Solving the Generalized Kaup–Kupershmidt Equation

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Abstract. In this paper we apply the Cole-Hopf transformation to find soliton solutions and the simplified Hirota’s method to find one and two soliton solutions for the generalized Kaup–Kupershmidt equation.

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1. INTRODUCTION

Hirota’s method has been one of the most successful direct techniques for constructing exact solutions of various nonlinear PDEs from soliton theory. In this paper we make use of a simplified version of this method [2] without ever using the bilinear forms. Furthermore, the simplified method can easily be implemented in any symbolic manipulation package.

This paper is organized as follows: We first find a soliton solution for the general fKdV by using a Cole-Hopf transformation [1][3]. In the next section we apply the simplified Hirota’s method [2] to find multisoliton solutions for the generalized Kaup–Kupershmidt equation.

2. SOLITON SOLUTIONS TO THE GENERAL fKdV BY THE COLE-HOPF TRANSFORMATION

The general fifth order KdV equation reads

\[ u_t + \omega u_{xxxx} + \alpha uu_{xxx} + \beta u_x uu_{xx} + \gamma u^2 u_x = 0, \]  

(2.1)
where \( \alpha, \beta, \gamma \) and \( \omega \) are arbitrary real parameters. To obtain a soliton solution, we apply the generalized Cole-Hopf transformation

\[
(2.2) \quad u = A \frac{\partial^2}{\partial x \partial t} \ln f(x, t) + B,
\]

where \( A \) and \( B \) are some constants and

\[
f(x, t) = 1 + \exp(\theta), \quad \theta = kx - ct + \delta
\]

Upon substitution of (2.2) into (2.1), we obtain the equation

\[
(A\omega_k^7 + AB\alpha k^5 + AB^2\gamma k^3 - Ack^2) e^\theta + (-57A\omega_k^7 + A^2\alpha k^7 + A^2\beta k^7 - 9AB\alpha k^5 + 2A^2B\gamma k^5 + 3AB^2\gamma k^3 - 3Ack^2) e^{2\theta} + (302A\omega_k^7 - 11A^2\alpha k^7 - 5A^2\beta k^7 - A^3\gamma k^7 - 10AB\alpha k^5 + 2A^2B\gamma k^5 + 2AB^2\gamma k^3 - 2Ack^2) e^{3\theta} + (-302A\omega_k^7 + 11A^2\alpha k^7 + 5A^2\beta k^7 - A^3\gamma k^7 - 10AB\alpha k^5 - 2A^2B\gamma k^5 - 2AB^2\gamma k^3 + 2Ack^2) e^{4\theta} + (57A\omega_k^7 - A^2\alpha k^7 - A^2\beta k^7 + 9AB\alpha k^5 - 2A^2B\gamma k^5 - 3AB^2\gamma k^3 + 3Ack^2) e^{5\theta} + (-A\omega_k^7 - AB\alpha k^5 - AB^2\gamma k^3 + Ack^2) e^{6\theta} = 0.
\]

Equating the coefficients of \( e^\theta, e^{2\theta}, ..., e^{6\theta} \) to zero, we obtain an algebraic system. Solving this system yields:

- **First solution**:

\[
A = \frac{6\alpha + 3\beta - 3 \sqrt{(2\alpha + \beta)^2 - 40\omega\gamma}}{\gamma}, \quad B = \frac{k^2 (-2\alpha - \beta + \sqrt{(2\alpha + \beta)^2 - 40\omega\gamma})}{4\gamma},
\]

\[
c = -\frac{k^5 \left(12\omega \gamma + \beta (-2\alpha - \beta + \sqrt{(2\alpha + \beta)^2 - 40\omega\gamma})\right)}{8\gamma}.
\]

\[
(2.3) \quad u(x, t) = \frac{k^2 (R - 2\alpha - \beta) \left( \cosh \left( kx + \frac{k^5}{8\gamma} \left((R - 2\alpha - \beta)\beta + 12\omega\gamma\right) t + \delta \right) - 5 \right)}{4\gamma \left( \cosh \left( kx + \frac{k^5}{8\gamma} \left((R - 2\alpha - \beta)\beta + 12\omega\gamma\right) t + \delta \right) + 1 \right)},
\]

where \( R = \sqrt{(2\alpha + \beta)^2 - 40\omega\gamma} \).

- **Second solution**:

\[
A = \frac{3(2\alpha + \beta + \sqrt{(2\alpha + \beta)^2 - 40\omega\gamma})}{\gamma}, \quad B = -\frac{k^2 \left( 2\alpha + \beta + \sqrt{(2\alpha + \beta)^2 - 40\omega\gamma} \right)}{4\gamma},
\]
Solving the generalized Kaup–Kupershmidt equation

\[
c = \frac{k^5 \left( \beta \left( 2\alpha + \beta + \sqrt{(2\alpha + \beta)^2 - 40\gamma \omega} \right) - 12\gamma \omega \right)}{8\gamma}.
\]

(2.4)

\[
u(x, t) = -\frac{k^2(R + 2\alpha + \beta) \left( \cosh \left( kx - \frac{k^5}{8\gamma}(\beta(R + 2\alpha + \beta) - 12\omega \gamma)t + \delta \right) - 5 \right)}{4\gamma \left( \cosh \left( kx - \frac{k^5}{8\gamma}(\beta(R + 2\alpha + \beta) - 12\omega \gamma)t + \delta \right) + 1 \right)},
\]

where \( R = \sqrt{(2\alpha + \beta)^2 - 40\omega \gamma}. \)

3. The generalized Kaup–Kupershmidt equation

This equation reads

\[
u_t + 10abu_{xxx} + 25abu_x u_{xx} + bu^2u_x + 20a^2bu_{xxxxx} = 0,
\]

where \( a \neq 0 \) and \( b \neq 0. \)

The well-known Kaup–Kupershmidt equation \([5][2]\) (KK equation)

\[
u_t + 10au_{xxx} + 25u_x u_{xx} + 20u^2 u_x + u_{xxxxx} = 0,
\]

is obtained from (3.1) for the values \( a = 1/20 \) and \( b = 20. \)

From Eqs. (2.3) and (2.4) we obtain the following solutions of (3.1):

\[
u_1(x, t) = -\frac{5}{2} a k^2 \left( 1 - \frac{6}{\cosh \left( kx - \frac{5}{4}a^2bk^5t + \delta \right) + 1} \right).
\]

(3.3)

\[
u_2(x, t) = 20a k^2 \left( 1 - \frac{6}{\cosh \left( kx - 220a^2bk^5t + \delta \right) + 1} \right).
\]

(3.4)

Solution (3.3) has the form (2.2) and it corresponds to values \( A = 30a, B = -5ak^2/2 \) and \( c = -\frac{5}{4}a^2bk^5. \) From (3.3) and (3.4) we obtain periodic solutions if we change \( k \) by \( \sqrt{-1}k \) and \( \delta \) by \( \sqrt{-1}\delta. \) These solutions are:

\[
u_3(x, t) = \frac{5}{2} a k^2 \left( 1 - \frac{6}{\cos \left( kx - \frac{5}{4}a^2bk^5t + \delta \right) + 1} \right).
\]

(3.5)

\[
u_4(x, t) = -20a k^2 \left( 1 - \frac{6}{\cos \left( kx - 220a^2bk^5t + \delta \right) + 1} \right).
\]

(3.6)

3.1. Simplified Hirota’s method. We will apply this method to find multisolitons solutions for Eq. (3.1). Using the Cole-Hopf transformation

\[
u = 30a \frac{\partial^2}{\partial x \partial x} \ln f,
\]

we get a fourth degree equation in \( f = f(x, t) \) and its derivatives,

\[
P^4(20a^2bfxxxxx + fxx) + \]

\[
P^3(50a^2bfxxx fxxx - 120a^2bfxx fxxxx - 140a^2bfxfxxxxx - 2fx fxx - ft fxx) + \]

where
\[ f^2(-200a^2bf_{xx}f_{xxx}^2 - 150a^2b f_{xx}^2 f_{xxxx} + 450a^2bf_{xx}f_{xxxx} + 540a^2bf_{xx}^2 f_{xxxxx} + 2f_{xx}^2) + \\
 f^1 \cdot 150a^2b(3f_{xx}f_{xxx}^2 - 8f_{xx}f_{xxxx}) + 300a^2bf_x^3(4f_{xx}f_{xxx} - 3f_{xx}^2) = 0. \]

Obviously, this last equation can be written as

\[ f^4 \mathcal{L}(f) + f^3 \mathcal{N}_1(f,f) + f^2 \mathcal{N}_2(f, f, f) + f \mathcal{N}_3(f, f, f, f) + \mathcal{N}_4(f, f, f, f, f) = 0, \]

where the linear operator \( \mathcal{L} \) and the nonlinear operators \( \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4 \) are defined as

\[ \mathcal{L}(f) = 20a^2bf_{xxxxxxx} + f_{xxx}, \]
\[ \mathcal{N}_1(f,g) = 50a^2bf_{xxx}g_{xxxx} - 120a^2bf_{xx}g_{xxxx} - 140a^2bf_{x}g_{xxxxx} - 2f_{x}g_{xx} - f_{t}g_{xx}, \]
\[ \mathcal{N}_2(f, g, h) = -200a^2bf_{xx}g_{xxx}h_{xxx} - 150a^2bf_{xx}g_{xx}h_{xxx} + 450a^2bf_{xx}g_{xxx}h_{xxxx} + 540a^2bf_{xx}g_{xx}h_{xxxx} + 2f_{xx}g_{xx}, \]
\[ \mathcal{N}_3(f, g, h, \phi) = 150a^2b(3f_{xx}g_{xx}h_{xxx} - 8f_{xx}g_{xx}h_{xxxx}), \]
\[ \mathcal{N}_4(f, g, h, \phi, \varphi) = 300a^2b(4f_{xx}g_{xx}h_{xxx} - 3f_{xx}g_{xx}h_{xxxx}). \]

for auxiliary functions \( f, g, h, \phi \) and \( \varphi \). We seek a solution of (3.8) in the form

\[ f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t). \]

We substitute (3.13) into (3.8) and equate the coefficients of different powers of \( \epsilon \) to zero. The following perturbation scheme follows:
\(O(\epsilon^1): \mathcal{L}(f^{(1)}) = 0.\)

\(O(\epsilon^2): \mathcal{L}(f^{(2)}) = -N_1(f^{(1)}, f^{(1)})\)

\(O(\epsilon^3): \mathcal{L}(f^{(3)}) = f^{(1)}N_1(f^{(1)}, f^{(1)}) - N_1(f^{(1)}, f^{(2)}) - N_1(f^{(2)}, f^{(1)}) - \)

\(N_2(f^{(1)}, f^{(1)}, f^{(1)})\)

\(O(\epsilon^4): \mathcal{L}(f^{(4)}) = -N_1(f^{(1)}, f^{(1)})(f^{(1)})^2 + N_1(f^{(1)}, f^{(2)})f^{(1)} + N_1(f^{(2)}, f^{(1)})f^{(1)} + \)

\(2N_2(f^{(1)}, f^{(1)}, f^{(1)})f^{(1)} + f^{(2)}N_1(f^{(1)}, f^{(1)}) - N_1(f^{(1)}, f^{(3)}) - N_1(f^{(2)}, f^{(2)}) - \)

\(N_1(f^{(3)}, f^{(1)}) - N_2(f^{(1)}, f^{(1)}, f^{(2)}) - N_2(f^{(1)}, f^{(2)}, f^{(1)}) - N_2(f^{(2)}, f^{(1)}, f^{(1)}) - \)

\(N_3(f^{(1)}, f^{(1)}, f^{(1)}).

\(O(\epsilon^5): \mathcal{L}(f^{(5)}) = -N_1(f^{(2)}, f^{(1)})(f^{(1)})^2 - 3N_2(f^{(1)}, f^{(1)}, f^{(1)})(f^{(1)})^2 + \)

\(N_1(f^{(1)}, f^{(3)})f^{(1)} + N_1(f^{(2)}, f^{(2)})f^{(1)} + N_1(f^{(3)}, f^{(1)})f^{(1)} + \)

\(2N_2(f^{(1)}, f^{(1)}, f^{(2)})f^{(1)} + 2N_2(f^{(1)}, f^{(2)}, f^{(1)})f^{(1)} + 2N_2(f^{(2)}, f^{(1)}, f^{(1)})f^{(1)} + \)

\(3N_3(f^{(1)}, f^{(1)}, f^{(1)})f^{(1)} + (f^{(1)})^3 - 2f^{(2)}f^{(1)} + f^{(3)}N_1(f^{(1)}, f^{(1)}) + \)

\((f^{(2)} - (f^{(1)})^2)N_1(f^{(1)}, f^{(2)}) - N_1(f^{(1)}, f^{(4)}) + f^{(2)}N_1(f^{(2)}, f^{(1)}) - \)

\(N_1(f^{(2)}, f^{(3)}) - N_1(f^{(3)}, f^{(2)}) - N_1(f^{(4)}, f^{(1)}) + 2f^{(2)}N_2(f^{(1)}, f^{(1)}, f^{(1)}) - \)

\(N_2(f^{(1)}, f^{(1)}, f^{(3)}) - N_2(f^{(1)}, f^{(2)}, f^{(2)}) - N_2(f^{(1)}, f^{(3)}, f^{(1)}) - \)

\(N_2(f^{(2)}, f^{(1)}, f^{(2)}) - N_2(f^{(2)}, f^{(2)}, f^{(1)}) - N_2(f^{(3)}, f^{(1)}, f^{(1)}) - \)

\(N_3(f^{(1)}, f^{(1)}, f^{(2)}) - N_3(f^{(1)}, f^{(1)}, f^{(2)}) - N_3(f^{(2)}, f^{(2)}, f^{(1)}) - \)

\(N_4(f^{(1)}, f^{(1)}, f^{(1)}).

Noticeably, the number of terms in the right hand side (RHS) of the equations grows rapidly as the order in \(\epsilon\) increases.

3.1.1. The One-Soliton solution. To find the one-soliton solution, take

\((3.19)\quad f^{(1)} = \exp(\theta), \quad \text{with} \quad \theta = kx - ct + \delta.\)

Equation (3.14) gives the dispersion law \(c = 20a^2bk^5\). To solve (3.15), we first compute its RHS,

\((3.20)\quad -N_1(f^{(1)}, f^{(1)}) = 150a^2k^7 \exp(2\theta).\)
Thus, \( f^{(2)} \) will be of the form \( f^{(2)} = \lambda \exp(2\theta) \). Calculating the left hand side (LHS) of (3.15),

\[
\mathcal{L}(f^{(2)}) = 2400a^2bk^7e^{-40a^2bk^5+2xk+2\delta} = 2400\lambda a^2bk^7 \exp(2\theta).
\]

and equating it with (3.20) we get \( \lambda = 1/16 \) so

\[
f^{(2)} = \frac{1}{16} \exp(2\theta).
\]

It is straightforward to check that \( f^{(n)} = 0 \) for \( n \geq 3 \). Therefore, using (3.7) and (3.13) with \( \epsilon = 1 \), the one-soliton solution of (3.1) generated by

\[
f = 1 + \exp(\theta) + \frac{1}{16} \exp(2\theta) = 1 + \exp(kx - 20a^2bk^5t + \delta) + \frac{1}{16} \exp(2(kx - 20a^2bk^5t + \delta))
\]
is

\[
u(x, t) = \frac{480ak^2(4 + e^{kx - 20a^2bk^5t + \delta} + 16e^{-(kx - 20a^2bk^5t + \delta)})}{(16 + e^{kx - 20a^2bk^5t + \delta} + 16e^{-(kx - 20a^2bk^5t + \delta)})^2}
\]

3.1.2. The Two-Soliton Solution. For the two-soliton solution, we start with

\[
f^{(1)} = \exp(\theta_1) + \exp(\theta_2),
\]

where \( \theta_i = k_ix - c_it + \delta_i, \; i = 1, 2 \). From (3.14) we get \( c_i = 20a^2bk_i^5, \; i = 1, 2 \).

To find \( f^{(2)} \), the RHS of (3.15) has to be calculated. We obtain

\[
-N_1(f^{(1)}, f^{(1)}) = 150a^2bk_1^7e^{2\theta_1} + 150a^2bk_2^7e^{2\theta_2} + 50ae^{\theta_1 + \theta_2}k_1k_2(k_1 + k_2)(2k_1^4 - k_2^2k_1^2 + 2k_1^2) \exp(\theta_1 + \theta_2)
\]

Obviously, \( f^{(2)} \) must be of the form

\[
f^{(2)} = p \exp(2\theta_1) + q \exp(2\theta_2) + r \exp(\theta_1 + \theta_2)
\]

In contrast to what happened for the KdV, the Lax, and SK equations, the terms in \( \exp(2\theta_i) \) no longer drop out. We now substitute (3.26) into (3.15). Computation of the LHS yields

\[
\mathcal{L}(f^{(2)}) = 2400a^2bk_1^7pe^{2\theta_1} + 2400a^2bk_2^7qe^{2\theta_2} + 100a^2bk_1k_2r(k_1 + k_2)^3(k_1^2 + k_2k_1 + k_2^2) \exp(\theta_1 + \theta_2)
\]

Equating (3.25) and (3.27) gives \( p = q = \frac{1}{16} \) and

\[
r = \frac{2k_1^4 - k_2^2k_1^2 + 2k_1^2}{2(k_1 + k_2)^2(k_1^2 + k_2k_1 + k_2^2)}
\]

Therefore,

\[
f^{(2)} = \frac{1}{16} \exp(\theta_1) + \frac{1}{16} \exp(\theta_2) + \frac{2k_1^4 - k_2^2k_1^2 + 2k_1^2}{2(k_1 + k_2)^2(k_1^2 + k_2k_1 + k_2^2)} \exp(\theta_1 + \theta_2)
\]
Proceeding in a similar way with (3.16) we find

\[ f^{(3)} = s \left( \exp(2\theta_1 + \theta_2) + \exp(\theta_1 + 2\theta_2) \right), \]

(3.30)

\[ s = \frac{(k_1 - k_2)^2 (k_1^2 - k_1 k_2 + k_2^2)}{16(k_1 + k_2)(k_1^2 + k_1 k_2 + k_2^2)}. \]

(3.31)

To find \( f^{(4)} \), equation (3.17) has to be solved, which leads to

\[ f^{(4)} = s^2 \exp(2\theta_1 + 2\theta_2) = \frac{(k_1 - k_2)^4 (k_1^2 - k_1 k_2 + k_2^2)^2}{256(k_1 + k_2)^2 (k_1^2 + k_1 k_2 + k_2^2)^2} \exp(2\theta_1 + 2\theta_2). \]

(3.32)

After verifying that all \( f^{(n)} \) will be zero, for \( n \geq 5 \), the form of \( f \) for \( \epsilon = 1 \) will be

\[ f = \exp(\theta_1) + \exp(\theta_2) + \frac{1}{16} \exp(2\theta_1) + \frac{1}{16} \exp(2\theta_2) + r \exp(\theta_1 + \theta_2) + s \left( \exp(2\theta_1 + \theta_2) + \exp(\theta_1 + 2\theta_2) \right) + s^2 \exp(2\theta_1 + 2\theta_2), \]

(3.33)

where \( r \) and \( s \) are given by (3.28) and (3.31), respectively. The two-soliton solution of Eq. (3.1) could be obtained by substituting (3.33) into (3.7).

In a similar fashion, we may find three and four soliton solutions. For additional details, see [2].

4. Conclusions

We have obtained many solutions of the generalized KK equation by using three distinct methods. The Exp-function method is a promising method because it can establish a variety of solutions of distinct physical structures. This method allows us to obtain many new solutions for evolution equations.

REFERENCES


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