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Radiation Dominated and Matter Dominated Standard Cosmology Extended to Include Particle Creation

Antonio Zecca[1, 2]

Dipartimento di Fisica dell' Universita' degli Studi di Milano - Italy
GNFM Gruppo Nazionale per la Fisica Matematica Sesto Fiorentino (Fi) - Italy
antonio.zecca@mi.infn.it

Abstract

A cosmological model with cosmological constant is formulated by assuming the validity of the Standard Cosmological model where density and pressure are modified by the adjoint of terms proportional to $\dot{R}/R$. These terms represent the back reaction contribution of created particles. Their expression is induced from previous results on field quantization in Robertson-Walker space-time. The radiation dominated and matter dominated models are solved exactly in the flat space-time case. The solutions allow the determination of the parameter Ω_{0c} relative to particle creation, for different "reasonable" value of the vacuum energy parameter Ω_λ. There results a strong improvement of the values of Ω_{0c} previously determined in absence of cosmological term.

Keywords: Classical general relativity; Standard Cosmology; Quantum fields in curved space-time; Particle creation; Back reaction; Einstein field equation; Solution

1 Introduction

Particle production is an effect that follows from field quantization in curved space-time.The existence of such effect was already pointed out by Parker for spin $0, 1/2$ [1, 2, 3]. The argument, even if with problematic aspects [4, 5] has been widely developed. One can now refer to [5, 6] and more recently [7, 8] for systematic treatment of the argument. A consequence of particle production is the modification of the gravitational dynamics that has to be taken into account in the formulation of a cosmological model. A general approach to that problem is that of taking into account the back reaction of quantum field along the line of the standard exposition proposed in [6].

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[1]Retired from: *Dipartimento di Fisica dell' Universita' degli Studi di Milano - Italy*
[2]Address: Via Celoria, 16 - 20133 Milano, Italy.

According to that approach a back reaction of quantum origin is incorporated in the theory by considering the expectation value of the effective energy momentum tensor as the source of the Einstein field equation. The study is very complex being based on the determination of the effective action of the fields and often reduces to perturbative solution of the Einstein equation (See, e., g. [8, 9, 10].

In the present paper we do not follow that canonical line of research. Instead we propose a standard-like cosmological model that includes the back reaction of created particles and it is based on recent results on quantization of spin 0, 1/2, 1 fields in Robertson-Walker (RW) space-time [11, 12, 13]. According to those results the balance $n(t)$ of created and destroyed particles per unit time, fixed mode and spin, results of the form $n(t) = \pm 6\dot{R}/R$, $R(t)$ the radius of the Universe in RW space-time (Even if not yet proved it is guessed that $n(t)$ is the same for arbitrary spin). Unfortunately, according to those results, the number of created particles of arbitrary mode results to be divergent. Moreover it is not clear where particles are created. In spite of these limitations it is possible to take into account the back reaction of particle production by assuming the validity of the Standard Cosmological Model where, however, density and pressure ρ, p are heuristically defined by the substitution

$$\begin{aligned} \rho \ &\rightarrow \rho + \rho_c = \rho + \alpha(t)\dot{R}/R \\ p \ &\rightarrow p + p_c = p + \beta(t)\dot{R}/R \end{aligned} \tag{1}$$

The expressions $\rho_c = \alpha(t)\dot{R}/R$ and $p_c = \beta(t)\dot{R}/R$ represent the contribution of the density and pressure of created particles. $\alpha(t)$, $\beta(t)$ should be known functions or in principles determinable by the observational data. If taken for grant such cosmological model should be more easily solvable than the canonical one mentioned above. Such model has been already studied in [14, 15]. (An analogous study has been performed in Lemaître-Tolman-Bondi space-time [16, 17]). In [14] it has been assumed $\alpha(t) \propto R^{-3}$, $\beta(t) \propto R^{-2}$ based on the naive assumption that the created (destroyed) particles are uniformly distributed in a sphere of radius R. The corresponding Einstein equation has been solved in different approximations. In [15] the problem has been reconsidered and discussed with $\alpha(t)$, $\beta(t)$ assigned functions. The choice $\alpha(t) = const.$, $\beta(t) = const.$ has been assumed for the study of the matter and radiation dominated cases that have been explicitly solved. The solution allows in both cases a numerical evaluation of the role of the created particles by a direct numerical evaluation of the parameter Ω_{0c} that results to fall into the expected range. In both studies [14, 15] the cosmological constant term has been neglected, the primary object being there of testing the relevance of Ω_{0c}.

In the present paper we formulate the Einstein equation with matter, radiation, cosmological term and back reaction due to particle creation. We study the Einstein cosmological equation in the matter dominated and radiation dominated cases by assuming the functions $\alpha(t)$, $\beta(t)$ to be constant in time. (The choice is somewhat supported by the remark in [15] according to which the only non constant expression of $\alpha(t)$ so far tested, that is $\alpha(t) \propto R^{-3}$, gives non acceptable value for Ω_{0c}).

We study the model for the matter dominated and radiation dominated case separately. The complete solution is explicitly determined, in both cases, in the flat space-time case. If evaluated at the present time, the solutions allow a discussion on the numerical relation between the created particle and the vacuum energy parameters Ω_{0c}, Ω_λ, respectively. There results a strong improvement of the value of Ω_{0c} previously determined in both cases [15]. Possible solutions are also found for negative Ω_{0c} values that would imply the dominance of annihilated particles with respect to the created ones at the present age of the Universe.

2 General cosmological model

We formulate the general scheme for homogeneous and isotropic universe. The appropriate space-time context is therefore [18, 19] the Robertson-Walker (RW) space-time whose geometry is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - R^2(t)\Big[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\Big], \quad K = 0, \pm 1 \quad (2)$$

with coordinates $x^0 = t$, $x^k = r, \theta, \varphi$. The cosmological equation is assumed to be the Einstein field equation with canonical Energy Momentum tensor of a perfect fluid of density and pressure $D = D(t)$, $P = P(t)$. Also the contribution of a cosmological term is taken into account. Formally

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\, T_{\mu\nu} - \lambda g_{\mu\nu} \quad (3)$$

$$T^\nu_\mu = diag\{D, -P, -P, -P\} \quad (4)$$

On account of the symmetry assumptions, the density D and pressure P of the Universe have a dependence on the coordinates of the type $D = D(t)$, $P = P(t)$. By explicit (3), (4) in the metric (2) and by taking into account the consistency condition $\nabla_\mu T^{\mu\nu} = 0$, one is left with ($\dot{R} = \partial R/\partial t$)

$$3\Big(\frac{\dot{R}^2}{R^2} + \frac{K}{R^2}\Big) = 8\pi GD - \lambda \qquad (\mu = \nu = t) \quad (5)$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{K}{R^2} = -8\pi GP - \lambda \qquad (\mu = \nu = r, \theta, \varphi) \quad (6)$$

$$3\frac{\dot{R}}{R}(D + P) + \dot{D} = 0 \qquad (\nu = t) \quad (7)$$

The only non trivial consistency condition (7) is in fact consequence of (5), (6) as one can easily see. There follows that it suffices to give $D(t)$ in (5) to solve the problem. Indeed $R(t)$ would follow from (5) that in turns gives $P(t)$ by solving (6).

We now suppose that the universe is filled with matter, radiation and particles created from universe expansion according to the considerations of the Introduction. Therefore we assume (e.g., [20])

$$D = \rho_{om}\Big(\frac{R_0}{R}\Big)^3 + \rho_{or}\Big(\frac{R_0}{R}\Big)^4 + \rho_{oc}\frac{\dot{R}}{R}\frac{R_0}{\dot{R}_0} \tag{8}$$

where R_0, ρ_{0m}, ρ_{0r}, ρ_{0c} are respectively the radius of the Universe and the different densities at the present time. By defining $z = R/R_0$ and by further setting

$$\begin{aligned}
&\Omega_{om} = \frac{8\pi G}{3H_0^2}\rho_{0m}, \;\; \Omega_{or} = \frac{8\pi G}{3H_0^2}\rho_{0r} \;\; \Omega_{oc} = \frac{8\pi G}{3H_0^2}\rho_{0c}, \;\; \Omega_K = -\frac{K}{H_0^2 R_0^2}, \\
&\Omega_\lambda = -\frac{\lambda}{H_0^2} \qquad (\Omega_K + \Omega_{0m} + \Omega_{0r} + \Omega_{0c} + \Omega_\lambda = 1)
\end{aligned} \tag{9}$$

($H_0 = \dot{R}_0/R_0$ the Hubble "constant" at the present time), the equation (5) can be transformed into

$$\Big(\frac{\dot{z}}{H_0}\Big)^2 - \frac{\dot{z}}{H_0}\Omega_{0c}z + \Big(\Omega_{0r} + \Omega_{0m} + \Omega_{0c} + \Omega_\lambda - 1 - \frac{\Omega_{0m}}{z} - \frac{\Omega_{0r}}{z^2} - \Omega_\lambda z^2\Big) = 0 \tag{10}$$

The last equation can be integrated in the form $t = \int_0^t dz/\dot{z}$ or

$$t = \int_0^z \frac{\frac{2}{H_0}dz}{\Omega_{0c}z \pm \sqrt{(\Omega_{0c}z)^2 + 4\Big[1 - \Omega_{0m} - \Omega_{0r} - \Omega_{0c} - \Omega_\lambda + \frac{\Omega_{0m}}{z} + \frac{\Omega_{0r}}{z^2} + \Omega_\lambda z^2\Big]}} \tag{11}$$

The integral (11) has been already considered in [15] in the cases

$$\Omega_\lambda = \Omega_K = \Omega_{0m} = 0, \quad \Omega_{0r} + \Omega_{0c} = 1 \tag{12}$$

$$\Omega_\lambda = \Omega_K = \Omega_{0r} = 0, \quad \Omega_{0m} + \Omega_{0c} = 1 \tag{13}$$

that correspond to the flat radiation dominated and matter dominated cases in absence of cosmological term. By the analytical solution of (11) it was possible also to determine the value of Ω_{0c} that results $\Omega_{0c} \approx 0.98$ both in case (12) and (13). To be complete the integral (11) could be explicitly performed also in the curved space-time radiation dominated case, that is for $\Omega_\lambda = \Omega_{0m} = 0$, $\Omega_K + \Omega_{0r} + \Omega_{0c} = 1$. The result however has so a complex dependence on Ω_{0r}, Ω_K, Ω_{0c} that makes a numerical study very cumbersome. We prefer to consider situations where also the vacuum energy term is present.

3 Radiation dominated model

We first study the cosmological model proposed in the previous section in the flat radiation dominated case with cosmological term. One has therefore to solve the scheme

$$\Omega_K = \Omega_{0m} = 0, \quad \Omega_{0c} + \Omega_\lambda + \Omega_{0r} = 1 \tag{14}$$

$$t = \frac{1}{H_0} \int_0^{z^2} \frac{dv}{\Omega_{0c} v \pm \sqrt{(\Omega_{0c}^2 + 4\Omega_\lambda) v^2 + 4\Omega_{0cr}}} \quad (v = z^2) \tag{15}$$

After a rationalization of the integrand one can perform the integration and finally to obtain

$$4H_0 t = -\frac{\Omega_{oc}}{2\Omega_\lambda} \log \left| \frac{\Omega_\lambda z^4 + \Omega_{0r}}{\Omega_{0r}} \right| \mp \frac{\Omega_{oc}}{\Omega_\lambda} \tanh^{-1} \frac{\Omega_{0c} z^2}{\sqrt{(\Omega_{0c}^2 + 4\Omega_\lambda) z^4 + 4\Omega_{or}}}$$
$$\pm \frac{\sqrt{\Omega_{0c}^2 + 4\Omega_\lambda}}{\Omega_\lambda} \log \frac{\sqrt{(\Omega_{0c}^2 + 4\Omega_\lambda) z^4 + 4\Omega_{or}} + z^2 \sqrt{\Omega_{0c}^2 + 4\Omega_\lambda}}{2\sqrt{\Omega_{0r}}} \tag{16}$$

At the present time $t = t_0,\ z = 1$ so that from (16) one obtains

$$4t_0 H_0 = f_\pm(x, y), \qquad x \equiv \Omega_{oc},\ y \equiv \Omega_\lambda \tag{17}$$

$$f_\pm = -\frac{x}{4y} \log \left| \frac{1-x}{1-x-y} \right|^2 \mp \frac{x}{y} \tanh^{-1} \frac{x}{\sqrt{(x-2)^2}} \pm$$
$$\pm \frac{\sqrt{x^2+4y}}{y} \log \frac{\sqrt{(x-2)^2} + \sqrt{x^2+4y}}{2\sqrt{1-x-y}} \tag{18}$$

It is interesting to know from (16), (17) what values $\Omega_{0c} = x$ takes for given Ω_λ at the present time with $t_0 \simeq 13.7 \times 10^9 yr,\ H_0^{-1} \simeq 9,77 \times 10^9 yr$ and $4t_0 H_0 \approx 5.5$. To that end it is useful the graphic representation of the functions $f_\pm$. In Fig. 1 one can see that the behavior of $f_+(x, y)$ is suitable in order to have possible solutions of (17). Instead we see from the representation of $f_-(x, y)$ in Fig. 2 that $f_-(x, y)$ is negative in the interval of interest of the variables so that the equation $4t_0 H_0 = f_-(x, y)$ has no solution in this case. For what concerns the solution of $4t_0 H_0 \simeq 5.5 = f_+(x, y)$ it is useful to consider Fig. 3 that represents $f_+(x, y)$ for fixed value of $y \equiv \Omega_\lambda$. In correspondence to some value of Ω_λ in the interval of interest (see e. g., [18]) one finds

$$\begin{array}{lll} \Omega_\lambda = 0.6 & \Omega_{0c} \simeq 0.38 & (\Omega_{0r} \simeq 0.02) \\ \Omega_\lambda = 0.8 & \Omega_{0c} \simeq 0.18 & (\Omega_{0r} \simeq 0.02) \\ \Omega_\lambda = 1.0 & \Omega_{0c} \simeq -0.02 & (\Omega_{0r} \simeq 0.02) \\ \Omega_\lambda = 1.2 & \Omega_{0c} \simeq -0,22 & (\Omega_{0r} \simeq 0.02) \\ \Omega_\lambda = 1.4 & \Omega_{0c} \simeq -0.42 & (\Omega_{0r} \simeq 0.02) \end{array} \tag{19}$$

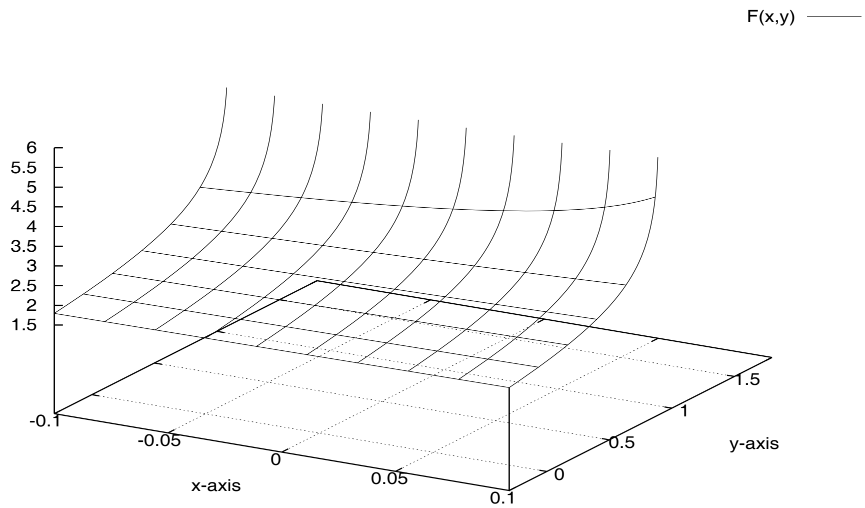

Figure 1: Qualitative behavior of $F(x,y) = f_+(x,y),\ x \equiv \Omega_{0c},\ y \equiv \Omega_\lambda$.

There results that the value of $\Omega_{0r} = 1 - \Omega_{0\lambda} - \Omega_{0c}$ seems to be constant at the order of the present numerical approximations and Ω_{0c} is decreasing for increasing Ω_λ. Moreover Ω_{0r} can take negative values. This means that for such values ρ_{0c} is negative and if $\dot{R}_0$ is positive the annihilated particles dominate over the created ones at the present age of Universe expansion. (See also the final Remark).

4 Matter dominated model

The second case of interest that we solve corresponds to the flat matter dominated model with cosmological term. From (9), (11) we are left with the solution of the scheme

$$\Omega_K = \Omega_{0r} = 0, \quad \Omega_{0c} + \Omega_\lambda + \Omega_{0m} = 1 \tag{20}$$

$$t = -\frac{1}{2H_0}\int_0^z \frac{\Omega_{0c}v^2 \mp v\sqrt{(\Omega_{0c}^2 + 4\Omega_\lambda)v^4 + 4\Omega_{0m}v}}{\Omega_\lambda v^3 + \Omega_{0m}}dv \tag{21}$$

The integration of (21) finally gives

$$3H_0 t = -\frac{\Omega_{oc}}{2\Omega_\lambda}\log\left|\frac{\Omega_\lambda z^3 + \Omega_{0m}}{\Omega_0}\right| \mp \frac{\Omega_{oc}}{\Omega_\lambda}\tanh^{-1}\frac{\Omega_{0c}z^{3/2}}{\sqrt{(\Omega_{0c}^2 + 4\Omega_\lambda)z^3 + 4\Omega_{om}}}$$
$$\pm\frac{\sqrt{\Omega_{0c}^2 + 4\Omega_\lambda}}{\Omega_\lambda}\log\frac{\sqrt{(\Omega_{0c}^2 + 4\Omega_\lambda)z^3 + 4\Omega_{om}} + z^{3/2}\sqrt{\Omega_{0c}^2 + 4\Omega_\lambda}}{2\sqrt{\Omega_{0m}}} \tag{22}$$

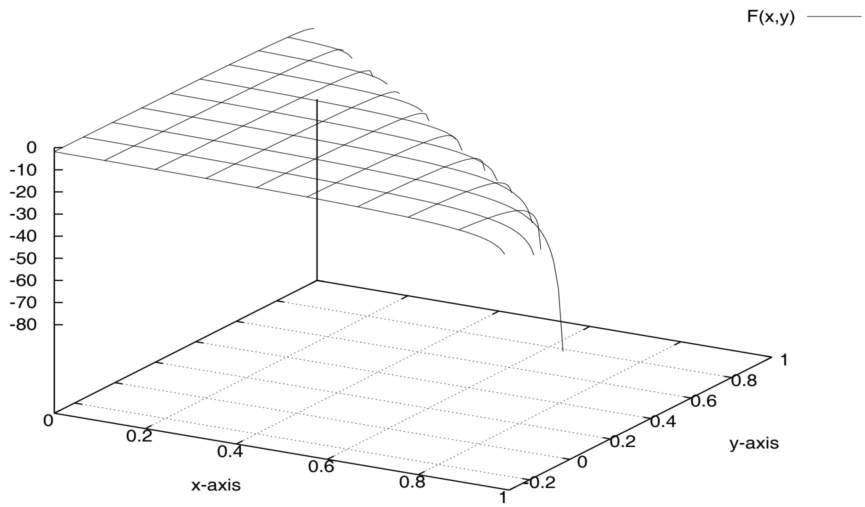

Figure 2: Qualitative behavior of $F(x,y) = f_-(x,y),\ x \equiv \Omega_{0c},\ y \equiv \Omega_\lambda$.

To estimate Ω_{0c} we again confine to $t = t_0,\ z = 1$. From (22) we have

$$3t_0H_0 = g_\pm(x,y), \qquad x \equiv \Omega_{oc},\ y \equiv \Omega_\lambda \tag{23}$$

$$g_\pm = -\frac{x}{4y}\log\left|\frac{1-x}{1-x-y}\right|^2 \mp \frac{x}{y}\tanh^{-1}\frac{x}{\sqrt{(x-2)^2}} \pm \pm\frac{\sqrt{x^2+4y}}{y}\log\frac{\sqrt{(x-2)^2}+\sqrt{x^2+4y}}{2\sqrt{1-x-y}} \tag{24}$$

so that $g_\pm(x,y)$ results to be exactly the function of the previous Section: $g_\pm(x,y) = f_\pm(x,y)$.

Therefore, as in the previous Section, $3t_0H_0 = g_-(x,y)$ has no solutions, while $3t_0H0 \simeq 4.11 = g_+(x,y) = f_+(x,y)$ can be solved again graphically from Fig. 3. In the interval of interest one has in the present case:

$$\begin{array}{lll} \Omega_\lambda = 0.6 & \Omega_{0c} \simeq 0.32 & (\Omega_{0m} \simeq 0.08) \\ \Omega_\lambda = 0.8 & \Omega_{0c} \simeq 0.12 & (\Omega_{0m} \simeq 0.08) \\ \Omega_\lambda = 1.0 & \Omega_{0c} \simeq -0.08 & (\Omega_{0m} \simeq 0.08) \\ \Omega_\lambda = 1.2 & \Omega_{0c} \simeq -0,28 & (\Omega_{0m} \simeq 0.08) \\ \Omega_\lambda = 1.4 & \Omega_{0c} \simeq -0.48 & (\Omega_{0m} \simeq 0.08) \end{array} \tag{25}$$

Also here $\Omega_{0m} = 1 - \Omega_\lambda - \Omega_{0c}$ remains constant in the numerical approximation done and Ω_{0c} can take negative values.

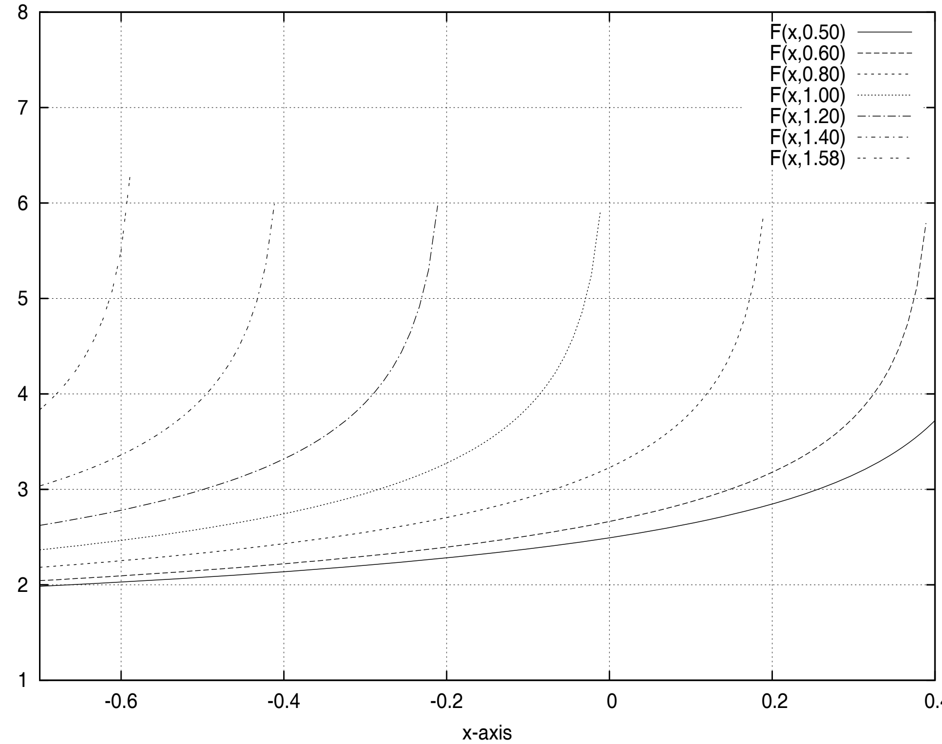

Figure 3: Behavior of $F(x,y) = f_+(x,y)$, $x \equiv \Omega_{0c}$ for different fixed values of $y \equiv \Omega_\lambda$.

5 Concluding Remarks

In the previous Sections the Standard Cosmological model with cosmological term has been extended to include the back reaction of particle production due to Universe expansion. To that end the density function of the Universe has been re-normalized by the adjoint of a term of the form $\alpha \dot{R}/R$. Such expression has been borrowed from the results on field quantization in curved space-time developed in previous papers [11, 12, 13, 16, 17].

The scheme has been solved exactly in the two cases corresponding to the radiation dominated and matter dominated flat cosmology. In correspondence to the solutions it has been possible to calculate the numerical value of the parameter Ω_{0c} relative to particles creation by Universe expansion at the present time. Both in the radiation dominated and matter dominated case such value is greatly reduced, as expected, if compared with the value in absence of cosmological term obtained in [15].

Another feature of the result is that, a priori, negative values of Ω_{0c} are possible. This corresponds to the fact that annihilated particles can dominate over the created ones in some phase of the Universe expansion. It seems of interest to compare our conclusion with some results originated from thermodynamical considerations. If one assumes that the Universe, considered as an open system, is subjected to an adiabatic transformation as a consequence of particle creation, then, by the second law of thermodynamics, one obtains that the total number of particles of the Universe cannot decrease [21, 22]. Therefore, if one takes for grant such application of thermodynamics to the Universe, the negative values of Ω_{0c} should be ruled out.

The solution of the general model proposed in this paper concerns only the flat space-time case and it has been separately given for the matter and the radiation dominated case. For what concerns the curved space-time cases, the eq. (11) could indeed be integrated explicitly in the radiation dominated case e. g., for $\Omega_{0m} = 0,\ \Omega_{0c} + \Omega_{0r} + \Omega_\lambda + \Omega_K = 1$. There results however so a complex and cumbersome dependence of the solutions on the parameters, to make a numerical analysis beyond the object of the present paper.

Finally, also the case with simultaneously $\Omega_{0m} \neq 0,\ \Omega_{0r} \neq 0$ is of interest. We are not able to furnish the analytical solution to this case, but we expect from it a further decreasing of the value of Ω_{0c}.

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