The Spin Bivector and Zeropoint Energy in Geometric Algebra

K. Muralidhar

Physics Department, National Defence Academy
Khadakwasla, Pune-411023, Maharashtra, India
kundetimuralidhar@gmail.com

Abstract

In a classical argument, treating action variable of an oscillator as a transformed angular momentum, it has been shown that the fluctuations induced on an electron by zeropoint fields produce rotations defined by an imaginary zeropoint angular momentum. The electron spin bivector is identified with this zeropoint angular momentum. The magnitude of spin is estimated from the relation between average zeropoint angular momentum and average zeropoint energy. The bivector nature of electron spin angular momentum is shown purely on classical grounds using geometric algebra. The spin equation and spin precession are presented in the light of geometric algebra.

Keywords: Classical spin, Zeropoint energy, Geometric algebra

1. Introduction

Historically, the spin angular momentum was hypothesized by Uhlenbeck and Goudsmit [1] and independently by Bichowsky and Urey [2]. However, the existence of spin can be derived from the fundamental postulates of quantum mechanics and the property of symmetry transformations. The spin angular momentum is a kinematic property of massive elementary particles and it corresponds to rotation group symmetry SU(2). The Dirac theory of electron automatically includes spin as a quantum relativistic effect. The Dirac electron executes very rapid oscillations in addition to the uniform rectilinear motion and this oscillatory motion is called zitterbewegung [3]. Thus within the wave packet associated with the electron there exists a superposition of violent oscillations each with angular frequency equal to $2m_e c^2 / \hbar$, where, $m_e$ is the mass of electron, $c$ is the velocity of light and $\hbar$ is the Plank’s constant. This frequency may be
regarded as the frequency of rotation of the *zitterbewegung* within the wave packet. Many authors over the years proposed several phenomenological models to explain the spin angular momentum of electron and its dynamical behavior. Huang [4] investigated the Dirac electron and analyzed the spin as an angular momentum of *zitterbewegung* circulatory motion. Barducci, Casalbuon and Lusanna [5] investigated path integrals for fermions in Grassmann algebra and considered those paths as spin. Barut and Brachen [6] proposed the spin as the orbital angular momentum associated with the internal oscillatory system. In the approach of geometric algebra, using a multivector valued Lagrangian, Barut and Zanghi [7] arrived at the bivector form of classical internal spin of Dirac electron and opined the spin as the angular momentum of *zitterbewegung* and in the extensions of semiclassical theories of Dirac electron the spin was identified with a bivector and the point particles execute circular motion by absorbing energy from vacuum field [8, 9]. In the multivector formalism of Pauli theory Hestenes and Gurtler [10] showed that the spin angular momentum is fundamentally a bivector quantity and concluded that in the absence of magnetic fields, the Schrödinger theory is identical with the Pauli theory and the constant imaginary factor $i\hbar$ is exactly twice the spin angular momentum. Further in a similar manner, in the hidden geometric structure of Dirac theory, Hestenes [11-13] observed the spin as the imaginary factor $i\hbar$ in the Dirac equation. The spin has been identified as a local circulatory motion of electron so that spin is connected to the *zitterbewegung* rotation within the wave packet. More recently by considering the physical space generated from a pair of annihilation and creation operators, the classical origin of spin half fermions was suggested by Baylis, Cabrera and Keselica [14]. The spin was recognised as a physical (but intrinsic) rotation with rotation rate equal to the *zitterbewegung* frequency. The magnitude of spin has been determined from the g-factor. In the stochastic electrodynamics approach to zeropoint field, we have an explanation to the origin of quantum behavior of matter [15, 16]. As such, it is expected that the nature of spin may be explored in the same treatment. In the theory of classical stochastic electrodynamics of microscopic phenomena Moor and Ramirez [17] discussed phenomenological aspects of spin. Considering angular momentum fluctuations, the electron spin was studied by de la Pena and Jauregui [18] and in this model the spin has been assumed to be hidden in the averaging process and hence the set of realizations of the field are divided into two mutually exclusive and complementary sub ensembles. Then the average of each sub ensemble may contain the components of spin angular momentum. Considering two independent circular polarization states of the electric field vector of zeropoint field, they arrived at an expression for angular momentum and obtained the spin value $\hbar/2$ multiplied by a factor $3/4$.

The core of semiclassical theories of electron spin, though the spin was derived as a bivector quantity, contains the quantum behaviour in the form of transforming quantum mechanical equations into spacetime algebraic formalism and the magnitude of spin was either introduced or derived from the known quantities from quantum mechanics. However, all the above mentioned theoretical models of classical spin firmly suggest that the origin of spin arises mainly due to the rotation of the electron produced by the absorption of energy from the
fluctuating zeropoint field. In the case of stochastic theory the magnitude of spin was derived with an additional multiplicative factor.

In the previous article on classical origin of quantum spin [19] we have shown that a charged particle immersed in the ZPF may be considered to possess complex oscillations and on purely mathematical grounds such complex oscillations are described in terms of modes of rotation $r_+(t)$ and $r_-(t)$. The imaginary part of complex rotations is considered in the classical explanation of quantum spin. The basic reason of considering such complex rotations is explained in the present paper. The main aim of the present paper is to show the bivector nature of spin angular momentum of electron, its geometrical meaning as spin bivector without any quantum background and to derive the correct magnitude of spin. This is achieved by considering the action variables of a periodic system and considering the zero point energy of electron as perturbation to a classical Hamiltonian of an oscillator, it has been shown that the energy absorbed by an electron from zero point fields actually produces rotations rather than oscillations and such rotations correspond to imaginary zeropoint angular momentum which is derived in section 2. As the zero point fields are random fields, the average action variable at zero temperature in relation to the average zero-point energy is presented in section 3. Certain essential aspects of geometric algebra are presented in section 4. The spin is identified with this angular momentum and the geometric meaning of its imaginary nature in terms of geometric algebra is used to express spin bivector. The bivector nature of spin is explored in section 5 and conclusions are presented in section 6.

2. Zero point angular momentum and rotation modes of electron

In the periodic motion of a system, a set of integration constants $\alpha$ that appear in the Hamilton Jacobi equation can be suitably defined as an action variable [20]. For a conservative system the momentum can be expressed in terms of integration constants $\alpha$, as $p = p(x, \alpha)$ and this equation traces out a curve in two dimensional phase space. The periodic motion of a system represents oscillations (librations), if the system point traces out a closed orbit in the phase space. The values of position coordinate $x$ are bounded and the initial position lies in two zeros of kinetic energy. However, when the system point traces out a periodic curve or an open orbit in two dimensional phase space the system represents rotations such that a $2\pi$ rotation keeps the system unchanged. The values of $x$ are unbounded and increase indefinitely. In either case one can introduce an action variable $I$ as a transformed constant angular momentum [21].

$$I = \frac{1}{2\pi} \oint p \, dx$$  \hspace{1cm} (1)

Where, the integration is to be carried out over a complete period of oscillation or rotation. An electron motion in fluctuating random zeropoint field can be approximated as simple harmonic. In addition to zero point fields, there exists thermal radiation and the energy of the electron oscillator can be considered as
sum of two parts, one as thermal energy $E_k$ and the other the zero point energy $\Delta E_0$ associated with the electron. Then Hamiltonian of the electron oscillator is $H = E_k + \Delta E_0 + V(x)$, where potential energy $V(x) = \frac{1}{2} m_e \omega_0^2 x^2$. Since $\Delta E_0$ arises from the fluctuating fields, the Hamiltonian describes the motion of a nonlinear harmonic oscillator. First let us consider the Hamiltonian in the absence of zero point fields, $H_0 = E = E_k + V(x)$ and the momentum $p = \pm \sqrt{2m_e (E - V)}$. The phase space diagram of the system gives a closed elliptical orbit under the condition $E - V < 0$, the harmonic oscillator is said to be in oscillation or libration mode and the area enclosed by the curve divided by $2\pi$ is equal to the action variable $I$. When the kinetic energy is too large i.e., under the condition $E - V > 0$, there is a possibility that the system point traces out a periodic curve or open orbit and the system is said to be in rotation mode. An important situation arises when the maximum energy of the oscillator equals the potential energy. The system point represents two saddle points in the phase space corresponding to two zeros of kinetic energy and the curve passing through these points is called separatrix. At this juncture any small external perturbation throws the system into either oscillation mode or into rotation mode.

Now consider a transformation $p \rightarrow ip$. With this transformation the Hamiltonian becomes $H_0 = E = -E_k + V(x)$ and gives $p = \pm \sqrt{2m_e (V - E)}$. This leads to opposite conditions for oscillations and rotations of the periodic system under consideration. In the case $E - V < 0$, the system point traces a periodic curve or an open orbit and the system executes rotations. On the contrary when $E - V > 0$, the system point traces a closed orbit and the system is in oscillation mode. In other words the transformation $p \rightarrow ip$ converts oscillation mode into rotation mode. Such transformation of inverting oscillations into rotations can be obtained by replacing real momentum with imaginary momentum is well used to obtain inverted potential [22]. Interestingly when $E - V = 0$, the system point again represents the two saddle points and any external perturbation throws the system either into oscillation mode or rotation mode.

The zero point energy associated with the electron may be treated as an external perturbation to the Hamiltonian $H_0$ and considering the total Hamiltonian $H$ the momentum $p = \pm \sqrt{2m_e (E - \Delta E_0 - V)}$. Now let us consider the case when $E - V = 0$, the momentum becomes $\pm \sqrt{2m_e (-\Delta E_0)} = \pm ip_0$. This result yields an interesting aspect of the system; the zero point energy gives imaginary momentum $\pm ip_0$ and according to the discussion above the system is thrown into rotation mode. Thus the fluctuations induced on an electron by the zero point fields invariably produce rotations defined by an imaginary momentum and depending on the sign of momentum we have either counterclockwise or clockwise rotations respectively. The total Hamiltonian $H$ is then a function of total momentum having components of linear momentum $p$ related to kinetic energy and the imaginary momentum $ip_0$ related to zero point energy. Introducing total momentum as $P = p + ip_0$ and its conjugate $P^*$ such that $PP^* = P^2$, the Hamiltonian $H$ can be written in a linear form.
Spin bivector and zeropoint energy

\[ H = \frac{p^2}{2m_e} + V \]  

(2)

This equation represents a combination of two types of orbits, one in the normal phase space plane formed by \( p \) and \( x \) which forms a real plane when \( p_0 \) is zero and the other in the phase space formed by \( ip_0 \) and \( x \) which is a complex phase space plane normal to the real plane and passing through \( x \)-axis when \( p \) is zero. The area enclosed by the orbit in real phase space plane divided by \( 2\pi \) gives the normal angular momentum. At the zero temperature the oscillator represents an orbit in the complex phase space plane only and the direction of the curve above the line joining the saddle points represents counterclockwise rotation and the direction of the curve below the line represents the clockwise rotation. These curves between the saddle points are like two open orbits with directions opposite. Let us represent the area enclosed by the upper curve and the line joining the saddle points divided by \( 2\pi \), corresponding to the rotation in the counterclockwise direction, by action \( I_p \). Similarly the area enclosed by the lower curve and the line joining the saddle points divided by \( 2\pi \), corresponding to the rotation in the clockwise direction, by action \( I_m \). The magnitude of \( I_p \) and \( I_m \) are equal but their orientations are opposite, \( I_p = -I_m \). Then the sum \( I_p + I_m \) gives the resultant area zero but the difference \( I_p - I_m = 2I_p \) and \( I_m - I_p = 2I_m \). Thus there are two types of resultant orientations of the planes in the complex phase space and if the rotation rate is \( \omega_0 \) for counter clockwise or clockwise rotations, the frequency of rotation associated with each resultant area is \( 2\omega_0 \). The resultant area \( 2I_p = -2I_m \) is related to the zero point energy associated with the electron. Denoting the \( I_0 = 2I_p \), the action variable corresponding to zero point energy of the electron can be expressed as

\[ I_0 = \pm i \frac{1}{2\pi} \int p \, dx \]  

(3)

Thus the action variable connected with the zero point energy of the periodic system exhibiting rotations is an imaginary quantity. The positive and negative signs correspond to counterclockwise and clockwise rotations respectively. The integral \( \int p \, dx \) represents the area enclosed by closed curve in phase space and thus \( I_0 \) represents the zeropoint angular momentum of the electron.

3. Average zeropoint angular momentum

Consider the Hamiltonian \( H \) of a periodic system as a function of position, momentum and temperature \( T \), \( H = H(x, p, T) = E(x, p, T) \). At a constant temperature the system point traces out a closed path in phase space and the system would execute a study periodic motion with constant energy \( E(\omega, T) \), where \( \omega \) is the frequency of periodic motion. However, when temperature is variable, the system is not closed and its energy is not conserved. Let us consider that \( T \) varies slowly during the time period \( \tau \) and satisfies the condition \( \tau (dT/dt) \)
<< T. Then the rate of instantaneous change of energy is very small and if this rate is averaged over a complete time period, the rapid oscillations of its value are smoothed out and the resulting value of energy determines the rate of steady slow variation of energy of the system with temperature. The quantity which remains constant during the motion of the system with slowly varying temperature is called adiabatic invariant. Such invariant for the periodic system is the action variable. The average action variable \( \langle I(\omega) \rangle \) of the periodic motion is related to the average energy \( \langle E(\omega, T) \rangle \) at temperature T and is given by [21]

\[
\langle I(\omega) \rangle = \frac{\langle E(\omega, T) \rangle}{\omega}.
\]

In the statistical equilibrium of the system in thermal contact one can consider the equality of field and matter oscillators. The average \( \langle E(\omega, T) \rangle \) represents the time average and since time average is equal to an ensemble average over a large number of individual random energy values, \( \langle E(\omega, T) \rangle \) can be taken as a stochastic average. Similarly the average of action variable is \( \langle I(\omega) \rangle \) which also represents stochastic average. Substituting \( T = 0 \) in Eq.(4), the average action variable \( \langle I_0 \rangle = \langle I_0(\omega) \rangle \) can then be obtained from the energy at zero temperature, \( \langle \Delta E_0(\omega) \rangle = \langle E(\omega, 0) \rangle \).

\[
\langle I_0 \rangle = \frac{\langle \Delta E_0(\omega) \rangle}{\omega} \tag{5}
\]

The value of \( \langle I_0 \rangle \) must be a universal constant with dimensions of angular momentum and remains as an adiabatic invariant. An average zero point energy of electron within the framework of stochastic electrodynamics is given by [19].

\[
\langle \Delta E_0 \rangle = \frac{\hbar \omega_0}{\pi} \tag{6}
\]

It is to be noted here that this energy depends on the cut-off frequency and thus Eq.(6) is a generalized formula and may be applicable to any charged particle. Combining Eqs.(5) and (6) gives the average zero point angular momentum of electron \( \langle J_0 \rangle = \hbar/\pi \). In a different procedure by considering the charged particle as a Brownian particle and expressing Fokker-Plank equations in terms of action angle variables and an approximation to a stationary solution at a constant frequency of the oscillator, the average component of adiabatic invariant was obtained as \( \langle J_1 \rangle = \hbar/2 \) which is in accordance with the adiabatic hypothesis of quantum theory [23] and the average \( \langle J_1 \rangle \) value differs from the value obtained here because of the adaptation of different averaging methods. A complete derivation considering both damping and radiation reaction terms in the equation of motion of electron gives the same expression as in Eq.(6) for average zero point energy. Even in the relativistic treatment a similar result with a factor 2 in the denominator was reported [24].

4. Geometric algebra

The necessary aspects of geometric algebra and spacetime algebra required
in the next section are outlined here. A geometric product $ab$ of two vectors $a$ and $b$ has a canonical decomposition into an inner product and an outer product.

$$ab = a.b + a \wedge b$$  \hspace{1cm} (7)

The inner product $a.b$ is a scalar and a symmetric product. The outer product $a \wedge b$ is an asymmetric product and is called a bivector which represents an oriented plane. If unit vectors $\sigma_i$ and $\sigma_j$ compose a right handed orthonormal basis for the oriented plane then the $\sigma_i \sigma_j = i$ is defined as a bivector and the square of this bivect is $-1$. In this view, the geometrical meaning of unit imaginary is that it represents a unit directed area and it can be seen as a generator of rotations in geometric algebra. A set of unit vectors $\{\sigma_k \mid k = 1, 2, 3\}$ form a right handed orthonormal basis in the Euclidean space and the products $\sigma_i \sigma_j$ ($i \neq j$) are treated as bivectors which specify the orientation of planes in space and the product $\sigma_i \sigma_j \sigma_k = i$ is called the unit right handed pseudoscalar. The set of elements $\{1, \sigma_i, \sigma_j, \sigma_k\}$ form a space group $G(3)$ in geometric algebra with dimensions eight. Geometric algebra is a graded linear space and the elements of this algebra are called multivectors. The spin operators in quantum theory are given by $2\sigma_k = \hbar \sigma_k$, where $\sigma_k$ are the familiar spin matrices and these matrices satisfy the commutation relations, $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk} \sigma_k$, anti commutation relations $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$ and all of the matrices square to identity matrix, $\sigma_k^2 = 1$. The set of orthonormal unit vectors $\{\sigma_k\}$ in space also obey the same commutation relations just like the Pauli matrices. A correspondence between unit imaginary $i$ in the complex plane or in conventional quantum theory and the pseudoscalar may be obtained by identifying $i$ with a bivector $i\sigma_3 = \sigma_1 \sigma_2$. Squaring a bivector results in a scalar and multiplication of two bivectors gives a third bivector. The product of two bivectors $\sigma_i \sigma_2$ and $\sigma_2 \sigma_3$, representing two orthogonal planes in three dimensions, is equal to $\sigma_i \sigma_j$ a bivector representing a plane orthogonal to both initial bivectors. One can denote three basis bivectors as $B_1 = i\sigma_1 = \sigma_2 \sigma_3$, $B_2 = i\sigma_2 = \sigma_3 \sigma_1$ and $B_3 = i\sigma_3 = \sigma_1 \sigma_2$. These basis bivectors satisfy the relation $B_i B_j = -\delta_{ij} - \epsilon_{ijk} B_k$ and the product $B_1 B_2 B_3 = +1$. The product of a bivector and pseudo scalar $i$ is a vector, $iB_3 = -\sigma_3$. This gives the definition of cross product of two vectors $a$ and $b$, $a \times b = -i a \wedge b$. With this identity, the angular momentum in vector algebra, a cross product of radius vector $r$ and linear momentum $p$, can be expressed as a bivector product $L = r \wedge p$ which represents an oriented plane. Thus the angular momentum is basically a bivector quantity.

A rotation of a vector is achieved by successive reflections in planes normal to two unit vectors denoted by $n$ and $m$ and the angle of rotation is twice the angle $\theta$ between $n$ and $m$. A rotation of a vector $a$ on to $b$ is represented by the relation.

$$b = RaR^\dagger$$  \hspace{1cm} (8)

Where $R = nm$ is called a rotor and $R^\dagger = mn$. The product $n.m = \cos \theta$ and $n \wedge m = \sin \theta$. Since $(n \wedge m)(n \wedge m) = -\sin^2 \theta$, a unit bivector $B$ can be defined as $B = m \wedge n$ and $B^2 = -1$. In terms of this unit bivector, the rotor can be expressed as

$$R = \cos \theta - B \sin \theta = e^{-B\theta}$$  \hspace{1cm} (9)

As the angle of rotation is $2\theta$ and it is required to rotate $a$ by $\theta$, the appropriate rotor is
If the unit bivecor $B = i\sigma_3$, then the rotation is described by the plane $\sigma_1\sigma_2$ and $\theta$. Using the above relation a set of orthonormal unit vectors $\{\sigma_i\}$ can be rotated into another set of orthonormal unit vectors $\{e_k\}$.

$$e_k = R\sigma_k R^\dagger \quad (11)$$

The time dependence of a rotor can be achieved by defining rate of rotation $\omega$, the rotor then can be expressed as

$$R = \exp(-i\omega t/2) = \exp(-\Omega t/2) \quad (12)$$

Where, $\Omega$ is the dual of $i\omega$ and is called angular velocity bivector. The product $RR^\dagger = 1$ and differentiating this product gives $\dot{R}R^\dagger = -R\dot{R}^\dagger$. Using this result on $R$ can find the rotor equation.

$$\dot{R} = -\frac{i}{2} \Omega R \quad (13)$$

Where, dot represents differentiation with respect to time. This equation is used at several instances in physics and in particular it is a generator of spin equation. More details on geometric algebra are given in the reference [8].

5. The spin bivector

In the Eq.(3) the zero point momentum $p_0$ and position $x$ can be chosen along the directions defined by unit vectors $e_1$ and $e_2$. Then the product $p_0dx$ can be expressed as $p_0 \times dx = e_1 e_2 p_0dx$ and in geometric algebra the cross product $e_1 e_2$ is expressed in terms of an outer product times a pseudo scalar, $e_1 e_2 = -ie_1 \wedge e_2$ and $I_0$ can be viewed as a bivector representing an oriented plane. Thus the fundamental bivector nature of angular momentum is obtained with the presence of unit imaginary and a geometrical meaning may be attributed to the unit imaginary as generator of rotations. These rotations can be identified with the rotational motion of electron. Hence it is proposed that the zero point angular momentum $I_0$ represents the electron spin bivector and the oriented plane defined by the bivecor $e_1 \wedge e_2$ specifies the spin rotation plane. The magnitude of the area defined by the integral in Eq.(3) is identified with the magnitude of spin vector $s$.

$$I_0 = \pm e_1 \wedge e_2 |s| \quad (14)$$

The positive and negative signs represent the two types of orientations of the spin rotation plane. Since the zero-point fields are random fields, it may be assumed that the orientations of spin rotation plane are also at random. The magnitude of $I_0$ is $|s| \sin \theta$, where $\theta$ is the angle between $e_1$ and $e_2$. An average over all these projections gives $\langle I_0 \rangle = |s| \langle \sin \theta \rangle$. As seen in section 2 the rotations are basically half rotations and hence $\theta$ varies only over a range $0$ to $\pi$. Then the average $\langle I_0 \rangle$ can be obtained by taking the average of $\sin \theta$ over this range, $\langle \sin \theta \rangle = 2/\pi$. Now, the magnitude of spin can be expressed as $|s| = \langle \pi /2 \rangle \langle I_0 \rangle$. The average zero point angular momentum $\langle I_0 \rangle$ is related to the average zero point energy $\langle \Delta E_0(\alpha_0) \rangle$ by the Eq.(5) and the magnitude of spin becomes
Substituting $\langle \Delta E_0(\omega_0) \rangle$ from Eq. (6) into the above equation gives $|s| = h/2$. The magnitude of spin represents the average of all projected areas on a plane defined by orthonormal unit vectors and let these unit vectors be $\sigma_1$ and $\sigma_2$. Since the zero point angular momentum is a bivector, the spin bivector $S$ can be defined as $S = \pm \sigma_1 \wedge \sigma_2 |s| = \pm \sigma_1 \sigma_2 |s|$. Thus in the language of geometric algebra the spin bivector and spin vector can be expressed as

$$S = \pm i \sigma_3 |s| = \pm i s$$

and

$$s = \pm (h/2) \sigma_3.$$  (16)

The positive and negative signs now represent the spin up and spin down directions and the spin up direction is chosen along $\sigma_3$ unit vector.

The rotor connected with the spin rotation can be formed from the spin bivector. Since the spin frequency $\omega_s = 2\omega_0$, the spin rotor can be defined as

$$R = \exp(-i \sigma_3 \omega_s t/2) = \exp(-i \sigma_3 \phi/2)$$

and

$$R^\dagger = \exp(i \sigma_3 \phi/2).$$  (18)

Such that $RR^\dagger = 1$. This shows the physical significance that $i \sigma_3$ is a generator of rotations in a plane orthogonal to spin vector. The Lie algebra can be represented by bivector algebra, the spin half description is represented by rotor and hence the rotor group is referred to as spin group [25]. Now it is possible to define an even multivector spinor $\psi$ which is having one to one correspondence with the spin wave function in quantum mechanics. The spinor in the multivector form can be expressed as

$$\psi = \rho^{1/2} R$$

Where, $\rho$ is the probability density and the spinor in this form shows symmetry with $4\pi$ rotation and asymmetry with $2\pi$ rotation confirming the well known properties of spin wave function. The spinor can also be expressed as a superposition of spin up and spin down parts $\psi = \psi_p + \psi_m$ and can be expressed in the standard form in the following manner.

$$\psi_p = \rho_p \exp(-i \sigma_3 \phi/2) ; \quad \psi_m = \rho_m \exp(-i \sigma_3 \phi/2) (-i \sigma_2)$$

And

$$\langle \psi \psi' \rangle = \psi_+^* + \psi_-^* = \rho$$

(22)

If $\{e_k\}$ represents a set of orthonormal unit vectors in Euclidean space, the properties of rotor allows us to express $e_k$ in terms of $\sigma_k$ with the relation $e_k = R \sigma_k R^\dagger$ and the spin components can be obtained from

$$s_k = (h/2) R \sigma_k R^\dagger = (h/2) e_k$$

and the expectation values of spin are obtained from the spinor relation.

$$\langle s_+ \rangle = h/2 \langle \psi \sigma_+ \psi' \rangle = \frac{\hbar}{2}.$$

Thus the bivector formalism of spin clearly gives the results equal to those of quantum mechanics. Differentiating the rotor $R$ defined in Eq.(24) with respect to time gives the general rotor equation $2R = -i \sigma_3 \omega_s R = -i \omega_s R$. This rotor equation in general satisfies the spin equation. The spin bivector is now defined as $S = (\hbar/2) R i \sigma_3 R^\dagger$ Differentiating $S$ with respect to time and using the rotor equation, one
can obtain the spin equation.

\[ \dot{S} = i\omega S = \Omega_S S \]  \hspace{1cm} (25)

Where, \( \Omega_S \) is the dual of \( i\omega \). This equation shows the frequency of spin plane rotation is \( \omega_S \) and it is the well known spin equation derived in the classical theories of magnetic moment by imposing an assumption that the particle velocity is not necessarily collinear with the momentum \([26]\). Such assumption is only possible when there is a deviation from mean path of electron. Then the momentum can be resolved into parallel and perpendicular components to the velocity vector. For small angles between momentum and velocity, a cross product of perpendicular component of momentum \( p_\perp \) with velocity \( v \) yields a torque term \(-v \times p_\perp\) which is related to the rate of change of spin. The classical theory gives the spin precession frequency \( \omega_S = eB/m_e \) where \( e \) is the charge of electron and \( B \) is the constant magnetic field. Then the spin precession equation in a constant magnetic field is expressed as

\[ \dot{S} = \frac{ge}{2m_e c} (iB) S \]  \hspace{1cm} (26)

Where, \( g \) is the gyromagnetic ratio and for electron its value \( g = 2 \). Though the classical theory of electron gives a good approach to spin, to obtain relativistic corrections it is required to consider spin in spacetime. The relativistic theory of spin is well discussed by Hestenes in his real Dirac theory \([11\text{-}13]\) and the algebra used is the spacetime algebra.

6. Conclusions

We find the spin bivector of electron as the zeropoint angular momentum due to rotations of electron and the correct value of spin is estimated from the average zeropoint angular momentum and average zeropoint energy of electron. The constant \( \hbar \) is a scale factor in zero point radiation and naturally chosen Planck’s constant and hence we find no quantum connection throughout and also in the magnitude of spin. In the language of geometric algebra, electron spin vector and spin bivector are presented and the bivector formalism of spin clearly gives the results equal to those of quantum mechanics. The classical theory of spin presented here is not only applicable to electron but also to other spin-half particles.

References

Spin bivector and zeropoint energy


Received: January, 2012