Exact Solutions of the Equation of One – Dimensional Motion of a Pion Meson Particle in an Atom Using Two Different Approaches

Shoukry El-Ganaini 1,2,*

1 Mathematics Department, Faculty of Science at Dawadmi Shaqra University, Saudi Arabia.
2 Mathematics Department, Faculty of Science, Damanhour University, Egypt
* e-mail address: ganaini5533@hotmail.com

Abstract. In this paper, using He’s semi – inverse method and the first integral method for an analytic treatment of the equation that describes a one-dimensional motion of a Pion Meson particle in an atom. The new derived exact solutions are solitary wave, triangular periodic, and rational. The used approaches are powerful, effective and can be applied to other integrable equations as well as non integrable ones.

Keywords:- exact solution; He’s semi – inverse method; first integral method; equation of a one-dimensional motion of a Pion Meson particle in an atom.

1. Introduction

It is an interesting topic to search for new solutions of Partial Differential Equations(PDEs.) A lot of systematic methods have been developed to obtain exact solutions for PDEs., such as the inverse scattering method, the Backlund and the Darboux transformations, the homogeneous balance method [1-6], the variable separation method [7,8] and the generalized hyperbolic function [9]. Among them, variational approaches, such as He’s semi – inverse method [10] is a powerful tool to the search
for variational principles for nonlinear physical problems directly from field equations without using the Lagrange multiplier and provides physical insight into the nature of the solution of the problem. Based on this formulation, a solitary solution can be obtained using the Ritz method. Unlike some known methods such as the first integral method, He’s semi-inverse method is a powerful mathematical tool to the construction of variational formulations for physical problems. Feng in his pioneering work [11] introduced the first integral method for a reliable treatment of nonlinear partial differential equations (NPDEs). This method was further developed by the same author in [11-17] and some other mathematicians[18-23]. The interest in the present work is to implement He’s semi-inverse method and the first integral method to stress their power in handling NPDEs, so that we can apply them for solving various types of these equations.

In this paper, we will use two different approaches namely, He’s semi-inverse method and the first integral method to study the equation of a one-dimensional motion of a Pion Meson particle in an atom.

2. He’s semi-inverse method

Consider a general NPDE for $u(x,t)$ to be in the form

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, \ldots) = 0. \quad (2.1)$$

where $P$ is a polynomial in its arguments.

A.Jabbari et.al. in [24] have written the He’s semi-inverse method in the following steps:

**Step 1.** Seek solitary wave solutions of Eq.(2.1) by taking $u(x,t) = u(\xi)$, $\xi = x - ct + \epsilon$, where $\epsilon$ is an arbitrary constant, and transform Eq.(2.1) to the ordinary differential equation (ODE)

$$Q(U', U'', U''', \ldots) = 0. \quad (2.2)$$

where the prime denotes the derivation with respect to $\xi$.

**Step 2.** If possible, integrate Eq.(2.2) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

**Step 3.** According to He’s semi-inverse method, we construct the following trial functional

$$J(u) = \int L \, d\xi, \quad (2.3)$$

where $L$ is an unknown function of $u$ and its derivatives.

**Step 4.** By the Ritz method, we can obtain different forms of solitary wave solutions, such as

$$u(\xi) = A \text{ sech}(B\xi), \quad u(\xi) = A \text{ csch}(B\xi), \quad u(\xi) = A \tanh(B\xi), \quad u(\xi) = A \coth(B\xi)$$

and so on. For example in this paper, we search a solitary wave solution in the form
Exact solutions

\[ u(\xi) = A \sec h(B\xi), \]

where \( A \) and \( B \) are constants to be further determined. Substituting Eq. (2.4) into Eq. (2.3) and making \( J \) stationary with respect to \( A \) and \( B \) results in

\[ \frac{\partial J}{\partial A} = 0 \]

\[ \frac{\partial J}{\partial B} = 0 \]

Solving Eqs. (2.5) and (2.6) we obtain \( A \) and \( B \). Hence the solitary wave solution (2.4) is well determined.

3. The first integral method

Consider a general NPDE in the form

\[ P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, \ldots) = 0. \]

where \( P \) is a polynomial in its arguments. Raslan in [25] summarized for using the first integral method in the following steps:

**Step 1.** Using a wave variable \( \xi = x - ct \), then Eq. (3.1) can be written in the following nonlinear Ordinary Differential Equation (NODE)

\[ Q(U', U'', U'''\ldots) = 0. \]

where the prime denotes the derivation with respect to \( \xi \).

**Step 2.** Suppose that the solution of ODE (3.2) can be written as

\[ u(x, t) = f(\xi). \]

**Step 3.** We introduce a new independent variables

\[ X(\xi) = f(\xi), \quad Y = f'(\xi) \]

which leads a system of NODEs

\[ \begin{cases} X(\xi) = Y(\xi), \\ Y(\xi) = F(X(\xi), Y(\xi)) \end{cases} \]

**Step 4.** According to the qualitative theory of ODEs [26], if we can find the integrals to (3.5) under the same conditions, then the general solution to (3.5) can be solved directly. However, in general, it is really difficult to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are?

We will apply the Division Theorem to obtain one first integral to (3.5) which reduces (3.2) to a first order integrable ODE.

An exact solution to (3.1) is then obtained by solving this equation.
Let us now recall the Division Theorem:
Suppose that \( P(w, z), Q(w, z) \) are polynomials in \( C(w, z) \) and \( P(w, z) \) is irreducible
in \( C(w, z) \). If \( Q(w, z) \) vanishes at all zero points of \( P(w, z) \), then there exists a
polynomial \( G(w, z) \) in \( C(w, z) \) such that
\[
Q(w, z) = P(w, z)G(w, z)
\]
(3.6)

4. Using He's semi – inverse method

In this section, we apply He’s semi – inverse method to solve the equation of a
one-dimensional motion of a Pion Meson particle in an atom.
The equation describing a one-dimensional motion of a Pion Meson particle in an
atom is the following [27]
\[
u_{tt} - u_{xx} + m^2 u + \lambda u^3 = 0,
\]
(4.1)
where \( m \) is the mass of the Pion Meson, and the cubic term in (4.1) describes the
Pion self – interaction with the effective coupling constant \( \lambda \).
Using the wave variable \( \xi = x - ct + \epsilon \), then the Eq.(4.1) is carried to an ODE
\[
c^2 \ddot{u} - u'' + m^2 u + \lambda u^3 = 0
\]
(4.2)
where the prime denotes the derivation with respect to \( \xi \).

By He's semi – inverse method [10], we can obtain the following variational
formulation
\[
J = \int_0^\infty \left[ -\frac{(c^2 - 1)}{2} \left( \frac{u'}{2} \right)^2 + \frac{m^2 u^2}{2} + \frac{\lambda}{4} u^4 \right] d\xi
\]
(4.3)

By a Ritz – like method, we search for a solitary wave solution in the form
\[
u(\xi) = A \sec h(B\xi)
\]
(4.4)
where \( A \) and \( B \) are unknown constants to be further determined. Substituting
Eq.(4.4) into Eq.(4.3), we have
\[
J = \int_0^\infty \left[ -\frac{A^2 B^2 (c^2 - 1)}{2} \sec h^2 (B\xi) \tanh^2 (B\xi) + \frac{A^2 m^2}{2} \sec h^2 (B\xi) + \frac{A^4 \lambda}{4} \sec h^4 (B\xi) \right] d\xi
\]
(4.5)
\[
= -\frac{A^2 B (c^2 - 1)}{6} + \frac{A^2 m^2}{2B} + \frac{A^4 \lambda}{6B}.
\]
Making \( J \) stationary with \( A \) and \( B \) yields
**Exact solutions**

\[
\frac{\partial J}{\partial A} = -\frac{1}{3} AB(-1 + c^2) + \frac{A m^2}{B} + \frac{2A^3 \lambda}{3B} = 0
\]

\[
\frac{\partial J}{\partial B} = -\frac{1}{6} A^2(-1 + c^2) - \frac{A^3 m^2}{2B^2} - \frac{A^4 \lambda}{6B^2} = 0
\]

From Eqs.(4.6) and (4.7), we get

\[
A = \frac{i \sqrt{2} m}{\sqrt{\lambda}}, \quad B = \frac{m}{\sqrt{1-c^2}}, \quad c^2 - 1 < 0.
\]

The soliton solutions are, therefore, obtained as follows

\[
u(x,t) = \frac{i \sqrt{2} m}{\sqrt{\lambda}} \text{sech} \left( \frac{m}{\sqrt{1-c^2}} (x - ct + \epsilon) \right).
\]

**5. Using first integral method**

In this section, we apply the first integral method to solve the equation of a one-dimensional motion of a Pion Meson particle in an atom.

Using the wave variable \( \xi = x - ct + \epsilon \), then the Eq.(4.1) is carried to an ODE

\[
c^2 u'' - u'' + m^2 u + \lambda u^3 = 0
\]

where the prime denotes the derivation with respect to \( \xi \).

Using (2.4) and (2.5), we get

\[
\begin{cases}
X'(\xi) = Y(\xi), \\
Y'(\xi) = \left( \frac{m^2}{1-c^2} \right) X(\xi) + \left( \frac{\lambda}{1-c^2} \right) X^3(\xi).
\end{cases}
\]

According to the first integral method, we suppose that \( X(\xi) \) and \( Y(\xi) \) are nontrivial solutions of (5.2) and (5.3) and

\[
Q(X,Y) = \sum_{i=0}^{m} a_i(X) Y^i = 0
\]

is an irreducible polynomial in the complex domain \( C[X,Y] \) such that

\[
Q(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X) Y^i = 0,
\]

where \( a_i(X) \) (\( i = 0, 1, 2, \ldots, m \)) are polynomials of \( X \) and \( a_m(X) \neq 0 \).

Eq.(5.4) is called the first integral to (5.2) and (5.3). Due to the Division Theorem, there exists a polynomial \( h(X) + g(X)Y \), in the complex domain \( C[X,Y] \) such that
\[
\frac{dQ}{d\xi} = \frac{dQ}{dX} \frac{dX}{d\xi} + \frac{dQ}{dY} \frac{dY}{d\xi} = (h(X) + g(X) Y) \sum_{i=0}^{m} a_i(X) Y^i .
\] (5.5)

Here we take two different cases, assuming that \( m = 1 \) and \( m = 2 \) in (5.4).

**Case I:**

Suppose that \( m = 1 \), by equating the coefficients of \( Y^i \)

\[
a'_1(X) = g(X) a_1(X),
\] (5.6)
\[
a'_0(X) = h(X) a_1(X) + g(X) a_0(X),
\] (5.7)
\[
a_1(X) \left[ \frac{m^2}{1-c^2} X(\xi) + \left( \frac{\lambda}{1-c^2} \right) X^3(\xi) \right] = h(X) a_0(X),
\] (5.8)

Since \( a_i(X)(i=0,1) \) are polynomials, then from (5.6) we deduce that \( a_1(X) \) is constant and \( g(X) = 0 \). For simplicity, take \( a_1(X) = 1 \).

Balancing the degrees of \( h(X) \) and \( a_0(X) \), we conclude that \( \text{deg}(h(X)) = 1 \) only.

Suppose that \( h(X) = AX + B \), and \( A \neq 0 \), then we find \( a_0(X) \):

\[
a_0(X) = \frac{A}{2} X^2 + BX + D
\] (5.9)

Substituting \( a_0(X), a_1(X) \) and \( h(X) \) in Eq.(5.8) and setting all the coefficients of powers \( X \) to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain:

\[
D = - \left( \frac{1}{1-c} \cdot \frac{1}{1+c} \right) \frac{m^2}{2 \sqrt{\lambda}} , \quad B = 0 , \quad A = - \left( \frac{1}{1-c} \cdot \frac{1}{1+c} \right) \sqrt{\lambda} ,
\] (5.10)

\[
D = \left( \frac{1}{1-c} \cdot \frac{1}{1+c} \right) \frac{m^2}{2 \sqrt{\lambda}} , \quad B = 0 , \quad A = \left( \frac{1}{1-c} \cdot \frac{1}{1+c} \right) \sqrt{\lambda} ,
\] (5.11)

Using Eqs.(5.10) and (5.11) into Eq.(5.4), we obtain
Exact solutions

\( Y_1(\xi) = \frac{1}{2} \sqrt{\frac{1}{1-c} + \frac{1}{1+c}} \sqrt{\lambda} X^2(\xi) + \frac{\sqrt{1-c} + 1}{2 \sqrt{\lambda}} m^2 \), \hspace{1cm} (5.12)

\( Y_2(\xi) = - \frac{1}{2} \sqrt{\frac{1}{1-c} + \frac{1}{1+c}} \sqrt{\lambda} X^2(\xi) - \frac{\sqrt{1-c} + 1}{2 \sqrt{\lambda}} m^2 \), \hspace{1cm} (5.13)

Combining (5.12) and (5.13) with (5.2) and (5.3), we obtain the exact solution to (5.1) as

\[ u_1(\xi) = \frac{m \tan \left[ \frac{m \left( i \sqrt{2} \left( - \frac{1}{1+c} \xi + 2 \left(-1+c\right) \sqrt{1+c} \sqrt{\lambda} c_1 \right) \right)}{2 \left(-1+c\right) \sqrt{1+c} \sqrt{\lambda}} \right]}{\sqrt{\lambda}}, \hspace{1cm} (5.14) \]

\[ u_2(\xi) = \frac{m \tan \left[ \frac{m \left( - \frac{i \xi}{\sqrt{2} \left(-1+c\right) \sqrt{1+c} \sqrt{\lambda} c_1 \right) \right)}{\sqrt{\lambda}} \right]}{\sqrt{\lambda}}, \hspace{1cm} (5.15) \]

where \( c_1 \) is an arbitrary integration constant.

Thus the exact travelling wave solutions to the equation (4.1) can be written as

\[ u_1(x, t) = \frac{m \tan \left[ \frac{m \left( i \sqrt{2} \left( x-ct+\varepsilon \right) + 2 \left(-1+c\right) \sqrt{1+c} \sqrt{\lambda} c_1 \right) \right]}{2 \left(-1+c\right) \sqrt{1+c} \sqrt{\lambda}} \right]}{\sqrt{\lambda}}, \hspace{1cm} (5.16) \]

\[ u_2(x, t) = \frac{m \tan \left[ \frac{m \left( - \frac{i \left( x-ct+\varepsilon \right)}{\sqrt{2} \left(-1+c\right) \sqrt{1+c} \sqrt{\lambda} c_1 \right) \right)}{\sqrt{\lambda}} \right]}{\sqrt{\lambda}}, \hspace{1cm} (5.17) \]

**Case II:**

Suppose that \( m=2 \), in Eq.(5.4), by equating the coefficients of \( Y^i, (i=0,1,2,3) \) on both sides of Eq.(5.5), we have

\[ a_2(X) = g(X) a_2(X), \hspace{1cm} (5.18) \]

\[ a_1^i(X) = h(X) a_2(X) + g(X) a_1(X), \hspace{1cm} (5.19) \]

\[ a_0^i(X) = -2 a_2(X)^i + h(X) a_1(X) + g(X) a_0(X), \hspace{1cm} (5.20) \]

\[ a_1(X) \left[ \left( \frac{m^2}{1-c^2} \right) X(\xi) + \left( \frac{\lambda}{1-c^2} \right) X^3(\xi) \right] = h(X) a_0(X). \hspace{1cm} (5.21) \]

Since \( a_2(X) \) is a polynomial of \( X \), then from (5.18) we deduce that \( a_2(X) \) is constant and \( g(X) = 0 \). For simplicity, take \( a_2(X) = 1 \).

Balancing the degrees of \( h(X) \), \( a_0(X) \) and \( a_1(X) \), we conclude that \( \deg(h(X)) = 0 \).
Suppose that \( \deg(h(X)) = 0 \), \( h(X) = A \) and \( A \neq 0 \),
then we find \( a_1(X) \) and \( a_0(X) \) as:

\[
a_1(X) = AX + B, \quad (5.22)
\]

\[
a_0(X) = \left(-\frac{\lambda}{2(1-c^2)}\right)X^4 + \left(\frac{-m^2}{1-c^2}\right)X^2 + ABX + D. \quad (5.23)
\]

where \( D \) and \( F \) are an arbitrary integration constants.

Substituting \( a_0(X), a_1(X) \) and \( h(X) \) in Eq.(5.21) and setting all the coefficients of
powers \( X \) to be zero, then we obtain a system of nonlinear algebraic equations and
by solving it, we obtain

\[
B = 0, \quad D = 0, \quad c = -\frac{\sqrt{A^2 - 4m^2}}{A}, \quad (5.24).
\]

\[
B = 0, \quad D = 0, \quad c = \frac{\sqrt{A^2 - 4m^2}}{A} \quad (5.25).
\]

Using Eqs.(5.24) and (5.25) into Eq.(5.4), we obtain:

\[
Y_1(\xi) = -\frac{A}{2}X(\xi) - \frac{\sqrt{2}\sqrt{A}}{4m}X^2(\xi), \quad (5.26)
\]

\[
Y_2(\xi) = -\frac{A}{2}X(\xi) + \frac{\sqrt{2}\sqrt{A}}{4m}X^2(\xi). \quad (5.27)
\]

Combining (5.26) and (5.27) with (5.2) and (5.3), we obtain the exact solution to (5.1) as

\[
u_3(\xi) = -\frac{2}{\exp \left( \frac{A}{2} \xi \right)} \exp \left[ \frac{A}{2} \xi \right] \exp \left[ 2m c_1 \right] \sqrt{A}, \quad (5.28)
\]

\[
u_4(\xi) = \frac{2}{\exp \left( \frac{A}{2} \xi \right)} \exp \left[ \frac{A}{2} \xi \right] \exp \left[ 2m c_1 \right] \sqrt{A}. \quad (5.29)
\]

where \( c_1 \) is an arbitrary integration constant.

Thus the exact travelling wave solutions to the equation (4.1) can be written as
These solutions are all new exact solutions.

6. Conclusion

In the present work, we have established variational formulation for the equation of a one-dimensional motion of a Pion Meson particle in an atom by He’s semi-inverse method to find solitary wave solutions. Also, the first integral method is implemented and applied for finding new exact solutions to the considered equation. The solutions obtained are triangular periodic, solitary waves and rationals. These new solutions may be important for the explanation of some practical physical problems. The first integral method described herein is not only efficient but also has the merit of being widely applicable. Therefore, the used methods are capable of greatly minimizing the size of computational work compared with other existing techniques.

References


[24] A. Jabbari, H. Kheiri and A. Bekir, Exact solutions of the coupled Higgs equation and the Maccari system using He’s semi – inverse method and \( \left( \frac{G'}{G} \right) \)- expansion method, Computers and Mathematics with Applications 62(2011) 2177-2186.


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