Quantum Statistics of Ideal Parafermi Gases: 
The Parafermi Distribution

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Abstract

The fundamentals of the quantum statistics of ideal gases of identical parafermions are developed. The wave function, the quantum distribution as well as the state equation of the parafermi gas are given. A comparison with the usual Bose and Fermi statistics is made.

Keywords: Partition Function, parafermi gases, Pauli Exclusion Principal, Quantum State

1. Introduction

Recently there has some interest in studying parafermionic theories [2]. These are two dimensional quantum models exhibiting fractional spin symmetries and are involved in many areas of two dimensional Physics. Examples of parafermionic models having fractional spin conserved charges, are numerous. The well common ones are given by minimal models and too particular the $Z_k \times Z_k$ ones [1]. Other examples of parafermionic systems which can be quoted also are given by the so called exotic supersymmetric models involved in the context of integrable models and used recently in elaborating new string models. The last kind of examples we want to report here are the massive $\Phi_{1,3}$ deformation of the $c = 1 - \frac{3}{M(2M+1)}$ minimal models exhibiting among other things one state particles of spin $\frac{1}{M}$ [5].

In this paper we study an other type of parafermionic systems different of the above mentioned models. Our system is an ideal gas of a large number $N$ of identical particles of spin $s = \frac{1}{M} \ (\text{mod} \ 1)$ where $M$ can take the values $M = 2, 3, 4, \ldots$. Then we propose to develop the fundamentals of quantum
statistics of this gas. Of immediate interest are the wave function of the gas, its quantum distribution at the equilibrium or again its state equation. Our references in performing this study are the known cases $M = 2$ and $M = \infty$ describing respectively the Fermi and the Bose gases. Among the obtained results, we construct the wave function $\Psi(1, 2, 3, \ldots, N)$ of the parafermi gas. We show that $\Psi(1, 2, 3, \ldots, N)$ obeys a generalized Pauli exclusion principle according to which no more than $(M - 1)$ particles of spin $s = \frac{1}{M} \mod 1$ can live on the same quantum state. The usual Pauli exclusion principle associated with $M = 2$ appears then as a special case. We also analyse the quantum distribution of the parafermi gas and derive its state equation as well as some of its low temperature properties.

The presentation of this paper is as follows. In section 2 we study the wave function $\Psi(1, 2, 3, \ldots, N)$ of the parafermi gas. Its partition function as well as its quantum distribution are analysed in section 3. In section 4, we derive the state equation of the parafermi gas and discuss some low temperature properties. A comparison with the Bose and the Fermi gases is given. In the last section we make a discussion and give our conclusion.

2. Generalities on the wave function of identical particles

We start by considering a gas of $N$ identical two-dimensional particles enclosed within a container of volume $V$. Let $\zeta_i$ denote collectively all the coordinates of the $i$-th particle namely its two position coordinates $(\tau, \sigma)$ and its spin coordinates $s$. Let $k_i$ be an index labelling the possible quantum states $\Psi_{k_i}(\zeta_i)$ of this single particle. Each possible value of $k_i$ corresponds to a specification of the Energy $\varepsilon_{k_i}$ and the momentum $p_{k_i}$ of the particle as well as the direction of the spin orientation $\pm s$. The state of the whole gas is then described by the set of quantum numbers $\{k_i, 1 \leq i \leq N\}$, which characterize the wave function $\Psi$ of the gas in this state.

$$\Psi \equiv \Psi_{\{k_1, k_2, \ldots, k_N\}}(\zeta_1, \zeta_2, \ldots, \zeta_N)$$

(1)

To write down explicitly Eq. (1) in terms of individual quantum states $\Psi_{k_i}(\zeta_i)$, one should take into account the indistinguishability of the particles of the gas since it imposes definite symmetry requirements on $\Psi$. Interchange of any two particles in Eq(1) does not lead to a new state of the gas. In symbols, we have:

$$\Psi(\zeta_1, \zeta_2, \ldots, \zeta_i, \ldots, \zeta_N) = w(i, j)\Psi(\zeta_1, \zeta_2, \ldots, \zeta_j, \ldots, \zeta_i, \ldots, \zeta_N)$$

(2)

where we have omitted the subscript $\{k_1, k_2, k_3, \ldots, k_N\}$ for the sake of brevity and where $w(i, j)$ is an unphysical global phase of module unity: $|w(i, j)| = 1$. In term of the spin $s$ of the particles of the gas, the phase $w(i, j)$ is solved as:

$$w(i, j) = \exp(2i\pi s v(i, j)) = q^{v(i,j)}$$

(3)
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where \( q = \exp(2i\pi s) \) and \( v(i,j) \) is an integer characterizing the interchange of the \( i \)-th and \( j \)-th particles. Thus, according to the values of the spin \( s \), various cases are of interest.

2.1. The wave function of Bose gas

Since the spin of bosons is integral. The phases \( w(i,j) \) reduce to the unity and then the wave function of the Bose gas is completely symmetric under the interchange of particles of the gas. Introducing the permutation group \( S_N \) and the operator \( S \)

\[
S = (N!)^{-1} \sum_{p \in S_N} P
\]

(4)

generating the subgroup of completely symmetric permutations, the Bose gas wave function reads as:

\[
\Psi_+(\zeta_1, \ldots, \zeta_N) = \sqrt{N!}S[\Psi(\zeta_1)\Psi(\zeta_2)\ldots\Psi(\zeta_N)]
\]

(5)

2.2. The wave function of the Fermi gas

In this case, the wave function \( \Psi \) is completely antisymmetric under the permutation \( P \) of any odd number \( n \leq N \) of fermions since we have

\[
w(p) = (-)^p
\]

(6)

Here also, we can express \( \Psi_+(\zeta_1, \zeta_2, \ldots, \zeta_N) \) in terms of the one particle wave functions \( \Psi(\zeta_i); i = 1, 2, \ldots, N \) by help of the completely antisymmetric operator \( A \) defined by

\[
A = (N!)^{-1} \sum_{p \in S_N} (-)^{v(p)} P
\]

(7)

Thus we have

\[
\Psi_-(\zeta_1, \ldots, \zeta_N) = \sqrt{N!}A[\Psi(\zeta_1)\Psi(\zeta_2)\ldots\Psi(\zeta_N)]
\]

(8)

or equivalently by using the Slater determinant [4]

\[
\Psi_{\{k_1, \ldots, k_N\}}(\zeta_1, \ldots, \zeta_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix}
\Psi_{k_1}(\zeta_1) & \Psi_{k_1}(\zeta_2) & \ldots & \Psi_{k_1}(\zeta_N) \\
\Psi_{k_2}(\zeta_1) & \Psi_{k_2}(\zeta_2) & \ldots & \Psi_{k_2}(\zeta_N) \\
\vdots & \vdots & \ddots & \vdots \\
\Psi_{k_N}(\zeta_1) & \Psi_{k_N}(\zeta_2) & \ldots & \Psi_{k_N}(\zeta_N)
\end{vmatrix}
\]

(9)
Note that the well known Pauli exclusion principle manifests itself though the antisymmetry property of the Fermi wave function Eqs.(8,9). Note moreover that both the symmetric $\Psi_+^0$ and antisymmetric $\Psi_-^0$ wave functions can be expressed in condensed form, by using the convention notation, as:

$$\Psi_{\pm}(\zeta_1, ..., \zeta_N) \sim \begin{vmatrix} \Psi_{k_1}(\zeta_1) & \Psi_{k_1}(\zeta_2) & \cdots & \Psi_{k_1}(\zeta_N) \\ \Psi_{k_2}(\zeta_1) & \Psi_{k_2}(\zeta_2) & \cdots & \Psi_{k_2}(\zeta_N) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{k_N}(\zeta_1) & \Psi_{k_N}(\zeta_2) & \cdots & \Psi_{k_N}(\zeta_N) \end{vmatrix}$$  \(10\)

The subscripts + and - refer to the Slater permanent and the Slater determinant respectively.

2.3. The wave function of the parafermi gas

This is the situation we are interested in here. The spin $s$ of the parafermions is no longer an integer nor a half odd integer, fact which renders unobvious the building of the parafermi wave function $\Psi_q$ in terms of the single wave functions $\Psi_{k_i}(\zeta_i)$. An intuitive way of constructing $\Psi_q$ is to postulate the following expression:

$$\Psi_{\{k_1, ..., k_N\}}(\zeta_1, \zeta_2, ..., \zeta_N) = \sqrt{N!} J(q) [\Psi_{k_1}(\zeta_1)\Psi_{k_2}(\zeta_2)\cdots\Psi_{k_N}(\zeta_N)]$$  \(11\)

which reduces to Eqs(5) and (8,9) once the parameter $q$ is set equal to one or minus one respectively, i.e.

$$J(1) = S \quad and \quad J(-1) = A$$  \(12\)

Eqs (12) are certainly not enough to fix the explicit form of $J(q)$ especially if one knows that $J(q)$ is not obliged to fulfill the same properties as those of $S$ and $A$ starting from hermiticity. A possible solution of $J(q)$ might be obtained by saying that the wave function $\Psi$ of parafermi gas under study is given by the $N \times N$ quantum determinant of the non commutative analysis [7]. However this hypothetic solution is not yet disponible. To our knowledge a general formula for the quantum space determinant or again $J(q)$ is still lacking. Nevertheless a natural solution, which could be referred to the quantum spaces determinants mentioned above, may be written down. Our idea in identifying $J(q)$ is as follows: First we suppose that the wave function $\Psi_q$ of the parafermi gas is given by a deformed Slater determinant generalizing Eqs(10):

$$\Psi_q(\zeta_1, ..., \zeta_N) \sim \begin{vmatrix} \Psi_{k_1}(\zeta_1) & \Psi_{k_1}(\zeta_2) & \cdots & \Psi_{k_1}(\zeta_N) \\ \Psi_{k_2}(\zeta_1) & \Psi_{k_2}(\zeta_2) & \cdots & \Psi_{k_2}(\zeta_N) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{k_N}(\zeta_1) & \Psi_{k_N}(\zeta_2) & \cdots & \Psi_{k_N}(\zeta_N) \end{vmatrix}$$  \(13\)
For a 2x2 matrix, the above deformed determinant is defined as:

\[
\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad + qbc
\]

where \( q \) is a root of unity chosen as:

\[
q = \exp(2i\pi/M) ; \quad M = 2, 3, 4, \ldots
\]  

(15)

Note that the spin \( s \) of the particles considered here is related to the integer \( M \) as:

\[
s = \frac{1}{M} \quad (mod 1)
\]  

(16)

Fermions and bosons are then associated to the boundary values of \( M \) namely \( M = 2 \) and \( M \) equals to infinity. Particles with spin \( s = \frac{1}{M} \) (mod 1) are the parafermions we are interested in here. They range between fermions and bosons and are supposed to obey a generalized Pauli exclusion principle according to which no more than \( (M-1) \) identical parafermions of spin \( s = \frac{1}{M} \) can live on the same quantum state. This is our second assumption in the building of the wave function \( \Psi_q \). In symbols, this hypothesis reads for \( M \leq N \) as:

\[
J(q)[\Psi_k(\zeta_1)\Psi_k(\zeta_2)\ldots\Psi_k(\zeta_M)\Psi_{kM+1}(\zeta_{M+1})\ldots\Psi_{kN}(\zeta_N)] = 0
\]  

(17)

where we have set \( k_1 = k_2 = \ldots = k_M = k \) in eq(11). A natural solution of this constraint equation is given by

\[
J(q) \sim \sum_{p \in S} q^{v(p)} P
\]  

(18)

The integer \( v(p) \), which by the way is the same as in Eq(6), turns out to be intimately linked with the identity

\[
\sum_{n=0}^{M-1} q^n = 0
\]  

(19)

solving the generalized Pauli exclusion principle of spin \( s = \frac{1}{M} \) particles. To better understand how things work, we propose to consider in what follows the example of an ideal gas of identical particles of spin \( s = \frac{1}{3} \).
3. More on the wave function of a parafermi gas

For simplicity, take for the moment a system of \( N = 3 \) identical parafermions of spin \( s = \frac{1}{3} \). The wave function \( \Psi(\zeta_1, \zeta_2, \zeta_3) \) of this system reads according to Eq (11) as:

\[
\Psi_{(k_1, k_2, k_3)}(\zeta_1, \zeta_2, \zeta_3) = \sqrt{3!} J(q) [\psi_{k_1}(\zeta_1)\psi_{k_2}(\zeta_2)\psi_{k_3}(\zeta_3)] \tag{20}
\]

where the operator \( J(q) \) is given by:

\[
J(q) = 1 + qP_{12} + qP_{23} + q^2P_{(123)} + q^2P_{(321)} + q^3P_{13} \tag{21}
\]

In the above equation \( P_{12}, P_{23} \) and \( P_{13} \) are respectively the \( S_3 \) transposition operators of the first and the second, the second and the third and the first and the third particles. \( P_{(123)} \) and \( P_{(321)} \) are the two non trivial cyclic \( S_3 \) permutations. Note that the exponent \( v(p) \) appearing in Eq (10) are obtained by help of the neighbour particles transpositions \( P_{12} \) and \( P_{23} \) along with identities:

\[
P_{(123)} = P_{23}P_{12} ; \quad P_{(321)} = P_{12}P_{23}
\]

\[
P_{13} = P_{12}P_{23}P_{12} = P_{23}P_{12}P_{23} \tag{22}
\]

These relations show also that all the \( S_3 \) group elements can be expressed in terms of the basic group generators \( P_{12} \) and \( P_{23} \). As a consequence of this property, Eq (21) can be put in the following factorized form:

\[
J(q) = [1 + qP_{23}][1 + qP_{12} + q^2P_{(321)}] \tag{23}
\]

Putting Eq (21) in Eq (20), one gets the deformed Slater determinant giving the wave function of the three identical parafermions:

\[
\Psi_{(k_1, k_2, k_3)}(\zeta_1, \zeta_2, \zeta_3) \sim \Psi_{k_1}(\zeta_1)\Psi_{k_2}(\zeta_2)\Psi_{k_3}(\zeta_3) + q^3\Psi_{k_3}(\zeta_1)\Psi_{k_2}(\zeta_2)\Psi_{k_1}(\zeta_3) \\
+ q\Psi_{k_2}(\zeta_1)\Psi_{k_1}(\zeta_2)\Psi_{k_3}(\zeta_3) + q\Psi_{k_1}(\zeta_1)\Psi_{k_3}(\zeta_2)\Psi_{k_2}(\zeta_3) \\
+ q^2\Psi_{k_3}(\zeta_1)\Psi_{k_2}(\zeta_2)\Psi_{k_1}(\zeta_3) + q^2\Psi_{k_2}(\zeta_1)\Psi_{k_3}(\zeta_2)\Psi_{k_1}(\zeta_3) \tag{24}
\]

Putting Eq (23) instead of Eq (21) in Eq (20), one discovers another expression of the wave function:

\[
\Psi_{(k_1, k_2, k_3)}(\zeta_1, \zeta_2, \zeta_3) = \Psi_{k_1}(\zeta_1) \left| \begin{array}{ccc} \Psi_{k_2}(\zeta_2) & \Psi_{k_2}(\zeta_3) \\ \Psi_{k_3}(\zeta_2) & \Psi_{k_3}(\zeta_3) \end{array} \right| + q\Psi_{k_2}(\zeta_1) \left| \begin{array}{ccc} \Psi_{k_1}(\zeta_2) & \Psi_{k_1}(\zeta_3) \\ \Psi_{k_3}(\zeta_2) & \Psi_{k_3}(\zeta_3) \end{array} \right| + q^2\Psi_{k_3}(\zeta_1) \left| \begin{array}{ccc} \Psi_{k_1}(\zeta_2) & \Psi_{k_1}(\zeta_3) \\ \Psi_{k_2}(\zeta_2) & \Psi_{k_2}(\zeta_3) \end{array} \right| \tag{25}
\]
Comparing the two results, one obtains the following generalized Sarrus decomposition:

\[
\begin{vmatrix}
\Psi_{k_1}(1) & \Psi_{k_1}(2) & \Psi_{k_1}(3) \\
\Psi_{k_2}(1) & \Psi_{k_2}(2) & \Psi_{k_2}(3) \\
\Psi_{k_3}(1) & \Psi_{k_3}(2) & \Psi_{k_3}(3)
\end{vmatrix} = \Psi_{k_1}(1) \begin{vmatrix}
\Psi_{k_2}(2) & \Psi_{k_2}(3) \\
\Psi_{k_3}(2) & \Psi_{k_3}(3)
\end{vmatrix}
+ q\Psi_{k_2}(1) \begin{vmatrix}
\Psi_{k_1}(2) & \Psi_{k_1}(3) \\
\Psi_{k_3}(2) & \Psi_{k_3}(3)
\end{vmatrix}
+ q^2 \Psi_{k_3}(1) \begin{vmatrix}
\Psi_{k_1}(2) & \Psi_{k_1}(3) \\
\Psi_{k_2}(2) & \Psi_{k_2}(3)
\end{vmatrix}
\]

(26)

where we have set \(\Psi_{k_i}(\zeta_j) = \Psi_{k_i}(j)\) for short. In the case where \(k_1 = k_2 = k_3 = k\), the wave function Eq (20) reads as:

\[
\Psi_{k,k,k}(1,2,3) \sim 2(1 + q + q^2)\Psi_k(1)\Psi_k(2)\Psi_k(3)
\]

(27)

which vanishes identically because of the identity Eq(18). Thus, three particles of spin \(s = \frac{1}{3}\) cannot exist on the same quantum state \(\Psi_k\). This defines the generalized Pauli exclusion principle according to which no more than two spin \(\frac{1}{3}\) particles can live on the same quantum state \(\Psi_k\). Note finally that putting the constraint \(k_1 = k_2 = k_3 = k\) in Eq(25) instead of Eq (24), one gets the following spin independent factorization:

\[
\Psi_{k,k,k}(1,2,3) \sim (1 + q)(1 + q + q^2)\Psi_k(1)\Psi_k(2)\Psi_k(3)
\]

(28)

This way of writing shows that the wave function Eq.(26) vanishes for both \(s = \frac{1}{2}\) and \(s = \frac{1}{3}\). This property holds also for higher values of \(N\). The second illustrating example we want to give is a system of \(N = 4\) identical particles of spin \(s = \frac{1}{3}\). The wave function of this system reads as:

\[
\Psi_{\{k_1,k_2,k_3,k_4\}}(1,2,3,4) \sim \begin{vmatrix}
\Psi_{k_1}(1) & \Psi_{k_1}(2) & \Psi_{k_1}(3) & \Psi_{k_1}(4) \\
\Psi_{k_2}(1) & \Psi_{k_2}(2) & \Psi_{k_2}(3) & \Psi_{k_2}(4) \\
\Psi_{k_3}(1) & \Psi_{k_3}(2) & \Psi_{k_3}(3) & \Psi_{k_3}(4) \\
\Psi_{k_4}(1) & \Psi_{k_4}(2) & \Psi_{k_4}(3) & \Psi_{k_4}(4)
\end{vmatrix}
\]

(29)
It can be expanded by help of the generalized Sarrus decomposition as:

\[
\psi_{k_1,k_2,k_3,k_4}(1,2,3,4) \sim \psi_{k_1}(1) + q\psi_{k_2}(1) + q^2\psi_{k_3}(1) + q^3\psi_{k_4}(1)
\]

One can repeat this procedure until identifying the 24 terms of the wave function Eq(26). We find:

\[
J(q) \sim (1 + qP_{23} + qP_{34} + q^2P_{(234)} + q^2P_{(432)} + q^3P_{24})
\]

\[
(1 + qP_{12} + q^2P_{(321)} + q^3P_{(4321)})
\]

(31)

Doing the same thing once more, we get the remarkable factorization:

\[
J(q) \sim [1 + qP_{34}][1 + qP_{23} + q^2P_{(432)}][1 + qP_{12} + q^2P_{(321)} + q^3P_{(4321)}]
\]

(32)

Moreover, putting \(k_2 = k_3 = k_4 = k\), i.e three spin \(\frac{1}{3}\) particles in the same quantum state, one discovers that the wave function of the four particles Eq(29) vanishes identically as shown by the following:

\[
\psi_{k,k,k,k}(1,2,3,4) \sim (1 + q)(1 + q + q^2)[\psi_k(1) + (q + q^2 + q^3)\psi_k(1)]\psi_k(2)\psi_k(3)\psi_k(4)
\]

(33)

4. The parafermi distribution

Consider an ideal gas of \(N\) identical parafermions, of spin \(s = \frac{1}{M}\) with \(M > 1\), contained in a volume \(V\) in equilibrium at the temperature \(T\). Denote by \(H\) and \(\hat{N}\) respectively the hamiltonian and the operator number associated with this gas. In quantum field theory, the adequate formalism for the study of this system, the operators \(H\) and \(\hat{N}\) can be expressed as:

\[
\hat{N} = \sum_k \hat{N}_k
\]

(34)

\[
H = \sum_k \varepsilon_k \hat{N}_k
\]

(35)
where \( k \) labels the possible quantum states of a single particle and \( \varepsilon_k \) denote the energy of a particle in a state \( \Psi_k \). \( \hat{N}_k \) is the operator number counting the number of parafermions in the state \( \Psi_k \). Note that the decomposition Eq(35) is in accord with the gas ideality assumption; i.e gas of free particles. At the equilibrium, the distribution operator describing the statistics of this quantum system of parafermions reads as:

\[
\rho(\beta, \mu) = e^{\frac{\beta(H-\mu \hat{N})}{Z(\beta, \mu)}}
\]  

(36)

where \( \beta \) is the inverse of the temperature and \( \mu \) is the chemical potential [8]. \( Z(\beta, \mu) \) is the grand partition function of the quantum parafermi gas defined as:

\[
Z(\beta, \mu) = Tr e^{-\beta(H-\mu \hat{N})}
\]  

(37)

where the trace is taken on the Fock space of quantum states of the gas. Using the eigenvalues \( n_k \) of the operator number \( \hat{N}_k \), the above equation can be decomposed as a product of partition functions \( Z_k(\beta, \mu) \) of particles living on the same quantum state of energy \( \varepsilon_k \). We have:

\[
Z(\beta, \mu) = \prod_k Z_k(\beta, \mu)
\]  

(38)

where \( Z_k(\beta, \mu) \) is given by

\[
Z_k(\beta, \mu) = \sum_{n_k=0}^{M-1} e^{-\beta(\varepsilon_k-\mu)n_k}
\]  

(39)

as required by the generalized Pauli exclusion principle. Eq(39) may also be written as:

\[
Z_k(\beta, \mu) = \frac{1 - e^{-\beta M(\varepsilon_k-\mu)}}{1 - e^{-\beta(\varepsilon_k-\mu)}}
\]  

(40)

It is this form of \( Z_k(\beta, \mu) \) that will be used in what follows. Having calculated the partition function of the gas, we are now in position to derive its quantum distribution. The latter is given by the mean value \(< n_k >\) of the occupation operator number \( \hat{N}_k \) of the state \( \Psi_k \). Following ref [8], this mean value \(< n_k >\) can be defined as:

\[
<n_k> = \frac{1}{\beta} \frac{\partial \log Z_k}{\partial \mu}
\]  

(41)

Using Eq(40), the above relation implies:

\[
<n_k> = \frac{1}{e^{\beta(\varepsilon_k-\mu)}} \frac{M}{e^{\beta M(\varepsilon_k-\mu)}} - 1
\]  

(42)
This is the parafermi distribution. Note that choosing $M = 2$, the spin $s = \frac{1}{M} \mod 1$ takes half odd integer values and the parafermi distribution $< n_k >$ reduces to the well known Fermi one namely:

$$< n_k > = \left[ \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1} \right]$$

Choosing $M$ equals to infinity, the spin $s$ takes integer values and the parafermi distribution (41) coincide with the Bose one:

$$< n_k > = \left[ \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} \right]$$

provided that the chemical potential $\mu$ is less than or at the limit equal to the energy $\varepsilon_k$ of the single quantum state $\Psi_k$. Note moreover that there are various properties of eq.(42) one of them is its behaviour in vicinity of the zero absolute temperature. Putting $\beta$ equals to infinity, Eq.(42) reduces to:

$$< n_k^0 > = (M - 1) \Theta(\mu^0 - \varepsilon_k^0)$$

where $\Theta(x)$ is the usual Heveaside function taking the value one for $0 \leq x \leq 1$ and zero elsewhere. eq.(45) shows that, at zero temperature the mean occupation number of the quantum state $\Psi_k$ is $(M - 1)$ for $\varepsilon_k$ less than the parafermi energy $\mu^0$ and zero otherwise. An immediate consequence of this property is the spin dependance of the chemical potential $\mu^0$.

$$< \mu^0 > \approx \left[ \frac{d}{M - 1} \right]^{\frac{2}{3}}$$

where $d = \frac{N}{V}$ is the density of the gas. Indeed, in the thermodynamic limit $V \to \infty$, the constraint equation

$$N = \sum_{k=0}^{\infty} < n_k >$$

may be approached by the quasi classic integral

$$(2\pi \hbar)^3 d = 4\pi (2m)^{\frac{3}{2}} \int_0^{\infty} < n > \sqrt{\varepsilon} d\varepsilon$$

where we have used the identity $\varepsilon = \frac{p^2}{2m}$, the non relativistic expression of the kinetic energy of a single particle of the gas. At the zero temperature, Eq.(48) reduces to

$$3d\hbar^3 = 8\pi (2m)^{\frac{3}{2}} (M - 1)(\mu^0)^{\frac{3}{2}}$$
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as it can be checked by help of Eq.(45). A convenient way of writing $\mu^0$ is to compare it to the Fermi energy $\varepsilon_F^0$ of half odd integer. We have \[8\]

$$
\mu^0 = \frac{\varepsilon_F^0}{(M - 1)^{\frac{3}{2}}}
$$

(50)

Note that for $M = 2$, $\mu^0 = \varepsilon_F^0$ and for $M = \infty$, $\mu^0$ vanishes as expected. For a non vanishing temperature, a detailed study of the chemical potential is necessary.

5. The state equation of the parafermi gas

To start consider Eqs.(38 - 40) and introduce the thermodynamic potential $\Omega$ defined as:

$$
\beta \Omega = -\log Z(\beta \mu)
$$

(51)

To determine the state equation of the parafermi gas, we calculate the above relation by using two methods. The comparison of the two results gives the desired equation. Following ref \[8\], one establishes easily that $\Omega$ is equal to $(-PV)$; where $P$ is the pressure of the gas and $V$ is its Volume. On the other hand, using the thermodynamic along the non relativistic limits, Eq.(51) is expressed as:

$$
\Omega = \left(\frac{8}{3}\pi V(2m)^{\frac{3}{2}}\right)\hbar^{-3} \int_0^\infty \left(\frac{\partial \varepsilon^{\frac{3}{2}}}{\partial \varepsilon}\right) w(\varepsilon) d\varepsilon
$$

(52)

where $w(\varepsilon)$ is defined as:

$$
\beta w(\varepsilon) = \log \left[\frac{1 - e^{-\beta(\varepsilon - \mu)}}{1 - e^{-\beta M(\varepsilon - \mu)}}\right]
$$

(53)

Integrating Eq.(52) by parts and using the identity

$$
\frac{\partial w(\varepsilon)}{\partial t} = <n(\varepsilon)>
$$

(54)

together with the convention notation $E = <H>$; the mean of the energy operator, one finds:

$$
\Omega = -\frac{2E}{3}
$$

(55)

The state equation of the non relativistic gas

$$
P V = \frac{2E}{3}
$$

(56)
does make no reference to any explicit dependence on the spin. Thus Fermi, Bose and Parafermi gases have the same state equation. Note that for an ultra relativistic gas of particles of individual energies $\varepsilon = pc$, the above state equation becomes

$$PV = \frac{E}{3} \tag{57}$$

In the end of this study, we would like to note that the determination of the specific heat $C_v = (\frac{\partial E}{\partial T})_{T,N}$ of the parafermic gas needs the knowledge of the explicit temperature dependence of the mean energy $E$ of the gas. However this requires performing complicated integrals such that given by Eq.(52). To over pass this difficulty one has to use perturbative methods. Recall that in the determination of $C_v$ for a Bose gas, which by the way behaves as:

$$C_v \sim T^3 \tag{58}$$

one uses a perturbation around the Bose temperature $T_B$ [8]. For a Fermi gas, one uses a perturbation in the vicinity of the zero temperature. The behaviour

$$C_v \sim T \tag{59}$$

is obtained by help of the basic Sommerfield development of the fermions occupation mean number $<n_k>$ Eq.(43) around $T = 0$. We have:

$$<n_k> = \Theta(\mu - \varepsilon) - \frac{\pi^2}{6\beta^2}\delta'(\mu - \varepsilon) - \frac{7\pi^4}{360\beta^4}\delta''(\mu - \varepsilon) + O(\beta^{-6}) \tag{60}$$

Methods of calculus of the specific heat of the parafermi gas based on perturbative techniques will be developed in a future publication.

6. Conclusion

In this paper, we have studied the quantum statistics of a ideal parafermi gas. Among our main results we mention:

1- We have built the wave function of the parafermi gas. We have found that for a gas of free particles of spin $s = \frac{1}{M} (\mod 1)$, the wave function obey a generalized Pauli exclusion principle according to which no more than $(M - 1)$ particles can live together on the same quantum state.

2- We have analysed with details the quantum distribution $<n_k>$ of the parafermi gas introduced in [3]. The main feature of $<n_k>$ is its unified description of all gases of identical particles of spin $s = \frac{1}{M} (\mod 1)$ with $M = 2, 3, 4, ...$

3- We have shown that Fermi, Bose and Parafermi gases are described by the
same state equation. We have also given some low temperature properties of this quantum gas. We should mention finally that though we have not performed the calculation, we expect that the so called Bose temperature, defining the critical point of the Bose condensation, and the Fermi temperature, defining the upper value of the energy of single particles of the Fermi gas, are related to each others. This property as well as others remarkable ones will be considered elsewhere.

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