generating function in eq. (17) as

\[ \langle n(T) \rangle = \theta + \frac{\beta T^\mu}{\Gamma(\mu + 1)}, \tag{24} \]

\[ \langle n(T)^2 \rangle = \theta^2 + (2\theta + 1) \frac{\beta T^\mu}{\Gamma(\mu + 1)} + \frac{2(\beta T^\mu)^2}{\Gamma(2\mu + 1)}, \tag{25} \]

and

\[ \langle n(T)^3 \rangle = \theta^3 + (3\theta^2 + 3\theta + 1) \frac{\beta T^\mu}{\Gamma(\mu + 1)} + 6(\theta + 1) \frac{(\beta T^\mu)^2}{\Gamma(2\mu + 1)} + 6 \frac{(\beta T^\mu)^3}{\Gamma(3\mu + 1)}. \tag{26} \]

The averaged variance \( \overline{\sigma^2(T)} \) takes the same form as \( \sigma^2(T) \) as eq. (23). Their expressions under the Fokker-Planck approximation [33-35] are shown in Appendix C.

### 2.5 Fano and Allan factors

The nature of fluctuations is often discussed in conjunction with the Fano factor and/or the Allan factor [17-22]. Let the sequence of event numbers (counts) be denoted by \( \{ Z_n \} \). The Fano factor in the time interval \( T \) is defined as the event-number variance divided by the event-number mean:

\[ FF(T) \equiv \frac{Var(Z_n(T))}{E[Z_n(T)]} = \frac{\sigma_n^2(T)}{\langle n(T) \rangle} \tag{27} \]

The quantity \( FF(T) \) has been used as an indicator of random, coherent and squeezed states (i.e., super-Poisson \( FF(T) > 1 \), Poisson \( FF(T) = 1 \) and sub-Poisson \( FF(T) < 1 \) statistics) of light in the photon counting processes.
On the other hand, the Allan factor $AF(T)$ with the counting time $T$ is the ratio of the event-number Allan variance to twice the mean:

$$AF(T) = \frac{E[(Z_n(T) - Z_{n+1}(T))^2]}{2E(Z_n(T))} = 2FF(T) - FF(2T). \quad (28)$$

It has been used as an indicator of clustering of events.

When $p(n, 0) = \delta_{n,0}$ (i.e., the IC1 with $n_0 = 0$), one obtains

$$FF(T) = \frac{\beta T^\mu}{\Gamma(\mu + 1)} \left[ \frac{\mu B(\mu, 1/2)}{2^{2\mu - 1}} - 1 \right] + 1 \quad (29)$$

and

$$AF(T) = \frac{\beta (2T^\mu - (2T)^\mu)}{\Gamma(\mu + 1)} \left[ \frac{\mu B(\mu, 1/2)}{2^{2\mu - 1}} - 1 \right] + 1. \quad (30)$$

Namely, the FPP process ($0 < \mu < 1$) in the initial condition $p(n, 0) = \delta_{n,0}$ is subjected to the super-Poissonian statistics ($FF(T) > 1$). Also, the spiking event in the FPP has a tendency of clustering stronger than that of the Poisson process with $\mu = 1$ ($AF(T) > 1$). One must remind that when $\mu = 1$, i.e., in the Poisson process, $\langle n(T) \rangle = \sigma_n^2(T) = FF(T) = AF(T) = 1$.

When the IC2 is adopted, the effect of distribution of $n_0$ with the parameter $\theta$ appears in the expression of $FF(T)$ and $AF(T)$ in the small values of $T$ as shown in Fig. 2.

![Figure 2: Fano Factor $FF(T)$ (solid line) and Alan Factor $AF(T)$ (dotted line) as a function of counting time $T$ ($\mu = 0.5$ and $\beta = 0.5$). Fano Factor $FF(T)$ (dashed line, $\theta = 0.1$; dash-tree-dots line, $\theta = 1$) and Alan Factor $AF(T)$ (dash-dot line, $\theta = 0.1$; long-dashed line, $\theta = 1$) as a function of counting time $T$ ($\mu = 0.5$ and $\beta = 0.5$).](image)
2.6 Interspike Interval Distribution

In the case of the fractional Poisson process (FPP) \((\beta > 0 \text{ and } 0 < \mu < 1)\), the occurrence of spikes is independent phenomenon. And the interspike interval (ISI) distribution \(f_{\mu,\text{ISI}}(\tau)\) can be derived exactly. The resulting ISI distribution becomes

\[
f_{\mu,\text{ISI}}(\tau) = \beta \tau^{\mu-1} E_{\mu,\mu}(-\beta \tau^\mu),
\]

where \(E_{\mu,\nu}(x)\) is a generalized Mittag-Leffler function defined by

\[
E_{\mu,\nu}(x) \equiv \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\mu + \nu)} x^n.
\]

In the Poisson process with \(\mu = 1\), the pdf in eq.(31) reduces to the exponential distribution \(f_{1,\text{ISI}}(\tau) = \beta \exp(-\beta \tau)\) with \(M(m) \equiv \langle \tau^m \rangle = \frac{m!}{\beta^m} (m = 1, 2, \ldots)\) and \(\sigma_\tau^2 = \frac{1}{\beta^2}\). So, the coefficient of variation (CV) becomes \(C_v \equiv \sigma_\tau / \langle \tau \rangle = 1\). Also, the skewness coefficient (CS) becomes \(C_s \equiv M_{3,\tau}(3) / \sigma_\tau^3 = 2\).

In the FPP, on the other hand, all the integer-order moments \(M(m) (m = 1, 2, 3, \cdots)\) diverge \(M(m) \equiv \langle \tau^m \rangle = \infty\). Consequently, \(C_v\) and \(C_s\) are not determined. This is due to the fact that the ISI distribution has a fat tail. One should note here that the fractional order moment \(M(\rho) (0 < \rho < 1)\) exists. Actually, one obtains

\[
M(\rho) \equiv \langle \tau^\rho \rangle = \beta^{-\rho} \frac{\Gamma(1 - \frac{\rho}{\mu}) \Gamma(1 + \frac{\rho}{\mu})}{\Gamma(1 - \rho)}.
\]

Figure 3: The profile of the ISI distribution \(f_{\mu,\text{ISI}}(\tau)\): (a) \(\mu = 1\) (solid line); (b) \(\mu = 0.75\) (dotted line); (c) \(\mu = 0.50\) (dashed line); (d) \(\mu = 0.25\) (dash-dot line)
Figure 4: The feature of divergence of the fractional moment $\langle \tau^\rho \rangle$ for the ISI distribution $f_{\mu,\text{ISI}}(\tau)$: (a) $\mu = 0.99$ (solid line); (b) $\mu = 0.75$ (dotted line); (c) $\mu = 0.50$ (dashed line); (d) $\mu = 0.25$ (dash-dot line)

The divergence of the ordinary moments remind us that a two-sided probability distribution in a generalized Cauchy process\[36\], the Lévy process\[37\] and the multiplicative log-normal distribution\[38\]. In these cases, one can avoid divergence by adopting the maximum likelihood estimators (MLEs), i.e., $\langle \ln(x^2 + a^2) \rangle$ and $\langle (x^2 + a^2)^{-1} \rangle$. Although they have non-divergent nature, they are not convenient since the parameter $a$ is involved in the MLEs. To improve the deficiency, one can adopt the logarithmic moments $\langle \ln |x| \rangle_c$ (or cumulants $\langle \ln |x|^m \rangle_c$), which do not diverge for any values of parameters in these distributions. Thus, the logarithmic cumulants are examined here. The results are shown below.

$$LogC_1 \equiv \langle \ln \tau \rangle_c = \psi(1) - \frac{1}{\mu} \ln \beta,$$

where $\psi(z)$ is the di-gamma function [$\psi(1) = -\gamma$ ($\gamma = -0.57721566$ is the Euler’s constant)].

$$LogC_2 \equiv \langle (\ln \tau)^2 \rangle_c = \left( \frac{2}{\mu^2} - 1 \right) \psi'(1) - 2 \left( \frac{1}{\mu} \ln \beta \right) \psi(1) + \left( \frac{1}{\mu} \ln \beta \right)^2,$$

where $\psi'(z)$ is the tri-gamma function [$\psi'(1) = \pi^2/6$]. When the two quantities $\langle \ln \tau \rangle_c$ and $\langle (\ln \tau)^2 \rangle_c$ are measured, one can infer the parameters ($\beta, \mu$). The third order cumulant takes the form:

$$LogC_3 \equiv \langle (\ln \tau)^3 \rangle_c = \psi''(1) - 3 \left( \frac{2}{\mu^2} - 1 \right) \left( \frac{1}{\mu} \ln \beta \right) \psi'(1) + 3 \left( \frac{1}{\mu} \ln \beta \right)^2 \psi(1) - \left( \frac{1}{\mu} \ln \beta \right)^3,$$

where $\psi''(z)$ is the tetra-gamma function [$\psi''(1) = -2\zeta(3)$], and $\zeta(z)$ is the zeta function defined by $\zeta(z) = \sum_{n=1}^{\infty} (1/n^z)$. 

The odd order logarithmic moments for $x = \tau \beta^{3/\mu}$ reduce to simple expressions in terms of the poly-gamma functions (cf. Appendix B). It is easy to estimate the parameters $(\beta, \mu)$ by using (34) and (35).

3 Comparison with a doubly stochastic process

3.1 probability mass

Many theoretical works have been done based on the idea due to Cox [1], which is called “the doubly stochastic process (DSP)” or “the Cox process”. This approach is essentially equivalent to the idea of mixture distributions.

Let us consider the mixture distribution under the assumption that the parameter $\beta$ in the Poisson process is subjected to a known stochastic process. Consider here the case that the process is characterized by the Gamma distribution with two parameters $(x, y)$ as

$$f_g(\beta) = \frac{y^x}{\Gamma(x)} \beta^{x-1} e^{-y\beta}.$$  \hspace{1cm} (37)

In this case, the counting number distribution $p_1(n, t)$ becomes the negative binomial distribution:

$$p_1(n, t) = \int_0^{\infty} p_1(n, t|\beta) f_g(\beta) \, d\beta = \frac{y^x \Gamma(n + x)}{n! \Gamma(x)} \frac{t^n}{(t + y)^{n+x}}.$$ \hspace{1cm} (38)

The mean and the variance with the counting time $T$ for this case become

$$\langle n(T) \rangle = \frac{x}{y} T$$ \hspace{1cm} (39)

and

$$\sigma_n^2(T) = \frac{x T}{y^2} (T + y).$$ \hspace{1cm} (40)

Thus, the Fano and the Allan factor become

$$FF(T) = 1 + \frac{T}{y} \quad \text{and} \quad AF(T) = 1.$$ \hspace{1cm} (41)

In the doubly stochastic process described in eq.(38), the variance can take a larger value than the mean (i.e., the super-Poisson statistics, $FF(T) > 1$), and the clustering nature does not appear (i.e., $AF(T) = 1$).
3.2 ISI distribution

How is the properties of the corresponding ISI distribution of the doubly stochastic process in eq.(38). Since $f_E(\tau|\beta) = \beta e^{-\beta \tau}$, one obtains the ISI distribution under the Gamma law in eq.(37) as

$$f_{1,ISI}(\tau) = \int_0^\infty f_E(\tau|\beta) f_g(\beta) \, d\beta = \frac{x}{y(1 + \tau/y)^{1+x}}.$$ (42)

This is the Pareto distribution with the parameter set $(x, y)$. Interestingly, the pdf is identical with the power-law waiting time distribution which has been discussed in continuous-time random walks (CTRWs) [31, 34-35]. The mean and the variance are estimated for $x > 2$ as

$$\langle \tau \rangle = \frac{y}{(x-1)} \, (x > 1) \quad \text{and} \quad \sigma^2_\tau = \frac{xy^2}{(x-2)(x-1)^2} \, (x > 2).$$ (43)

When $0 < x < 2$, the mean and the variance diverge even though $\mu = 1$ due to the fat-tail of the Pareto distribution. When $x > 2$, the coefficient of variation (CV) reduces to

$$C_v = \frac{\sigma_\tau}{\langle \tau \rangle} = \sqrt{x}/(x-2) > 1 \, (x > 2).$$ (44)

It is clear that the CV $(C_v)$ takes larger value than 1 for $x > 2$. Also, the skewness coefficient $(C_s)$ becomes

$$C_s = \frac{M_{c,\tau}(3)}{\sigma^3_\tau} = \frac{2(x+1)\sqrt{x-2}}{\sqrt{x(x-3)}} \, (x > 3).$$ (45)

The fractional order moment for positive real values of $\rho (> 0)$ is obtained as

$$\langle \tau^\rho \rangle = xy^\rho B(x-\rho, 1+\rho),$$ (46)

where the parameter value of $x$ is limited in the range $x > \rho > 0$. In this way, introducing the distribution of $\beta$, the large variability is generated. Consequently, $C_s$ takes a value larger than 1, which is consistent with the experimental observation in neural systems.

3.3 MLEs and Logarithmic cumulants

This quantity diverges for $0 < x < \rho$. To avoid the divergence of the value of the moments and their related statistical quantities, one can take the maximum likelihood estimators (MLEs) for the ISI distribution. Since the pdf can be rewritten in the form of the exponential family as

$$f_{1,ISI}(\tau) = \exp \left( -(1+x) \ln(\tau + y) - \Phi(x,y) \right),$$ (47)
where $\Phi(x, y)$ is the information geometrical potential [39],

$$
\Phi(x, y) = -(\ln x + x \ln y).
$$

It is easy to verify that the maximum likelihood estimators (MLEs) for the distribution in eq.(48) become

$$
\langle \ln(\tau + y) \rangle = -\frac{\partial}{\partial x} \Phi(x, y) = \frac{1}{x} \ln y
$$

and

$$
\langle \frac{1}{\tau + y} \rangle = -\frac{1}{1 + x} \frac{\partial}{\partial y} \Phi(x, y) = \frac{x}{(1 + x)y}.
$$

It is not convenient to use these quantity since the parameter $y$ is contained in the both sides of the expressions of eqs.(49)-(50). The same situation can be seen in the case of a generalized Cauchy distribution [36]. To improve the inconvenience, one can adopt the logarithmic cumulants as

$$
\log C_1 \equiv \langle \ln \tau \rangle_c = \ln y - \psi(x) + \psi(1)
$$

$$
\log C_2 \equiv \langle (\ln \tau)^2 \rangle_c = \psi'(x) + \psi'(1)
$$

and

$$
\log C_3 \equiv \langle (\ln \tau)^3 \rangle_c = -\psi''(x) + \psi''(1).
$$

In this case, $\log C_1$ is a function of the two parameters $x$ and $y$. On the other hand, $\log C_2$ and $\log C_3$ are the functions depending only the parameter $x$. In any case, they take finite values for arbitrary set of parameter values $(x, y)$. In deriving the results, it is utilized the mathematical formula as

$$
\int_0^\infty \frac{\ln u}{(1 + u)^{1 + x}} du = -\frac{\psi(x) + \gamma}{x}.
$$

4 Discussions

4.1 Relation to theory and experiment (I)

Lowen and Teich [17-22] have been studied various fractal-point process based on phenomenological view points to describe various experimental features observed in many neural systems: (i) neurotransmitter secretion (NS) [22], (ii) auditory nerve fiber (ANF) [18,22], (iii) retinal ganglion cells (RGC) [21,22], (iv) lateral geniculate nucleus (LGN) [22], (v) visual center area striata) [22] as shown in Table 1.

Table 1 Observed values of $\alpha_A$ ($AF(T) \propto T^{\alpha_A}$ due to Lowen, Teich and their coworkers [21,22]) in neural systems (RGC and LGN)
Their mathematical description is based on the long-tailed relaxation, not based on the rate of occurrence of neural spike. It is reported that the distribution of inter spike interval (ISI) in their various experiments is reproduced well [18] by

$$f_{ISI}(\tau) = \frac{(AB)^{\ell/2}}{2K_\ell(2\sqrt{A/B})} \exp\left(-\frac{A}{\tau} - \frac{\tau}{B}\right) \cdot \tau^{-(\ell+1)}, \quad (55)$$

where $K_\ell(x)$ is the $K$-Bessel function of order $\ell$. Also, they have demonstrated numerically that the fractional power law $T^\alpha$ nature of the Fano FF($T$) and the Allan factor AF($T$) ($FF(T) \propto T^{\alpha_F}, AF(T) \propto T^{\alpha_A}$) can be reproduced by various Gaussian noise driven fractional point processes (GNDFPP) [21,22]. It is clear that the fractal nature of spike number fluctuation in their theory is an incorporated fractional noise. Also the ISI distribution is not derived from the GNDFPP.

By introducing the information geometrical potential $\Phi(A,B,\ell)$ as seen in section 3.3, we can see that the maximum likelihood estimators (MLEs) for the ISI distribution in eq.(55) are $\langle \tau^{-1} \rangle$, $\langle \tau \rangle$ and $\langle \ln \tau \rangle$. The expression of the $m$-th ($m = \pm 1, \pm 2, \cdots$) order moment is given by

$$\langle \tau^m \rangle = (AB)^{m/2} \frac{K_{\ell-m}(2\sqrt{A/B})}{K_{\ell}(2\sqrt{A/B})}. \quad (56)$$

Also, the logarithmic moment $\langle \ln \tau \rangle$ can be evaluated according to the formula

$$\langle \ln \tau \rangle = \frac{1}{2} \ln(AB) - \frac{1}{K_{\ell}(2\sqrt{A/B})} \cdot \frac{\partial}{\partial \ell} K_{\ell}(2\sqrt{A/B}). \quad (57)$$

These MLEs take finite values for any values of parameters. One can estimate the parameters ($A, B, \ell$) by using the MLEs which are expressed in terms of $K_\ell(z)$, in principle. It is not easy to evaluate the derivative of the $K$-Bessel function with respect to $\ell$. Inferring ($A, B, \ell$) without the mathematical difficulty, one can use the information on the set of moments ($\langle \tau^{-1} \rangle$, $\langle \tau \rangle$, $\langle \tau^2 \rangle$ and $\langle \tau^3 \rangle$) as shown in Appendix D.

The expression of the CV for eq.(55) becomes

$$C_v = \sqrt{AB\left(\frac{K_{\ell-2}(2\sqrt{A/B})K_{\ell}(2\sqrt{A/B})}{K_{\ell-1}(2\sqrt{A/B})} - 1\right)}^{\frac{1}{2}}. \quad (58)$$
Figure 5 shows the variation of the CV as a function of $2\sqrt{A/B}$ for $B = 2$. The CV takes both smaller and larger value than 1 depending on the parameters $(A, B)$.

Figure 5: The CV for the ISI distribution in eq.(55) for $\ell = 2$ (solid line), $\ell = 3$ (dotted line) and $\ell = 4$ (dashed line) with $B = 2$.

4.2 Relation to theory and experiment (II)

Tsubo et al reported [15] that an experimental ISI distribution in vitro can be reproduced well by the beta distribution of the second kind in the form:

$$f_{ISI}(\tau) = \frac{y^x}{B(z, x)} \frac{\tau^{z-1}}{(\tau + y)^{z+x}}.$$  \hspace{1cm} (59)

In spite of the success to identify the experimental ISI distribution, it is difficult to derive the corresponding counting number distribution.

The ISI distribution is derived under the assumption of the doubly Gamma mixtures. How is the nature of fluctuations in this case. Based on the pdf, one obtains the expressions of the mean and the variance as

$$\langle \tau \rangle = \frac{zy}{x-1} \text{ and } \sigma^2_\tau = \frac{y^2z(x + x - 1)}{(x-1)^2(x-2)}.$$  \hspace{1cm} (60)

Also, the expressions of the CV and the CS become

$$C_v = \sqrt{\frac{z + x - 1}{z(x - 2)}} \text{ and } C_s = \frac{2(2z + x - 1)\sqrt{x - 2}}{\sqrt{z(x + x - 1)}}.$$  \hspace{1cm} (61)

In this case, these statistical quantities take finite values provided that $x > 2, z + x - 1 > 0$. Otherwise, their values diverge. Let us examine the logarithmic cumulants:

$$LogC_1 \equiv \langle \ln \tau \rangle_c = \ln y - \psi(x) + \psi(z),$$  \hspace{1cm} (62)
\[ \text{Log}C_2 \equiv \langle (\ln \tau)^2 \rangle_c = \psi'(x) + \psi'(z) \] 

(63)

and

\[ \text{Log}C_3 \equiv \langle (\ln \tau)^3 \rangle_c = -\psi''(x) + \psi''(z). \] 

(64)

It is easy to see that they take finite for arbitrary values of \( x > 0, y > 0 \) and \( z > 0 \). One can determine the set of parameters \((x, y, z)\) from \( \text{Log}C_1, \text{Log}C_2 \) and \( \text{Log}C_3 \). When \( z = 1 \), the formula reduces exactly to the formula obtained in eqs.(51)-(53).

5 Conclusion

(1) In this paper, we have obtained the exact solution of the probability mass of counting number for the fractional Poisson process (FPP) in eq.(4) \((0 < \mu < 1)\) for the fixed initial condition \( p(n, 0) = \delta_{n,n_0} \) with the use of the inverse Lévy transform (cf. Ref.[40]). The feature of solution under the Poisson distribution of initial values \( p(n, 0) = \delta_{n,n_0}(\theta)^n \exp(-\theta)/n! \) is also elucidated.

(2) The ISI distribution for the FPP is obtained though there is a long-memory. The expressions of the logarithmic cumulants \( \langle \log \tau \rangle_c^j \), \( j = 1 - 3 \) are obtained for any parameter values of the model, although the ordinary integer order moments diverge. It is shown that the parameters of the model can be inferred by the \( \langle \log \tau \rangle_c^j \).

(3) The fractional power laws of the Fano and the Allan factor in neural counting statistics is obtained. Due to the existence of a long-memory, the tendency of clustering arises (i.e., the Allan factor take a larger value than 1: \( AF(T) > 1 \)).

(4) The above results are partially consistent with the observations of the fractal nature of fluctuation \( FF(T) \propto T^{\alpha_F} \) and \( AF(T) \propto T^{\alpha_A} \) \((0 < \alpha_F < 2 \text{ and } 0 < \alpha_A < 2)\) in various systems described by Lowen and Teich [17-22] and spike counting experiment by Liberman [2] in auditory nerve systems in the range of fractal dimension, \( 0 < \alpha_F = \alpha_A = \mu < 1 \). It is necessary to introduce a new mechanism in explaining the fractal nature in the range \( 1 < \alpha_F = \alpha_A < 2 \).

(5) The FP approximation gives rise to a good estimates of the lower order moments. Actually, the mean and the variance agree exactly with those in the fractional Poisson process.

(6) The nature of fluctuations associated with the fractional generalized birth process

\[ \frac{D^n}{D t^n} p_\mu(n, t) = [\alpha(n - 1) + \beta] p_\mu(n - 1, t) - [\alpha n + \beta] p_\mu(n, t), \] 

(65)

is under investigation, which will be published elsewhere.
ACKNOWLEDGEMENTS. This work is partially supported by the JSPS, No. 20500251 and No. 24650147.

References


Appendix A

Integral Kernel of Lévy Transform and Its Laplace Transform

The density of the one-sided Lévy distribution $\ell_\mu(z)$ is expressed in terms of the Gamma function as

$$\ell_\mu(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(1/\mu - s/\mu)}{\mu\Gamma(1 - s)} z^{-s} \, ds$$  \hspace{1cm} (A1)

In the form of Fourier series, (A1) is written in the form:

$$\ell_\mu(z) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(1 + n\mu)}{n!} (-1)^n \sin(\pi n\mu) z^{-1+n\mu}.$$  \hspace{1cm} (A2)

The cumulative distribution should be integrated in the range $[0, t]$ as

$$L_\mu(t) = \int_0^t \ell_\mu(z) \, dz.$$  \hspace{1cm} (A3)

Therefore, the kernel function in eq.(9) is expressed as

$$K_\mu(\tau, t) = \frac{d}{d\tau} \left[ 1 - L_\mu(t/\tau^{1/\mu}) \right] = \frac{t}{\mu \tau^{1+1/\mu}} \ell_\mu\left( \frac{t}{\tau^{1/\mu}} \right).$$  \hspace{1cm} (A4)

The Laplace transform of the kernel function (A4) is given by

$$\hat{K}_\mu(\tau, s) = \int_0^\infty e^{-st} K_\mu(\tau, t) \, dt = s^{\mu-1} \exp(-s^\mu \tau).$$  \hspace{1cm} (A5)

Appendix B

Derivation of Logarithmic Cumulants of the ISI distribution

The ISI distribution in eq.(31) is expressed in the dimension less form as

$$f_{\mu,\text{ISI}}(x) = x^{\mu-1} E_{\mu,\mu}(-x^\mu),$$  \hspace{1cm} (B1)

where the variable transformation from $\tau$ to $x$ is defined by

$$x = \beta^{1/\mu} \tau.$$  \hspace{1cm} (B2)
In this case, the fractional \((0 < \rho < 1)\) order moment in eq.(33) is expressed as
\[
\langle x^\rho \rangle = \frac{\Gamma(1-\rho/\mu)\Gamma(1+\rho/\mu)}{\Gamma(1-\rho)}. \tag{B3}
\]
By taking the derivative of eq.(B3) \(n\) times with respect to \(\rho\), and taking the limit \(\rho \to 0\), one obtains the logarithmic \(n\)-th order cumulants \((\text{Log}C_n; n=1,2,3\text{ and }4)\) as
\[
\text{Log}C_1 \equiv \langle \ln x \rangle_c = \psi(1), \tag{B4}
\]
\[
\text{Log}C_2 \equiv \langle (\ln x)^2 \rangle_c = \left(\frac{2}{\mu^2} - 1\right)\psi(1)', \tag{B5}
\]
\[
\text{Log}C_3 \equiv \langle (\ln x)^3 \rangle_c = \psi(1)'' \tag{B6}
\]
and
\[
\text{Log}C_4 \equiv \langle (\ln x)^4 \rangle_c = \left(\frac{2}{\mu^4} - 1\right)\psi(1)'' + 3\left(\frac{2}{\mu^2} - 1\right)^2(\psi(1)')^2 \tag{B7}
\]
Substituting (B2) into (B4), (B5) and (B6), one obtains the expressions of the logarithmic cumulants as shown in eqs.(34)-(36).

**Appendix C**

**Fokker-Planck approximation**

For large \(n >> 1\) (provided that \(n\) is a continuous variable in \([0, \infty]\)), one can take the Taylor expansion by the truncation up to the second order
\[
p_\mu(n-1,t) \approx p_\mu(n,t) - \frac{\partial}{\partial n} p_\mu(n,t) + \frac{1}{2} \frac{\partial^2}{\partial n^2} p_\mu(n,t) \tag{C1}
\]
one obtains the fractional Fokker-Planck equation as
\[
0D_\mu t p_\mu(n,t) = -\beta \frac{\partial}{\partial n} p_\mu(n,t) + \frac{1}{2} \beta \frac{\partial^2}{\partial n^2} p_\mu(n,t) \tag{C2}
\]
An approximate analytic probability density function (pdf) for \(\mu = 1\) is obtained in the form:
\[
p_1(n,t) \approx \frac{1}{\sqrt{2\pi \langle n(t) \rangle}} \exp \left[ -\frac{(n - \langle n(t) \rangle)^2}{2\langle n(t) \rangle} \right], \tag{C3}
\]
where \(\int_0^\infty p_1(n,t)dn \neq 1\) since \(n\) is defined in the region \([0, \infty]\). With the use of the inverse Lévy transform, \(p_\mu(n,t)\) is obtained in the form:
\[
\hat{p}_\mu[n, s] = \int_0^\infty \hat{K}_\mu[\tau, s]p_1(n, \tau) d\tau = \frac{s^{\mu-1}}{\beta \sqrt{1 + \frac{2}{\beta}s^\mu}} \exp \left[ n\left( 1 - \sqrt{1 + \frac{2}{\beta}s^\mu} \right) \right]. \tag{C4}
\]
It is easy to see that \( \int_{0}^{\infty} \hat{\rho}_{\mu}[n,s]dn \neq \frac{1}{n} \) (cf. Brakai (2001)).

By imposing the exact normalization condition on the pdf, \( \int_{0}^{\infty} p_{\mu}(n, t)dn = 1 \), the initial condition \( p_{\mu}(n, 0) = \delta_{n,0} \) and the boundary conditions \( p_{\mu}(0, t) = \frac{\partial}{\partial n} p_{\mu}(t, 0) = \frac{\partial}{\partial n} p_{\mu}(\infty, t) = 0 \), one obtains the first, the second and the third moments of the counting number for a counting time \( T \) as

\[
\langle n(T) \rangle = \frac{\beta T^\mu}{\Gamma(\mu+1)}, \langle n(T)^2 \rangle = \frac{2(\beta T^\mu)^2}{\Gamma(2\mu+1)} + \frac{\beta T^\mu}{\Gamma(\mu+1)}
\]

and \( \langle n(T)^3 \rangle = \frac{6(\beta T^\mu)^3}{\Gamma(3\mu+1)} + \frac{6(\beta T^\mu)^2}{\Gamma(2\mu+1)} + \frac{\beta T^\mu}{\Gamma(\mu+1)} \). \hspace{1cm} (C5)

As far as the counting statistics in the lower order quantities, up to the second moment, the FP approximation is satisfactory to describe the FPP model. Actually, the exact value of the third moment in the fractional Poisson process is obtained from eq.(22) with \( n_0 = 0 \) in the form:

\[
\langle n(T)^3 \rangle = \frac{6(\beta T^\mu)^3}{\Gamma(3\mu+1)} + \frac{6(\beta T^\mu)^2}{\Gamma(2\mu+1)} + \frac{(\beta T^\mu)}{\Gamma(\mu+1)} \). \hspace{1cm} (C6)

Interestingly, the mean and the variance estimated under the FP approximation with the boundary conditions agree the true values up to the second order moment. The third order moment takes a smaller value than the true one.

**Appendix D**

**Estimation of Parameters \((A, B, \ell)\)**

The expressions of moments are expressed in terms of the \(K\)-Bessel function \(K_\ell(z)\) in eq.(56). Noticing the recurrence relation of the \(K\)-Bessel function,

\[
K_{\ell-1}(z) - K_{\ell+1}(z) = -\frac{2\ell}{z} K_{\ell}(z)
\]

one obtains the matrix equation for the parameters \((A, B, \ell)\) as

\[
L \begin{pmatrix} A \\ B^{-1} \\ \ell \end{pmatrix} = \begin{pmatrix} 0 \\ -\langle \tau \rangle \\ -2\langle \tau^2 \rangle \end{pmatrix}, \hspace{1cm} (D2)
\]

where the matrix \(L\) is defined by

\[
L = \begin{pmatrix} \langle 1/\tau \rangle, -\langle \tau \rangle, -1 \\ 1, -\langle \tau^2 \rangle, -\langle \tau \rangle \\ \langle \tau \rangle, -\langle \tau^3 \rangle, -\langle \tau^2 \rangle \end{pmatrix}. \hspace{1cm} (D3)
\]

The parameters are estimated by using the observed values of moments \(\langle 1/\tau \rangle\), \(\langle \tau \rangle\), \(\langle \tau^2 \rangle\) and \(\langle \tau^3 \rangle\):

\[
\begin{pmatrix} \hat{A} \\ \hat{B}^{-1} \\ \hat{\ell} \end{pmatrix} = L^{-1} \begin{pmatrix} 0 \\ -\langle \tau \rangle \\ -2\langle \tau^2 \rangle \end{pmatrix}. \hspace{1cm} (D4)
\]

Received: May, 2012