Gravitational Collapse of a Charged Shell

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Abstract

The junction formalism is applied to study the gravitational collapse of charged spherical shell. The modeling is based on the radiation fluid singular hypersurface filled with physical vacuum.

The barotropic fluids with the linear equation of state are considered. An equation governing the behavior of the physical vacuum energy density is deduced. Our result show that, no charge can stop the gravitational collapse of the shell below the upper Reissner-Nordstrom radius.

Keywords: General Relativity, Stability, Equation of Motion.

1 Introduction

About one hundred years ago, Lorentz suggested a model of an extended electron as the body having pure charge and no matter [1]. His model had severe problems with stability because the electric repulsion should eventually lead to explosion of the configuration proposed. Since that time, many modifications of
this model have been done. After appearance of general relativity, Einstein raised
the particle problem on the relativistic level; subsequently, Lorentz-Poincare
model gave a new sounding in terms of repulsive gravitation [2]. This relativistic
model seems to be appropriate for a charged spineless particle (e.g., π-meson)
rather than for an electron (except the works [3]).

Bonnor and Cooperstock [4] used a charged sphere to model the electron,
with a radius smaller than $10^{-16}$ cm. More recently, Ponce de Leon [5] has
constructed a new type of extended source for the Reissner-Nordstrom (RN)
geometry, with classical electron radius. Varela [6] introduced a neutral perfect
fluid core bounded by a charged thin shell. Cohen [7] discussed the
electromagnetic equations with singular energy-momentum tensor, which is
suitable for the study of inner and outer spherically symmetric, static geometries
described in curvature coordinates.

The study of the dynamics of a shell separating two background in the
context of general relativity has been developed in a powerful and direct
formalism since the pioneer work of Israel [8] and applied to the charged shell by
Kuchar [9]. Israel [8] found a set of invariant boundary conditions connecting the
relation between the extrinsic curvature of a shell on its both sides and the matter
of this shell. Also, Poisson [10] discussed the linearization stability of thin shell
wormholes. The gravitational collapse of a charged spherical shell is described by
matching the Schwarzschild de-Sitter solution with RN solution across the
junction surface (which is considered as a particle surface).

This paper is organized as follows. In Section 2 the Darmois–Israel
formalism is briefly reviewed. Match an interior Schwarzschild de-Sitter solution
to an exterior RN solution, and the equations of motion of charged shell are given
in Section 3. A general conclusion is given in Section 4. Also adopt the units such
that $c = G = 1$.

2 The Darmois – Israel Formalism

Consider two distinct space-time manifolds $M_+$ and $M_-$ with metrics
given by $g^+_{\mu\nu}(x^+)$ and $g^-_{\mu\nu}(x^-)$, in terms of independently defined coordinate
systems $x^+$. The manifolds are bounded by hypersurfaces $\Sigma_+$ and $\Sigma_-$,
respectively, with induced metrics $g^\pm_{ij}$. The hypersurfaces are isometric, i.e.
$g^+_{ij}(\xi) = g^-_{ij}(\xi) = g_{ij}(\xi)$, in terms of the intrinsic coordinates, invariant under the
isometry. A single manifold $M$ is obtained by gluing together $M_+$ and $M_-$ at
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their boundaries, i.e. \( M = M_+ \cup M_- \), with the natural identification of the boundaries \( \Sigma = \Sigma_+ = \Sigma_- \). The basic vectors \( e^\mu = \frac{\partial}{\partial \xi^a} \) tangent to \( \Sigma \) have the components \( e^\mu_{a\pm} = \frac{\partial x^a}{\partial \xi^a} \), with respect to the two four dimensional coordinate systems in \( M^{\pm} \). The second fundamental forms (extrinsic curvature) associated with the two sides of the shell are:

\[
K^\pm_\gamma = -n^\gamma_j \left( \frac{\partial^2 x^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma^\gamma_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right) ; \Sigma
\]  

(1)

where \( n^\gamma_j \) are the unit normal 4-vector to \( \Sigma \) in \( M \), with \( n_\gamma n^\mu = 1 \) and \( n_\mu e^\mu_i = 0 \). The Israel formalism requires that the normal point from \( M_- \) to \( M_+ \).

For the case of a thin shell \( K_\gamma_\gamma \) is not continuous across \( \Sigma \), so that, the discontinuity in the second fundamental form is defined as \( [K_\gamma_\gamma] = K^+_\gamma_\gamma - K^-_\gamma_\gamma \). The Einstein equation determines the relations between the extrinsic curvature and the three dimensional intrinsic energy momentum tensor are given by The Lanczos equations,

\[
S_\gamma = -\frac{1}{8\pi} \left( [K_\gamma_\gamma] - [K] g_{\gamma_\gamma} \right)
\]  

(2)

where \([K]\) is the trace of \([K_\gamma_\gamma]\) and \( S_\gamma \) is the surface stress-energy tensor on \( \Sigma \). The first contracted Gauss- Kodazzi equation or the “Hamiltonian” constraint

\[
G_{\mu\nu} n^\mu n^\nu = \frac{1}{2} (K^2 - K_\gamma_\gamma K^{-3} R),
\]  

(3)

with the Einstein equations provide the evolution identity

\[
S^\theta K_\gamma = \left[ -T_{\mu\nu} n^\mu n^\nu - \frac{\Lambda}{8\pi} \right]_\gamma.
\]  

(4)

The convention, \([X] = X^+ - X^- \), and \( \overline{X} = \frac{1}{2} (X^+ + X^-) \), is used. The second contracted Gauss- Kodazzi equation or the “ADM” constraint,

\[
G_{\mu\nu} e^\mu_\gamma n^\nu = K^j_{\gamma j} - K_{\gamma j}
\]  

(5)

With the Lanczos equations gives the conservation identity

\[
S^{\mu}_{\gamma j} = \left[ T_{\mu\nu} e^\mu_\gamma n^\nu \right].
\]  

(6)

The surface stress-energy tensor may be written in terms of the surface energy density \( \sigma \), and surface pressure \( p : S^i_j = diag \cdot (-\sigma, p, p) \).
For spherically symmetric thin shell, the Lanczos equations reduce to

\[
\sigma = -\frac{1}{4\pi} \left[ K^\theta_\theta \right] \\
p = \frac{1}{8\pi} \left( [K^r_r] + [K^\theta_\theta] \right) .
\]

(7) \hspace{1cm} (8)

If the surface stress-energy terms are zero, the junction is denoted as a boundary surface. If surface stress terms are present, the junction is called a thin shell.

### 3 Equation of motion

The Schwarzschild de-Sitter and RN metrics are written as:

\[
ds_z^2 = -C_\pm(r)dt_z^2 + C_\pm^{-1}(r)dr_z^2 + r_z^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

(9) where \(C_\pm(r)\) are given by

\[
C_\pm(r) = 1 - \frac{2m_\pm}{r} - \chi^2 r^2
\]

(10) for the Schwarzschild de-Sitter space-time and

\[
C_\pm(r) = 1 - \frac{2m_\pm}{r} + \frac{Q^2}{r^2}
\]

(11) for the RN space-time, where \(m_\pm\) are the gravitational masses, \(Q\) is the charge. Also, \(\chi^2 = \frac{8\pi}{3} \epsilon\), and \(\epsilon\) is a vacuum energy density. The suffix `+` denotes a quantity evaluated just outside the shell and `-' just inside the shell. From (10), (11) we get

\[
m_+ - m_- = \frac{Q^2}{2r} + \frac{\chi^2}{2} r^3
\]

(12) Let \(r\) be the area radius, i.e. the radial coordinate such that \(A = 4\pi r^2\) is the area of the spheres of symmetry at constant \(r\). The area radius is continuous across \(\Sigma\), which is not true for the time coordinates. Let the equation of the shell be \(r_\pm = R_\pm(\tau)\), the history of the shell is described by the hypersurface \(x_\pm^a = x_\pm^a(\tau, \theta, \phi)\), in the regions \(M^\pm\), respectively; the function \(R(\tau)\) describes the time evolution of the shell. The intrinsic metric on \(\Sigma\) is written as

\[
ds^2 = -d\tau^2 + R^2(\tau)(d\theta^2 + \sin^2 \theta d\phi^2)
\]

where \(\tau\) is the proper time of the shell. Using the Einstein field equation in an orthogonal reference frame, the stress-energy tensor components are given by
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\[ T^0_0 = \varepsilon = \rho + \frac{1}{8\pi} \frac{Q^2}{R^4} \]  \hspace{1cm} (13)

\[ T^1_1 = \varepsilon = -p_r + \frac{1}{8\pi} \frac{Q^2}{R^4} \]  \hspace{1cm} (14)

\[ T^2_2 = T^3_3 = \varepsilon = -p_t - \frac{1}{8\pi} \frac{Q^2}{R^4} \]  \hspace{1cm} (15)

where \( \rho(R) \) is the energy density, \( P_r(R) \) is the radial pressure, and \( P_t(R) \) is the lateral pressure measured in the orthogonal direction to the radial direction. The four velocity is given by,

\[ U^\mu = (C^{-1}_+ \sqrt{C_+ + \dot{R}^2}, \dot{R}, 0, 0) \]

where the over dot denotes a derivative with respect to \( \tau \). The unit normal to the junction surface is

\[ n^\mu = (\dot{R} C^{-1}_+ \sqrt{C_+ + \dot{R}^2}, 0, 0). \]

Taking into account the transparency condition, \([G_{\mu\nu} U^\mu n^\nu] = 0\), the conservation identity, equation (6), provides the simple relationship:

\[ \frac{d}{d\tau} (\sigma A) + p \frac{dA}{d\tau} = 0 \]  \hspace{1cm} (16)

where \( A = 4\pi R^2 \) is the area of the shell. The first term represents the variation of the internal energy of the shell, and the second term is the work done by the shell’s internal force. The generalization of the Tolman-Oppenheimer-Volkov equation for such charged models yields [12],

\[ \frac{d}{dr} p - \frac{1}{8\pi r^2} \frac{dQ^2}{dr} = 0 \]

for \( r \leq R \)

hence, \( Q(r) = Q \theta(r - R) \), where \( \theta(r) \) is the Heaviside step function. Thus, there is no electric field inside the core, and charge is accumulated on the boundary only. The electromagnetic potential is zero inside the shell whereas outside it has to be \( A_i = (-Q/r, 0, 0, 0) \). Using equation (1), the non-trivial components of the extrinsic curvature are given by:

\[ K^0_\phi = K^\phi_0 = \varepsilon_\pm \frac{1}{R} \sqrt{C_\pm + \dot{R}^2} \]

\[ K^r_\phi = \varepsilon_+ \frac{1}{\sqrt{C_+ + \dot{R}^2}} \left( \frac{m_+}{R^2 - \frac{Q^2}{R^3} + \ddot{R}} \right) \]

\[ K^r_\tau = \varepsilon_- \frac{1}{\sqrt{C_- + \dot{R}^2}} \left( \frac{m_-}{R^2 - \dot{\chi}^2 R + \ddot{R}} \right) \]

where \( \varepsilon_\pm = \text{sign} \sqrt{R^2 + C_\pm} \) are sign factors. Therefore, the Lanczos equations are given by
\( \varepsilon_+ \sqrt{\dot{R}^2 + C_+} - \varepsilon_- \sqrt{\dot{R}^2 + C_-} = -4\pi\sigma R \) \hspace{1cm} (17)

\[ \varepsilon_+ \left( \frac{m_+ - Q^2}{R^2} + \ddot{R} \right) \sqrt{C_+ + \dot{R}^2} - \varepsilon_- \left( \frac{m_- - \chi^2 R + \ddot{R}}{R^2} \right) \sqrt{C_- + \dot{R}^2} = -4\pi \frac{d}{dR} (\sigma R) \] \hspace{1cm} (18)

with \( M = 4\pi R^2 \sigma \) is the effective rest mass. It is well known that, \( \varepsilon = 1 \) if \( R \) increases in the outward normal direction to the shell, and \( \varepsilon = -1 \) if \( R \) decreases. We deal here with the condition \( \varepsilon_+ = \varepsilon_- = 1 \) of the ordinary shell. Independently of the Einstein equations the equation of state for matter on the shell \( p = p(\sigma) \) should be added as well. The simplest equation of state we choose is the linear one of a barotropic fluid \( p = \zeta \sigma \), including as a private case, dust \( (\zeta = 0) \), radiation fluid \( (\zeta = \frac{1}{2}) \), bubble matter \( (\zeta = -1) \), and so on \([13]\). Then the constant \( \zeta \) remains to be arbitrary. From (16), (17) we obtain

\[ \zeta = \frac{\alpha}{4\pi} R^{-2(\zeta + 1)} \] \hspace{1cm} (19)

where \( \alpha \) is the integration constant determined by the specific shell’s matter. The value of \( \alpha \) is closely related to the value of surface mass density (or pressure) at fixed \( R \). Therefore equations (17), (18) can be written in the form

\[ \sqrt{\dot{R}^2 + C_+} - \sqrt{\dot{R}^2 + C_-} = -\alpha / R^{2\zeta + 1} \] \hspace{1cm} (20)

\[ \left( \frac{m_+ - Q^2}{R^2} + \ddot{R} \right) \sqrt{C_+ + \dot{R}^2} - \left( \frac{m_- - \chi^2 R + \ddot{R}}{R^2} \right) \sqrt{C_- + \dot{R}^2} = \alpha (2\zeta + 1) / R^{2(\zeta + 1)} \] \hspace{1cm} (21)

In static case \( (\ddot{R} = 0) \), these equations turn to be the equilibrium conditions:

\[ \sqrt{C_+} - \sqrt{C_-} = -\alpha / R^{2\zeta + 1} \] \hspace{1cm} (22)

\[ \left( \frac{m_+ - Q^2}{R^2} \right) \sqrt{C_+} - \left( \frac{m_- - \chi^2 R}{R^2} \right) \sqrt{C_-} = \alpha (2\zeta + 1) / R^{2(\zeta + 1)} \] \hspace{1cm} (23)

Further, it is well-known that the classical radius \( R_c \) of a charged particle can be defined as the radius at which the function \( C_+ \) approaches a minimum. Therefore, \( R_c = Q^2 / m_+ \), then
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\[ \sqrt{1 - \frac{m_+^2}{Q^2}} = \sqrt{1 - \frac{2m_+}{Q^2} - \left(\frac{\chi_+ Q^2}{m_+}\right)^2} = -\alpha_\epsilon \left(\frac{m_+}{Q^2}\right)^{2\zeta+1} \]  

(24)

\[ X_\epsilon^2 = \frac{m_+ m^3}{Q^6} + \alpha_\epsilon (2\zeta + 1) \left(\frac{m_+}{Q^2}\right)^{2\zeta+3} \sqrt{1 - \frac{2m_+}{Q^2} - \left(\frac{\chi_+ Q^2}{m_+}\right)^2} \]  

(25)

These equations including many cases with different values of \( \zeta \). Considering the choice \( \zeta = \frac{1}{2} \), (the case of radiation fluid) and eliminating \( \alpha_\epsilon \) by dividing the last two equations, to get

\[ X_\epsilon^2 - \frac{m_+ m_3}{Q^6} = -2\Psi \left(\frac{m_+}{Q^2}\right)^2 \sqrt{1 - \frac{2m_+}{Q^2} - \left(\frac{\chi_+ Q^2}{m_+}\right)^2} \]

with

\[ \Psi = \sqrt{1 - \frac{m_+^2}{Q^2}} - \sqrt{1 - \frac{2m_+}{Q^2} - \left(\frac{\chi_+ Q^2}{m_+}\right)^2}. \]

Then, the exact electromagnetic mass relation is

\[ X_\epsilon^2 = \frac{2m_+^2}{9Q^4} \left\{ 2 + \frac{m_+^2}{Q^2} - \frac{9m_+}{2Q^2} \pm \sqrt{\left(1 - \frac{m_+^2}{Q^2}\right)\left(4 - \frac{m_+^2}{Q^2} - \frac{9m_+}{2Q^2}\right)} \right\} \]

(26)

which determines the necessary stabilizing energy of Schwarzschild de-Sitter vacuum inside a charged particle within frameworks of the classical model having a radiation fluid singular surface.

Equation (17) can be written in the following dynamical form

\[ \dot{R}^2 + V(R) = 0 \]  

(27)

where,

\[ V(R) = 1 - \frac{M^2}{4R^2} - \left(\frac{m_+ - m_-}{M}\right)^2 - \left(\frac{m_+ + m_-}{R}\right)^2 - \frac{R^2}{4M^2} \Gamma \]

(28)

with

\[ \Gamma = \chi^4 R^4 + 2M^2 \chi^2 (1 - a) + 2\chi^2 Q^2 + \frac{Q^4}{R^4} - \frac{2M^2 Q^2}{R^4} (1 + a) \]

is the effective potential of the shell, with \( a = \frac{2R}{M^2} (m_+ - m_-) \), and \( M = 4\pi\sigma R^2 = \alpha/R \), is the mass of the radiation fluid shell in the static reference frame. This single dynamical equation (27), completely determines the motion of the shell. From the static solution, equation (22), the exterior gravitational mass is given by
\[ m_* = \frac{1}{2} \chi^2 R^3 + m_+ + \frac{1}{2R} (Q^2 + M^2) \pm \frac{M}{R} \sqrt{Q^2 + R^2 + 2m_- R} \]  \hspace{1cm} (29)

Linearizing around a stable solution situated at \( R = R_* \), we consider a Taylor expansion of \( V(R) \) around \( R_* \) to second order, given by

\[ V(R) = V(R_*) + V'(R_*)(R - R_*) + \frac{1}{2} V''(R - R_*)^2 + O(3) \]

where the prime denotes a derivative with respect to \( R \). The first and second derivatives of \( V(R) \) are, respectively, given by

\[ V'(R) = \frac{m_+ + m_-}{R^2} + \frac{M^2}{2R^3} - \frac{1}{M^2 R^3} \left\{ \frac{1}{2} \chi^4 R^4 + M^2 \chi^2 R^4 + H \right\} \]

\[ V''(R) = \frac{-2(m_+ + m_-)}{R^3} - \frac{3M^2}{2R^4} - \frac{1}{M^2 R^4} \left\{ \frac{3}{2} \chi^4 R^8 + M^2 \chi^2 R^8 + Z \right\} \]

with

\[ H = \chi^2 Q^2 R^4 - \frac{1}{2} Q^4 + M^2 Q^2 - (m_+ - m_-)(3 \chi^2 R^5 - Q^2 R) \]

\[ Z = \chi^2 Q^2 R^4 + \frac{3}{2} Q^4 - 3M^2 Q^2 - (m_+ - m_-)(6 \chi^2 R^5 + 2Q^2 R) \]

Evaluated at the static solution, through a long calculation, we find \( V(R_*) = 0 \) and \( V'(R) = 0 \). Since, the value of \( m_+ / Q \) is very small for the known elementary particles, then, using the Taylor expansion of equation (26), to get

\[ \varepsilon = \frac{3}{16 \pi} \frac{m_*^3}{Q^6} (m_+ - m_-) + O\left( \frac{m_*^5}{Q^8} \right). \]  \hspace{1cm} (30)

Let us display some numerical estimations of internal vacuum energy density in the case of \( m_- = 0 \). For an electron we have: \( m_* = 0.511 MeV = 6.76 \times 10^{-58} m \), \( Q = 1.38 \times 10^{-36} m, R_c = 2.8 \times 10^{-15} m \) hence \( \varepsilon = 2.5 KeV fm^{-3} \).

For \( \pi \)-mesons we have: \( m_* = 139.57 MeV, Q = 1.38 \times 10^{-36} m, R_c = 1.02 \times 10^{-17} m \) hence \( \varepsilon = 14 TeV fm^{-3} \).

We can see that, for particles of reasonably large mass the huge internal vacuum energy and electrical charge can be equilibrated by the surface gravitational forces providing the stability of the model. The gravitational defect of masses causes the property that the external mass of a particle appears to be much less than the internal energy.

4 Conclusion

In the framework of Darmois-Israel formalism, gravitational collapse of a charged spherical shell is constructed, by matching the external RN solution of the
Einstein-Maxwell field equations with the internal Schwarzschild de-Sitter solution across the singular surface.

We construct the classical model of an extended charged particle based on the equilibrium radiation fluid shell filled with vacuum energy. Such model can have equilibrium states at the radius equal to the classical radius of a charged particle. In the future we will study this model in the quantum level, to get the Wheeler-de Witt equation.

References


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