$W_k$– Algebra and Fractional Supersymmetry

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Abstract

Based on several previous works on fractional statistics and on the symmetry $U_q(sl2)$, we focus in this work to present some properties of the fractional supersymmetry by using the generalized Weyl-Heisenberg algebra $W_k$. Field theoretical representation is also discussed.

1 Introduction

Recently, there has been several interests in studying 2d field theoretical models exhibiting fractional supersymmetries [1, 2, 3, 4, 5, 6]. The latter are special symmetries of the infinite dimensional parafermionic invariance of 2d conformal coset models [7]. They may also be viewed as finite dimensional global symmetries extending the usual 2d supersymmetric algebra

$$\left (Q_{-\frac{1}{2}}\right )^2 = P_{-1}; \left (Q_{\frac{1}{2}}\right )^2 = P_1$$

(1)

generated by the supersymmetric charges $Q_{\pm \frac{1}{2}}$ and the energy momentum vector $P_{\pm 1}$. Fractional supersymmetries are related to the periodic representation of $U_q(sl2)$; $q^2 = 1$; for which the momentum vector $P_{\pm 1}$ is proportional to the center of the group representation. For the $k^{th}$ root of unity; $q^k = 1$, we can write

$$\left (Q_{-\frac{1}{4}}\right )^k = P_{-1}; \left (Q_{\frac{1}{4}}\right )^k = P_1$$

(2)

where $Q_{\pm \frac{1}{4}}$ are new charge operators carrying fractional spins.

Following the Zamalodchicov and Fateev $Z_3$ parafermionic symmetry and knowing that the operators charge is depending of the type of the vacuum state, the authors of [7] have postulated that the equation (2) must be rewriting as follow

$$P_{-1} = (Q^-_{-\frac{1}{3}}Q^-_0Q^-_{-\frac{2}{3}} + Q'^+_{-\frac{1}{3}}Q'^+_0Q'^+_{-\frac{2}{3}})S_0$$

$$+ (Q^-_{-\frac{2}{3}}Q^-_0Q^-_{-\frac{4}{3}} + Q'^+_{-\frac{2}{3}}Q'^+_0Q'^+_{-\frac{4}{3}})S_2$$

$$+ (Q^-_{-\frac{1}{3}}Q^-_0Q^-_{-\frac{2}{3}} + Q'^+_{-\frac{1}{3}}Q'^+_0Q'^+_{-\frac{2}{3}})S_{-2}$$

(3)

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where $Q_{-r}$; $r = 0, \frac{1}{3}, \frac{2}{3}$ are operators that act on projectors $S_0, S_1, S_2$
Thus, knowing that the equation (2) is used on the 2d fractional supersymmetry, the aim of this paper is to prove that the development of equation (2) lead to equation (3). Another focus of this paper is to give a representation of the equation (3) by using the power of the Generalized Weyl-Heisenberg Algebra.

2 Representation of the Fractional supersymmetry

A Generalized Weyl-Heisenberg Algebra $W_k$

The generalized Weyl-Heisenberg algebra $W_k$, with $k \in \mathbb{N} \setminus \{0, 1\}$, is generated by four linear operators $X_-$ (annihilation operator), $X_+$ (creation operator), $N$ (number operator) and $K$ ($Z_k$-grading operator). The operators $X_-$ and $X_+ = (X_-)^+$ are connected via the hermitian conjugation. These operators are acting on separable Hilbert space and satisfying the following relations:

$$[X_-, X_+] = \sum_{s=0}^{k-1} f_s(N) \Pi_s$$

$$[N, X_\pm] = \pm X_\pm$$

$$[N, X_+]_q = KX_+ - qX_+ K = 0$$

$$[N, X_-]_q = KX_- - qX_+ K = 0$$

$$[K, N] = 0, \quad K^k = 1$$  \hspace{1cm} (4)

The functions $f_s : N \mapsto f_s(N)$ must satisfy the constraint $(f_s(N))^+ = f_s(N)$ and the operators projection $\Pi_s$ are defined by

$$\Pi_s = \frac{1}{k} \sum_{s=0}^{k-1} q^{-st} K^t$$  \hspace{1cm} (5)

with

$$\sum_{s=0}^{k-1} \Pi_s = 1$$

$$\Pi_s \Pi_{s'} = \delta(s, s') \Pi_s$$

$$\Pi_s X_+ = X_+ \Pi_{s-1} \implies X_- \Pi_s = \Pi_{s-1} X_-$$

$q$ is a root of unity $q = \exp\left(\frac{2\pi i}{k}\right)$
2.1 The quantum algebra $U_q(sl_2)$

In few words, the $U_q(sl_2)$ quantum algebra with $q^k = 1$ is a particular case of the generalized Weyl-Heisenberg $W_k$. This algebra is satisfying the commutation relations

$$[X_-, X_] = [2N]q = \sum_{s=0}^{k-1} f_s(N)\Pi_s, \quad KX_+K^{-1} = qX_+, KX_-K^{-1} = \overline{q}X_-$$

(7)

where

$$f_s(N) = -[2s]_q = -\frac{sin^{4s}\frac{k}{2}}{sin^{2s}\frac{k}{2}}$$

(8)

Using these equations, we can prove that the operators $(X_-)^k$ and $(X_+)^k$ commute with $(X_-)$ and $(X_+)$. 

3 Heterotic fractional supersymmetries

3.1 Supercharges Representations

It is possible to build a correspondence between the operator $Q$, of the fractional supersymmetric algebra, and the generalized Weyl-Heisenberg algebra $W_k$. We define the supercharge operators $Q^\pm_0, Q^\pm_1$ and $Q^\pm_2$

$$Q_{-\frac{2}{3}}^- = X^-\Pi_0, \quad Q_{-\frac{1}{3}}^- = X^-\Pi_1, \quad Q_{-\frac{2}{3}}^- = X^-\Pi_2$$

$$Q_{-\frac{2}{3}}^+ = X^+\Pi_0, \quad Q_{-\frac{1}{3}}^+ = X^+\Pi_1, \quad Q_{-\frac{2}{3}}^+ = X^+\Pi_2$$

(9)

such that

$$Q_{-\frac{2}{3}}^-Q_{-\frac{2}{3}}^0Q_{-\frac{2}{3}}^-\Pi_0 = X_-X_-X_-\Pi_0 = \Pi_0X_-X_-X_-$$

$$Q_{-\frac{2}{3}}^-Q_{-\frac{1}{3}}^-Q_{-\frac{1}{3}}^-\Pi_1 = X_-X_-X_-\Pi_1 = \Pi_1X_-X_-X_-$$

$$Q_{-\frac{2}{3}}^-Q_{-\frac{1}{3}}^-Q_{-\frac{2}{3}}^-\Pi_2 = X_-X_-X_-\Pi_2 = \Pi_2X_-X_-X_-$$

(10)

and

$$Q_{-\frac{2}{3}}^+Q_{-\frac{2}{3}}^+Q_{-\frac{1}{3}}^+\Pi_0 = X_+X_+X_+\Pi_0 = \Pi_0X_+X_+X_+$$

$$Q_{-\frac{2}{3}}^+Q_{-\frac{1}{3}}^+Q_{-\frac{2}{3}}^+\Pi_1 = X_+X_+X_+\Pi_1 = \Pi_1X_+X_+X_+$$

$$Q_{-\frac{2}{3}}^+Q_{-\frac{2}{3}}^+Q_{-\frac{2}{3}}^-\Pi_2 = X_+X_+X_+\Pi_2 = \Pi_2X_+X_+X_+$$

(11)
3.2 Remarks and Properties:

1) $Q_{-\frac{2}{3}}^-$ and $Q_{+\frac{2}{3}}^+$ depend on $\Pi_0$ moreover $(Q_{-\frac{2}{3}}^-)^2 = 0$ et $(Q_{+\frac{2}{3}}^+)^2 = 0$ $Q_{-\frac{2}{3}}^-$ and $Q_{+\frac{2}{3}}^+$ depend on $\Pi_1$ moreover $(Q_{-\frac{1}{3}}^-)^2 = 0$ et $(Q_{+\frac{1}{3}}^+)^2 = 0$ $Q_{-\frac{1}{3}}^-$ depend on $\Pi_2$ moreover $(Q_{0}^-)^2 = 0$ et $(Q_{0}^+)^2 = 0$

2) $(Q_{-\frac{2}{3}}^-)^+ = Q_{-\frac{2}{3}}^+$, $(Q_{-\frac{1}{3}}^-)^+ = Q_{-\frac{1}{3}}^+$, $(Q_{0}^-)^+ = Q_{0}^+$

3) The allowed combinations are no null:

\[
\begin{align*}
Q_{-\frac{2}{3}}^- Q_0^- Q_{-\frac{2}{3}}^- &= X_\ldots X_\ldots \Pi_0 \\
Q_0^- Q_{-\frac{2}{3}}^- Q_{-\frac{1}{3}}^- &= X_\ldots X_\ldots \Pi_1 \\
Q_{-\frac{2}{3}}^- Q_{-\frac{1}{3}}^- Q_0^- &= X_\ldots X_\ldots \Pi_2 \\
Q_{-\frac{2}{3}}^- Q_{-\frac{1}{3}}^- Q_{-\frac{2}{3}}^- &= 0 \\
Q_0^- Q_{-\frac{2}{3}}^- Q_{-\frac{1}{3}}^- &= 0 \\
Q_{-\frac{2}{3}}^- Q_0^- Q_{-\frac{1}{3}}^- &= 0
\end{align*}
\]

(12)

\[
\begin{align*}
Q_{-\frac{2}{3}}^+ Q_0^+ Q_{-\frac{2}{3}}^+ &= X_\ldots X_\ldots \Pi_0 \\
Q_{-\frac{2}{3}}^+ Q_{-\frac{1}{3}}^+ Q_0^+ &= X_\ldots X_\ldots \Pi_1 \\
Q_0^+ Q_{-\frac{2}{3}}^+ Q_{-\frac{1}{3}}^+ &= X_\ldots X_\ldots \Pi_2 \\
Q_{-\frac{2}{3}}^+ Q_{-\frac{1}{3}}^+ Q_{-\frac{2}{3}}^+ &= 0 \\
Q_{-\frac{2}{3}}^+ Q_0^+ Q_{-\frac{1}{3}}^+ &= 0 \\
Q_{-\frac{1}{3}}^+ Q_0^+ Q_{-\frac{1}{3}}^+ &= 0
\end{align*}
\]

(13)

4) The $Q^-$ and $Q^+$ check the identity of Jacobi

\[
[[Q_{-\frac{2}{3}}^- , Q_{-\frac{2}{3}}^- , Q_0^- ] + [[Q_0^- , Q_{-\frac{2}{3}}^- , Q_{-\frac{2}{3}}^- ] + [[Q_{-\frac{1}{3}}^- , Q_{-\frac{2}{3}}^- , Q_0^- ] = 0
\]

\[
[[Q_{-\frac{2}{3}}^+ , Q_{0}^+ , Q_{-\frac{2}{3}}^+ ] + [[Q_0^+ , Q_{-\frac{2}{3}}^+ , Q_{-\frac{2}{3}}^+ ] + [[Q_{-\frac{1}{3}}^+ , Q_{-\frac{2}{3}}^+ , Q_0^+ ] = 0
\]
5) We have the following commutations relations

\[
\begin{align*}
[Q_{-\frac{2}{3}}, Q_{-\frac{2}{3}}^+] &= 0 \\
[Q_{-\frac{2}{3}}, Q_{0}^+] &= 0 \\
[Q_{-\frac{2}{3}}, Q_{+1}^-] &= X_+X_+\Pi_2 - X_+X_-\Pi_0 \\
[Q_{-\frac{2}{3}}, Q_{+1}^+] &= X_+X_+\Pi_0 - X_+X_-\Pi_1 \\
[Q_{-\frac{2}{3}}, Q_{0}^+] &= 0 \\
[Q_{-\frac{2}{3}}, Q_{-\frac{1}{3}}^+] &= 0 \\
[Q_{0}^-, Q_{-\frac{2}{3}}^+] &= 0 \\
[Q_{0}^-, Q_{-\frac{1}{3}}^+] &= 0 \\
[Q_{0}^-, Q_{0}^+] &= X_+X_+\Pi_1 - X_+X_-\Pi_2 \\
[Q_{0}^-, Q_{-\frac{1}{3}}^+] &= 0 \\
\end{align*}
\]

which implies

\[
[Q_{-\frac{2}{3}}, Q_{-\frac{1}{3}}^+] + [Q_{-\frac{2}{3}}, Q_{-\frac{2}{3}}^+] + [Q_{0}^-, Q_{0}^+] = 0
\]

6) If we put

\[
Q^- = Q_0^- + Q_{-\frac{1}{3}}^- + Q_{-\frac{2}{3}}^-
\]

and

\[
Q^+ = Q_0^+ + Q_{-\frac{1}{3}}^+ + Q_{-\frac{2}{3}}^+
\]

we find:

\[
\begin{align*}
(Q^-)^3 &= Q_{-\frac{2}{3}}^- Q_0^- Q_{-\frac{2}{3}}^- + Q_0^- Q_{-\frac{2}{3}}^- Q_{-\frac{1}{3}}^- + Q_{-\frac{2}{3}}^- Q_{-\frac{1}{3}}^- Q_{0}^- \\
&= X^-X^-X^- (\Pi_0 + \Pi_1 + \Pi_2) \\
&= P_{-1}
\end{align*}
\]

\[
\begin{align*}
(Q^+)^3 &= Q_{-\frac{2}{3}}^+ Q_0^+ Q_{-\frac{2}{3}}^+ + Q_0^+ Q_{-\frac{2}{3}}^+ Q_{-\frac{1}{3}}^+ + Q_{-\frac{2}{3}}^+ Q_{-\frac{1}{3}}^+ Q_{0}^+ \\
&= X^+X^+X^+ (\Pi_0 + \Pi_1 + \Pi_2) \\
&= P_{+1}
\end{align*}
\]
3.3 The Hamiltonian of the system

\[ 2H = P^- + P^+ = (Q^-)^3 + (Q^+)^3 \]  

(21)

knowing that

\[ (H)^+ = H \]
\[ [Q^-, H] = [Q^+, H] = 0 \]  

(22)

The generalization of an unspecified number of Q requires the same step of in top. Thus

\[ Q^-_k = X_- \Pi_{\frac{k}{2}}, Q^+_k = X_+ \Pi_{\frac{k}{2}} \]
\[ Q^- = \sum_{n=0}^{k-1} Q^-_n, Q^+ = \sum_{n=0}^{k-1} Q^+_n \]
\[ H = (Q^-_k)^k + (Q^+_k)^k \]

4 Field theoretical models

To construct field theoretical models exhibiting the algebra \( Q^k = P \) as symmetry, we shall proceed by analogy with \( D = 2(\frac{1}{2}, 0) \) supersymmetric theory by introducing the left spin \( \frac{1}{k} \) superspace \((z, \theta_1)\) where \( \theta_{\frac{1}{k}} \) are parafermionic variables carrying a spin \( \frac{1}{k} \) and obeying:

\[ (\theta_{\frac{1}{k}})^k = 0, \quad (\theta_{\frac{1}{k}})^{k-1} \neq 0, \quad (\frac{\partial}{\partial \theta_{\frac{1}{k}}})^k = 0, \quad (\frac{\partial}{\partial \theta_{\frac{1}{k}}})^{k-1} \neq 0, \quad (23) \]

4.1 The case \( k = 2 \)

We know that:

\[ \Phi = \varphi(z) + \theta_{\frac{1}{2}} \psi_{-\frac{1}{2}}(z) = \Phi_0 + \Phi_1 \]
\[ Q^-_{\frac{1}{2}} = \theta_{\frac{1}{2}} \partial_{-1} - \frac{\partial}{\partial \theta_{\frac{1}{2}}} = (Q^-_{\frac{1}{2}})_0 + (Q^-_{\frac{1}{2}})_1 \]
\[ D^-_{\frac{1}{2}} = \theta_{\frac{1}{2}} \partial_{-1} + \frac{\partial}{\partial \theta_{\frac{1}{2}}} = (D^-_{\frac{1}{2}})_0 + (D^-_{\frac{1}{2}})_1 \]  

(24)
such as

\[
(Q_{-\frac{1}{2}})_0\phi_0 = (Q_{-\frac{1}{2}})_0\varphi(z) = \theta_{\frac{1}{2}}\partial_{-1}\varphi(z), \\
(Q_{-\frac{1}{2}})_0\phi_1 = (Q_{-\frac{1}{2}})_0\theta_{-\frac{1}{2}}\psi(z) = \theta_{\frac{1}{2}}^2\partial_{-1}\psi(z) = 0, \\
(Q_{-\frac{1}{2}})_1\phi_0 = (Q_{-\frac{1}{2}})_1\varphi(z) = -\frac{\partial}{\partial \theta_{-\frac{1}{2}}}\varphi(z) = 0, \\
(Q_{-\frac{1}{2}})_1\phi_1 = (Q_{-\frac{1}{2}})_1\theta_{-\frac{1}{2}}\psi(z) = -\frac{\partial}{\partial \theta_{-\frac{1}{2}}}\theta_{\frac{1}{2}}\psi(z) = -\psi(z); \tag{25}
\]

and

\[
(D_{-\frac{1}{2}})_0\phi_0 = (D_{-\frac{1}{2}})_0\varphi(z) = \theta_{\frac{1}{2}}\theta_{-1}\varphi(z), \\
(D_{-\frac{1}{2}})_0\phi_1 = (D_{-\frac{1}{2}})_0\theta_{\frac{1}{2}}\psi(z) = \theta_{\frac{1}{2}}^2\theta_{-1}\psi(z) = 0, \\
(D_{-\frac{1}{2}})_1\phi_0 = (D_{-\frac{1}{2}})_1\varphi(z) = \frac{\partial}{\partial \theta_{\frac{1}{2}}}\varphi(z) = 0, \\
(D_{-\frac{1}{2}})_1\phi_1 = (D_{-\frac{1}{2}})_1\theta_{\frac{1}{2}}\psi(z) = \frac{\partial}{\partial \theta_{\frac{1}{2}}}\theta_{\frac{1}{2}}\psi(z) = \partial_{-1}\psi(z) \tag{26}
\]

moreover we see that:

\[
Q_{-\frac{1}{2}}Q_{-\frac{1}{2}} = (Q_{-\frac{1}{2}})_0(Q_{-\frac{1}{2}})_1 + (Q_{-\frac{1}{2}})_1(Q_{-\frac{1}{2}})_0 \\
(Q_{-\frac{1}{2}})_0(Q_{-\frac{1}{2}})_0 = Q_{-\frac{1}{2}}(Q_{-\frac{1}{2}})_1 = 0 \tag{27}
\]

and

\[
D_{-\frac{1}{2}}D_{-\frac{1}{2}} = (D_{-\frac{1}{2}})_0(D_{-\frac{1}{2}})_1 + (D_{-\frac{1}{2}})_1(D_{-\frac{1}{2}})_0 \\
D_{-\frac{1}{2}}D_{-\frac{1}{2}} = (D_{-\frac{1}{2}})_0(D_{-\frac{1}{2}})_1 + (D_{-\frac{1}{2}})_1(D_{-\frac{1}{2}})_0 \tag{28}
\]

### 4.2 The case \( k = 3 \)

\[
\theta_{\frac{1}{4}} = \theta; \phi = \varphi_0(z) + \theta\varphi_{-\frac{1}{4}}(z) + \theta^2\varphi_{-\frac{3}{4}}, \\
Q^- = \theta^2\partial_{-1} + (\frac{\partial}{\partial \theta})^2\theta + q^2\theta(\frac{\partial}{\partial \theta})^2, \\
Q^- = (Q_{-\frac{1}{3}})_{-\frac{2}{3}} + (Q_{-\frac{1}{3}})_{-\frac{1}{3}} + (Q_{-\frac{1}{3}})_0, \\
D^- = q(\theta)^2\partial_{-1} + q^2(\frac{\partial}{\partial \theta})^2\theta + q^2\theta(\frac{\partial}{\partial \theta})^2, \\
D^- = (D_{-\frac{1}{4}})_{-\frac{2}{4}} + (D_{-\frac{1}{4}})_{-\frac{1}{4}} + (D_{-\frac{1}{4}})_0, \\
Q^-D^- = qD^-Q^-; (Q^-)^3 = (D^-)^3 = \partial_{-1}; \tag{29}
\]
knowing that \( \theta \) is a parafermionic variable

\[
\begin{align*}
(\theta)^3 &= 0, & (\theta)^2 &\neq 0, \\
(\frac{\partial}{\partial \theta})^3 &= 0, & (\frac{\partial}{\partial \theta})^2 &\neq 0 \\
\frac{\partial}{\partial \theta}(\theta) &= 1, & \frac{\partial}{\partial \theta}(\theta)^2 &= (1 + q)\theta
\end{align*}
\] (30)

with \((q^3 = 1)\) such that

\[
\begin{align*}
(Q_{-\frac{1}{3}}^-)_z \phi_0(z) &= (\theta_{\frac{1}{3}})^2 \partial_{-1} \varphi_0(z); \\
(Q_{-\frac{1}{3}}^-)_z \phi_1(z) &= 0; \\
(Q_{-\frac{1}{3}}^-)_z \phi_2(z) &= 0; \\
(Q_{-\frac{1}{3}}^-)_z \phi_0(z) &= 0; \\
(Q_{-\frac{1}{3}}^-)_z \phi_1(z) &= -q^2 \varphi_1(z); \\
(Q_{-\frac{1}{3}}^-)_z \phi_2(z) &= 0; \\
(Q_{-\frac{1}{3}}^-)_z \phi_0(z) &= 0; \\
(Q_{-\frac{1}{3}}^-)_z \phi_1(z) &= 0; \\
(Q_{-\frac{1}{3}}^-)_z \phi_2(z) &= (-q)\theta_{\frac{1}{3}} \varphi_2(z); \\
\end{align*}
\] (31)

and

\[
\begin{align*}
(D_{-\frac{1}{3}}^-)_z \phi_0(z) &= q(\theta_{\frac{1}{3}})^2 \partial_{-1} \varphi_0(z), \\
(D_{-\frac{1}{3}}^-)_z \phi_1(z) &= 0, \\
(D_{-\frac{1}{3}}^-)_z \phi_2(z) &= 0, \\
(D_{-\frac{1}{3}}^-)_z \phi_0(z) &= 0, \\
(D_{-\frac{1}{3}}^-)_z \phi_1(z) &= -q \varphi_1(z), \\
(D_{-\frac{1}{3}}^-)_z \phi_2(z) &= 0, \\
(D_{-\frac{1}{3}}^-)_z \phi_0(z) &= 0, \\
(D_{-\frac{1}{3}}^-)_z \phi_1(z) &= 0, \\
(Q_{-\frac{1}{3}}^-)_z \phi_2(z) &= (-q)\theta_{\frac{1}{3}} \varphi_2(z); \\
\end{align*}
\] (32)
4.3 The case $k > 3$

\[
\phi = \varphi_0(z) + \theta_\frac{1}{k} \varphi_\frac{1}{k} + (\theta_\frac{1}{k}^2 \varphi_\frac{2}{k}(z) + \ldots + (\theta_\frac{1}{k})^{k-1} \varphi_{\frac{k-1}{k}}(z)
\]

\[
\phi = \phi_0 + \phi_1 + \phi_2 + \ldots + \phi_{k-1}
\]

\[
Q^- = [(\theta_\frac{1}{k})^{k-1} \partial_{-1} - (-1)^k q^k (\frac{\partial}{\partial \theta_\frac{1}{k}})k^{-1}(\theta_\frac{1}{k})^{-k-2} - (-1)^k q^{k-1}(\theta_\frac{1}{k})\left(\frac{\partial}{\partial \theta_\frac{1}{k}}\right)^{k-1}(\theta_\frac{1}{k})^{-3}
\]

\[
- \ldots - (-1)^k q^3(\theta_\frac{1}{k})^{-k-3}(\frac{\partial}{\partial \theta_\frac{1}{k}})^{k-1}(\theta_\frac{1}{k}) - (-1)^k q^2(\theta_\frac{1}{k})^{-k-2}(\frac{\partial}{\partial \theta_\frac{1}{k}})^{k-1}]A(q)
\]

where

\[
A(q) = \frac{(-1)^{\frac{k-1}{2}}}{((1+q+q^2+\ldots+q^{k-2})(1+q+q^2+\ldots+q^{k-3})\ldots(1+q))^{\frac{k-2}{2}}(q^{1-k}q^{2-k}q^{3-k})^\frac{1}{k}}
\]

\[
Q^- = Q_0^\frac{1}{k} + Q_1^\frac{1}{k} + \ldots + Q_{k-2}^\frac{1}{k} + Q_{k-1}^\frac{1}{k}
\]

\[
D^- = [q(\theta_\frac{1}{k})^{k-1} \partial_{-1} - (-1)^k q^2(\frac{\partial}{\partial \theta_\frac{1}{k}})^{k-1}(\theta_\frac{1}{k})^{-k-2} - (-1)^k \frac{\partial^2}{\partial \theta_\frac{1}{k}} (\frac{\partial}{\partial \theta_\frac{1}{k}})^{k-1}(\theta_\frac{1}{k})^{-3}
\]

\[
- \ldots - (-1)^k q^2(\frac{\partial}{\partial \theta_\frac{1}{k}})^{k-3}(\theta_\frac{1}{k})^{-k-3}(\theta_\frac{1}{k}) - (-1)^k q^2(\theta_\frac{1}{k})^{-k-2}(\theta_\frac{1}{k})^{-1}]B(q)
\]

and

\[
B(q) = \frac{(-1)^{\frac{k-1}{2}}}{((1+q+q^2+\ldots+q^{k-2})(1+q+q^2+\ldots+q^{k-3})\ldots(1+q))^{\frac{k-2}{2}}((q^{2-k})^\frac{1}{2})}
\]

\[
D^- = D_0^\frac{1}{k} + D_1^\frac{1}{k} + \ldots + D_{k-2}^\frac{1}{k} + D_{k-1}^\frac{1}{k}
\]

\[
(Q^-)^k = (D^-)^k = \partial_{-1}
\]

\[
Q^- D^- = qD^- Q^-
\]

Also, knowing that $\theta$ is a parafermionic variable

\[
(\theta_\frac{1}{k})^k = 0; (\theta_\frac{1}{k})^{k-1} \neq 0
\]

\[
(\frac{\partial}{\partial \theta_\frac{1}{k}})^k = 0; (\frac{\partial}{\partial \theta_\frac{1}{k}})^{k-1} \neq 0
\]

\[
\frac{\partial}{\partial \theta_\frac{1}{k}}(\theta_\frac{1}{k})^n = (1+q+\ldots+q^{n-1})(\theta_\frac{1}{k})^{n-1}; (n < k)
\]
5 The action $S$

5.1 The Case of $k = 3$

The expression of the action $S$ is:

$$ S = \int dzd\theta D\phi \partial_z \phi, $$

where the superLagrangian $L$ has the form:

$$ L = -\partial_z \phi_0(z) \partial_z \phi_0(z) + \phi_1(z) \partial_z \phi_2(z) + \phi_2(z) \partial_z \phi_1(z) $$

satisfying:

$$ \frac{\partial L}{\partial \phi_0} - \partial_z \frac{\partial L}{\partial (\partial_z \phi_0)} = -2 \partial_z \phi_0 = 0, $$

$$ \frac{\partial L}{\partial \phi_1} - \partial_z \frac{\partial L}{\partial (\partial_z \phi_1)} = 0, $$

$$ \frac{\partial L}{\partial \phi_2} - \partial_z \frac{\partial L}{\partial (\partial_z \phi_2)} = 0 $$

5.2 The case $k > 3$

The expression of the superLagrangian $L$ is given by

$$ L = -\partial_z \phi_0 \partial_z \phi_0 - \sum_{i=1}^{k-1} \phi_{-\frac{1}{2} + \frac{i}{k-1}} \partial_z \phi_{-\frac{1}{2} + \frac{i}{k-1}}(z) $$

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