Dirac-Hua Problem Including a
Coulomb-like Tensor Interaction

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Abstract

In this research, we have solved the Dirac equation with the spin and pseudo-spin symmetries for Hua potential including a Coulomb-like tensor interaction. The energy eigenvalues equation and the corresponding unnormalized wave functions have obtained in terms of the Jacobi polynomials. The parametric generalization of the Nikiforov-Uvarov method had used in the calculations.

Keywords: Dirac equation, Hua potential, Coulomb-like, Nikiforov-Uvarov

1 Introduction

It is well known that the exact energy eigenvalues of the bound state play an important role in quantum mechanics. In particular, the Dirac equation which describe the motion of a spin-1/2 particle has been used in solving many problems of nuclear and high-energy physics. Recently, there has been an increased in searching for analytic solution of the Dirac equation. For example, see [1-29].

Concepts of spin and pseudo-spin symmetries and a tensor potential have been found interesting applications in the field of nuclear physics [30-34]. On the other hand, tensor potentials were introduced into the Dirac equation with the substitution \( \hat{p} \rightarrow \hat{p} - imo\beta \cdot \hat{r}U(r) \) [35, 36]. In this way, a spin-orbit coupling term is added to the Dirac Hamiltonian. Recently, tensor couplings have been used widely in the studies of nuclear properties. In this regard, see [37-44].
The internuclear potential [45] has been introduced by Hua, of the form is

\[ V(r) = V_0 \left( \frac{1 - e^{-2ar}}{1 - qe^{-2ar}} \right)^2, \]  

(1)

where \( q, \alpha, r \) and \( V_0 \) are the potential constants. Tensor potential Coulomb-like [37, 38] is

\[ U(r) = -\frac{H}{r}, \quad H = \frac{Z_aZ_b\varepsilon^2}{4\pi\varepsilon_0}, \quad r \geq R_c. \]  

(2)

where \( R_c = 7.78 \text{ fm} \) is the Coulomb radius, \( Z_a \) and \( Z_b \) denote the charges of the projectile \( a \) and the target nuclei \( b \), respectively.

Our aim the reason for choosing the Hua potential in this work is study the Dirac equation for this potential including a coulomb-like tensor coupling under the spin and pseudo-spin symmetries. We have obtained the energy eigenvalues equation and the corresponding spinor wave functions by using the parametric generalization of the Nikiforov-Uvarov (NU) method.

2 NU method

We give a brief description of the conventional NU method [46]. This method is based on solving the second order differential equation of hypergeometric-type by means of special orthogonal functions

\[ \psi_n^\ast(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi_n(s) = 0, \]  

(3)

where \( \sigma(s) \) and \( \tilde{\sigma}(s) \) are polynomials, at the most of the second degree, and \( \tilde{\tau}(s) \) is a polynomials, at most of the first degree. If we take the following factorization \( \psi_n(s) = \phi(s)y_n(s) \), (3) becomes

\[ \sigma(s)y_n'(s) + \tau(s)y_n'(s) + \lambda y_n(s) = 0, \]  

(4)

where

\[ \pi(s) = \sigma(s) \frac{d}{ds} (\ln \phi(s)) \]  

(5)

\[ \tau(s) = \tilde{\tau}(s) + 2\pi(s), \quad \tau'(s) < 0, \]  

(6)

where \( \pi(r) \) is a polynomial of order at most one.

The \( y_n(s) \) which is a polynomial of degree can be expressed in terms of the Rodrigues relation

\[ y_n(s) = \frac{a_n}{\rho(s)} \frac{d^n}{ds^n} \left[ \sigma^n(s) \rho(s) \right], \]  

(7)

where \( a_n \) is a normalization constant and the weight function \( \rho(s) \) must satisfy the differential equation
The function \( \pi(s) \) and the parameter \( \lambda \) in the above equation are defined as
\[
\pi(s) = \left( \sigma'(s) - \tau'(s) \right) \pm \sqrt{\left( \frac{\sigma'(s) - \tau'(s)}{2} \right)^2 - \tilde{\sigma}(s) + q \sigma(s)},
\]
\[
\lambda = q + \pi'(s).
\]

The determination of \( q \) is the essential point in the calculation of \( \pi(s) \). It is simply defined by setting the discriminate of the square root which must be zero. The eigenvalues equation have calculated from the above equation
\[
\lambda = \lambda_n = -n \tau'(s) - \frac{n(n - 1)}{2} \sigma''(s). \quad n = 0, 1, 2, \ldots
\]

For a more simple application of the method, we develop a parametric generalization of the NU method valid for any potential under consideration by an appropriate coordinate transformation \( s = s(r) \). Thus, we obtain another generalized hypergeometric equation [39, 44]
\[
\left[ s^2 \left(1 - \alpha_i s\right)^2 \frac{d^2}{ds^2} + s \left(1 - \alpha_i s\right) (\alpha_i - \alpha_2 s) \frac{d}{ds} + [-\xi_2 s^2 + \xi_2 s - \xi_3] \right] \psi_n(s) = 0. \quad (12)
\]

We may solve this as follows. Comparing (12) with (3), yields
\[
\tilde{\tau}(s) = \alpha_i - \alpha_2 s, \quad \sigma(s) = s \left(1 - \alpha_3 s\right), \quad \tilde{\sigma}(s) = -\xi_2 s^2 + \xi_2 s - \xi_3. \quad (13)
\]

Substituting these into (9), we find
\[
\pi(s) = \alpha_4 + \alpha_5 s \pm \left[ (\alpha_6 - k \alpha_3) s^2 + (\alpha_7 + k) s + \alpha_8 \right]^{1/2}, \quad (14)
\]
with the following parameters
\[
\alpha_4 = \frac{1}{2} (1 - \alpha_i), \quad \alpha_5 = \frac{1}{2} (\alpha_2 - 2 \alpha_3), \quad \alpha_6 = \alpha_2^2 + \xi_1,
\]
\[
\alpha_7 = 2 \alpha_i \alpha_2 - \xi_2, \quad \alpha_8 = \alpha_2^2 + \xi_3. \quad (15)
\]

We obtain the parameter \( k \) from the condition that the function under the square root should be the square of a polynomial
\[
k_{1,2} = -\left( \alpha_7 + 2 \alpha_3 \alpha_9 \right) \pm 2 \sqrt{\alpha_2 \alpha_9}, \quad (16)
\]

where
\[
\alpha_9 = \alpha_3 \alpha_4 + \alpha_3^2 \alpha_8 + \alpha_6. \quad (17)
\]

For each \( k \) the following \( \pi \)'s are obtained. The function \( \pi(s) \) becomes
\[
\pi(s) = \alpha_4 + \alpha_5 s - (\sqrt{\alpha_9} + \sqrt{\alpha_8}) s - \sqrt{\alpha_6}, \quad (18)
\]
for the \( k \)-value
\[
k = -\left( \alpha_7 + 2 \alpha_3 \alpha_9 \right) - 2 \sqrt{\alpha_2 \alpha_9}. \quad (19)
\]

We also have from \( \tau(s) = \tilde{\tau}(s) + 2 \pi(s), \)
\[
\tau(s) = \alpha_i + 2 \alpha_4 - (\alpha_2 - 2 \alpha_3) s - 2(\sqrt{\alpha_9} + \sqrt{\alpha_8}) s - 2 \sqrt{\alpha_6}. \quad (20)
\]
Thus, we impose the following condition to fix the \( k \)-value
\[
\tau'(s) = -(\alpha_2 - 2\alpha_s) - 2(\sqrt{\alpha_6} + \sqrt{\alpha_5}) \\
= -2\alpha_5 - 2(\sqrt{\alpha_6} + \sqrt{\alpha_5}) < 0.
\] (21)

When (10) is used with (20) and (21) the following equation is derived
\[
-\alpha_5 + (2n + 1)(\sqrt{\alpha_6} + \sqrt{\alpha_5}) + 2\alpha_5\alpha_6 + 2\sqrt{\alpha_6}\alpha_6 = 0.
\] (22)

By using (8)
\[
\rho(s) = s^{\alpha_0 - 1}(1 - \alpha_5s)^{\alpha_1 - \alpha_0 - 1},
\] (23)
and together with (7), we have
\[
y_n(s) = P^n_{\alpha_0 - 1}^{\alpha_1 - \alpha_0 - 1}(1 - 2\alpha_5s),
\] (24)
where
\[
\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_6},
\] (25)
and
\[
\alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_6} + \sqrt{\alpha_5}),
\] (26)
and \( P_n^{(\alpha, \beta)} \) are Jacobi polynomials. By using (5), we get
\[
\phi(s) = s^{\alpha_0 - 1/2}(1 - \alpha_5s)^{-\alpha_2 - \alpha_1 - \alpha_0},
\] (27)
and the total wave function become
\[
\Psi(s) = s^{\alpha_2}(1 - \alpha_5s)^{-\alpha_2 - \alpha_1 - \alpha_0} P^n_{\alpha_0 - 1}^{\alpha_1 - \alpha_0 - 1}(1 - 2\alpha_5s),
\] (28)
where \( \alpha_{12} = \alpha_5 + \sqrt{\alpha_6} \) and \( \alpha_{13} = \alpha_5 - (\sqrt{\alpha_6} + \sqrt{\alpha_5}) \).

In some problems the situation appears where \( \alpha_5 = 0 \). For such problems, the solution given in (28) becomes as
\[
\Psi(s) = s^{\alpha_2} e^{\alpha_5s} L_n^{\alpha_0 - 1}(\alpha_1 s).
\] (29)
In some cases, one may need a second solution of (12). In this case, if the same procedure is followed, by using
\[
k = -(\alpha_1 + 2\alpha_5\alpha_6) + 2\sqrt{\alpha_6}\alpha_6,
\] (30)
the solution becomes
\[
\Psi(s) = s^{\alpha_2}(1 - \alpha_5s)^{-\alpha_2 - \alpha_1 - \alpha_0} P^n_{\alpha_0 - 1}^{\alpha_1 - \alpha_0 - 1}(1 - 2\alpha_5s),
\] (31)
and the energy spectrum is
\[
n[(n - 1)\alpha_5 + \alpha_5 - 2\alpha_5] \\
+ (2n + 1)(\sqrt{\alpha_6} - \alpha_5\sqrt{\alpha_6}) + \alpha_7 + 2\alpha_5\alpha_6 - 2\sqrt{\alpha_6}\alpha_6 + \alpha_5 = 0.
\] (32)

Pre-defined \( \alpha \) parameters are:
\[
\alpha_{10} = \alpha_1 + 2\alpha_4 - 2\sqrt{\alpha_6}, \quad \alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_6} - \alpha_5\sqrt{\alpha_6}),
\]
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\[ \alpha_{12}' = \alpha_4 - \sqrt{\alpha_8}, \quad \alpha_{13}' = \alpha_5 - (\sqrt{\alpha_9} - \alpha_3 \sqrt{\alpha_8}). \] (33)

3 Solution of the Dirac equation

According to [37-44], the Dirac equation of a nucleon with mass \( M \) moving in a scalar and a vector potentials including tensor interaction for spin-1/2 particles can be written as (\( \hbar = c = 1 \)),

\[ [\tilde{\alpha} \cdot \tilde{P} + \beta (M + V_s(r)) - i \beta \tilde{\alpha} \cdot \tilde{r} U(r)] \psi_{ak}(\vec{r}) = [E - V_s(r)] \psi_{ak}(\vec{r}), \] (34)

where \( E \) is the relativistic energy of the system, \( \tilde{P} = -i \tilde{\nabla} \) is the three-dimensional momentum operator and \( \alpha \) and \( \beta \) the \( 4 \times 4 \) Dirac matrices which have the following forms [47], respectively

\[ \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \tilde{P} = -i \hbar \tilde{\nabla}, \] (35)

where \( I \) denotes the \( 2 \times 2 \) identity matrix and \( \sigma \) are three-vector pauli spin matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (36)

For spherical nuclei, the nucleon angular momentum \( J \) and \( \hat{K} = -\beta(\hat{\sigma} \hat{L} + 1) \) commute with the Dirac Hamiltonian, where \( \hat{\sigma} \) and \( \hat{L} \) are the pauli matrix and orbital angular momentum, respectively.

The Dirac spinors can be written according to there angular momentum \( j \) and \( k \),

\[ \psi_{ak}(\vec{r}) = \begin{pmatrix} F_{ak}(r) Y^j_m(\theta, \phi) \\ i G_{ak}(r) Y^{\tilde{j}}_m(\theta, \phi) \end{pmatrix}, \] (37)

where \( F_{ak}(r) \) is upper and \( G_{ak}(r) \) is the lower radial wave functions of the Dirac spinors, \( Y^j_m(\theta, \phi) \) and \( Y^{\tilde{j}}_m(\theta, \phi) \) are the spin and pseudo-spin spherical harmonical functions, and \( n \) is the radial quantum number, and \( m \) is the projection of the total angular momentum on the z-axis. The eigenvalues of \( \hat{K} \) are \( k = \pm(j + (1/2)) \) with ‘ - ’ for aligned spin (\( s_{1/2}, p_{3/2} \), etc.) and ‘ + ’ for unaligned spin (\( p_{1/2}, d_{3/2} \), etc.).

Substituting (37) into (34) and using the following relations [48]

\[ (\hat{\sigma} \cdot \hat{\nabla}) (\hat{\sigma} \cdot \hat{B}) = \hat{A} \cdot \hat{B} + i \hat{\sigma} \cdot (\hat{A} \times \hat{B}), \] (38)

\[ (\hat{\sigma} \cdot \hat{P}) = \hat{\sigma} \cdot \hat{r} (\hat{\nabla} + i \hat{\sigma} \cdot \hat{L} / r), \] (39)

and properties

\[ (\hat{\sigma} \cdot \hat{L}) Y^{\tilde{j}}_m(\theta, \phi) = (k - 1) Y^{\tilde{j}}_m(\theta, \phi), \] (40)

\[ (\hat{\sigma} \cdot \hat{L}) Y^j_m(\theta, \phi) = -(k - 1) Y^j_m(\theta, \phi), \] (41)
\[
(\vec{\sigma} \cdot \vec{r}) Y_{jm}^i (\theta, \phi) = -Y_{jm}^i (\theta, \phi), \quad (42)
\]
\[
(\vec{\sigma} \cdot \vec{r}) Y_{jm}^i (\theta, \phi) = -Y_{jm}^i (\theta, \phi), \quad (43)
\]
we obtain the radial part of the Dirac equation as
\[
\left( \frac{d}{dr} + \frac{k}{r} - U(r) \right) F_{nk} (r) = \left[ E_{nk} + M - \Delta(r) \right] G_{nk} (r), \quad (44)
\]
\[
\left( \frac{d}{dr} + \frac{k}{r} + U(r) \right) G_{nk} (r) = \left[ M - E_{nk} + \Sigma(r) \right] F_{nk} (r), \quad (45)
\]
where \( \Delta \) and \( \Sigma \) have been assumed to be radial functions, i.e., \( \Delta(r) = V_s(r) - V_v(r) \) and \( \Sigma(r) = V_s(r) + V_v(r) \). By substituting \( G_{nk} (r) \) from (44) into (45) and \( F_{nk} (r) \) from (45) into (44), we have obtained the following two second-order differential equations for the upper and lower components,
\[
\begin{align*}
\left\{ \frac{d^2}{dr^2} &- \frac{k(k+1)}{r^2} + \frac{2k}{r} U(r) - \frac{dU(r)}{dr} - U^2(r) \\
&+ (E_{nk} + M - \Delta(r))(E_{nk} - M - \Sigma(r)) \\
&- \frac{d\Delta(r)}{dr} \left( \frac{d}{dr} + \frac{k}{r} \right) \right\} F_{nk} (r) = 0, \quad (46)
\end{align*}
\]
\[
\begin{align*}
\left\{ \frac{d^2}{dr^2} &- \frac{k(k-1)}{r^2} + \frac{2k}{r} U(r) + \frac{dU(r)}{dr} - U^2(r) \\
&+ (E_{nk} + M - \Delta(r))(E_{nk} - M - \Sigma(r)) \\
&+ \frac{d\Sigma(r)}{dr} \left( \frac{d}{dr} + \frac{k}{r} \right) \right\} G_{nk} (r) = 0. \quad (47)
\end{align*}
\]
In the above equations \( k(k+1) = l(l+1) \) and \( k(k-1) = \tilde{l} (\tilde{l} + 1) \).

### 3.1 Spin symmetry case

Substituting (1) and (2) into (46), considering pseudo-spin symmetry, taking \( \Delta(r) \) as the Hua potential and \( \Sigma(r) = C_{ps} = \text{const.} \left( d\Sigma(r) / dr = 0 \right) \) [49, 50], i.e., the equation have been obtained for the upper component of the Dirac spinor \( F_{nk} (r) \) becomes
\[
\begin{align*}
\left\{ \frac{d^2}{dr^2} &- \frac{(k + H)(k + H + 1)}{r^2} \\
&+ (E_{nk} - M)(M + E_{nk} - C_v) - V_s (E_{nk} + M - C_s) \left[ \frac{1 - e^{-2ar}}{1 - q e^{-2ar}} \right]^2 \right\} F_{nk} (r) = 0. \quad (48)
\end{align*}
\]
This equation is describes a particle of spin-1/2 such as the electron in the
Dirac-Hua problem

Dirac theory with Hua potential including a tensor coupling, that can not be solved analytically because of \((k + H)(k + H + 1)/r^2\) term, we used the approximation scheme suggested by Greene and Aldrich \[51\]

\[
\frac{1}{r^2} \approx 4\alpha^2 \left[ C_0 + \frac{e^{-2\alpha r}}{(1-qe^{-2\alpha r})^2} \right], \tag{49}
\]

where the parameter \(C_0 = 1/12\) is dimensionless constant.

By using a transformation of the form \(s = e^{-2\alpha r}\), we rewrite it as follows

\[
\left\{ \frac{d^2}{ds^2} + \frac{1-qs}{s(1-ts)} \frac{d}{ds} + \frac{1}{(s(1-qs))^2} \left[ -(b_2C_0 q^2 - b_3 q^2 + V_0) s^2 \right. \right.
\]

\[
\left. + \left(-2b_1b_3q + 2b_2C_0 q - b_2 \right) s + b_1b_3 - b_3 V_0 - C_0 b_2 \right] \right\} F_{nk}(s) = 0. \tag{50}
\]

By comparing (50) with (12), we have been obtained the parameter set as

\[
\xi_1 = b_1C_0 q^2 - b_3 q^2 + V_0 b_1, \]

\[
\xi_2 = -2b_1b_3q + 2b_2C_0 q - b_2, \]

\[
\xi_3 = -(b_1b_3 - b_3 V_0 - C_0 b_2),
\]

\[
\alpha_1 = 1, \quad \alpha_2 = q, \quad \alpha_3 = q, \quad \alpha_4 = 0, \quad \alpha_5 = -\frac{q}{2}, \quad \alpha_6 = \frac{q^2}{4} + \xi_1,
\]

\[
\alpha_7 = -\xi_2, \quad \alpha_8 = \xi_3, \quad \alpha_9 = \xi_1 - q \xi_2 + q^2 \xi_3 + \frac{q^2}{4} = V_0 b_1(1-q)^2 + q(b_2 + \frac{q}{4}),
\]

\[
\alpha_{10} = 1 + 2\sqrt{\xi_3}, \quad \alpha_{11} = 2q + 2 \left( \sqrt{\xi_1 - q \xi_2 + q^2 \xi_3 + \frac{q^2}{4}} + \sqrt{\xi_3} \right),
\]

\[
\alpha_{12} = \sqrt{\xi_3}, \quad \alpha_{13} = -\frac{q}{2} - \left( \sqrt{\xi_1 - q \xi_2 + q^2 \xi_3 + \frac{q^2}{4}} + q \sqrt{\xi_3} \right). \tag{51}
\]

Using (14), (16) and (51), we calculate the parameters required for the method

\[
\pi(s) = -\frac{q}{2} s \pm \left\{ q^2 \left( \frac{1}{4} + b_2C_0 - b_3 \right) + V_0 b_1 - kq \right\} s^2
\]

\[
+ \left[ -(2b_1b_3q + 2b_2C_0 q - b_2) + k \right] s - (b_1b_3 - b_3 V_0 - C_0 b_2) \right\}^\frac{1}{2}, \tag{52}
\]

where

\[
k_{1,2} = -\left[ 2b_2 V_0(1-q) - b_2 \right]
\]

\[
\pm 2 \left\{ (b_1b_3 + b_3 V_0 + C_0 b_1) \left[ b_3 V_0(1-q)^2 + q \left( b_2 + \frac{q}{4} \right) \right] \right\}^\frac{1}{2}. \tag{53}
\]

Different \(k\)'s lead to the different \(\pi\)'s. For
\[
k = 2b \sqrt{\nu_0(1-\nu)} - b_2 - 2 \left[ -b_3 + b \sqrt{\nu_0} + C_0 b_2 \right] \left[ \sqrt{\nu_0(1-\nu)^2 + q(b_2 + \frac{q}{4})} \right]^{\frac{1}{2}}, \tag{54}
\]

\(\pi(s)\) becomes
\[
\pi(s) = -\frac{q}{2} \sqrt{\nu_0(1-\nu)^2 + q(b_2 + \frac{q}{4})} + q \sqrt{-b_3 + b \sqrt{\nu_0} + C_0 b_2}, \tag{55}
\]

and using (20), we obtain
\[
\tau(s) = 1 - 2qs - 2s \sqrt{\nu \sqrt{\nu_0(1-\nu)^2 + q(b_2 + \frac{q}{4})} + (1 - 2qs) \sqrt{-b_3 + b \sqrt{\nu_0} + C_0 b_2}}, \tag{56}
\]

where \(\tau'(s) < 0\).

Therefore, using (22) and (51), we write the eigenvalues equation as
\[
n^2 q + \left( q + 2 \sqrt{\nu \sqrt{\nu_0(1-\nu)^2 + q(b_2 + \frac{q}{4})} + q \sqrt{-b_3 + b \sqrt{\nu_0} + C_0 b_2}} \right) n
\]
\[
+ \left( 1 + 2q \sqrt{b_2 C_0 - b_3 + \sqrt{\nu_0}} b_1 \right) \left( \sqrt{\nu_0(1-\nu)^2 + q(b_2 + \frac{q}{4})} + q \sqrt{b_2 C_0 - b_3 + \sqrt{\nu_0} b_1} \right)
\]
\[-2b_2 C_0 q + b_2 + 2b_2 q - 2b_2 b_1 + \frac{q}{2} = 0. \tag{57}
\]

Now, let us give the corresponding upper Dirac spinor. Using (23), (24) and (27), we write the corresponding unnormalized eigenfunctions are obtained in terms of the functions,
\[
\rho(s) = s^{\frac{2}{q} \sqrt{\nu_0 - b_3 + b \sqrt{\nu_0} + \frac{q}{4}}} \left( 1 - qs \right)^{\frac{2}{q} \sqrt{\nu \sqrt{\nu_0(1-\nu)^2 + q(b_2 + \frac{q}{4})} + q \sqrt{-b_3 + b \sqrt{\nu_0} + C_0 b_2}}} \tag{58}
\]
and
\[
y_n(s) = P_n \left[ s^{\frac{2}{q} \sqrt{\nu_0 - b_3 + b \sqrt{\nu_0} + \frac{q}{4}}} \left( 1 - qs \right)^{\frac{2}{q} \sqrt{\nu \sqrt{\nu_0(1-\nu)^2 + q(b_2 + \frac{q}{4})} + q \sqrt{-b_3 + b \sqrt{\nu_0} + C_0 b_2}}} \right] (1 - 2qs), \tag{59}
\]
and
\[
\phi(s) = s^{\frac{1}{2} \sqrt{\nu_0 - b_3 + b \sqrt{\nu_0} + \frac{q}{4}}} \left( 1 - qs \right)^{\frac{1}{2} \sqrt{\nu \sqrt{\nu_0(1-\nu)^2 + q(b_2 + \frac{q}{4})} + q \sqrt{-b_3 + b \sqrt{\nu_0} + C_0 b_2}}}. \tag{60}
\]

and using (28), the corresponding wave functions to be
\[
F_{nk}(s) = B_n s^{\frac{1}{2} \sqrt{\nu_0 - b_3 + b \sqrt{\nu_0} + \frac{q}{4}}} \left( 1 - qs \right)^{\frac{1}{2} \sqrt{\nu \sqrt{\nu_0(1-\nu)^2 + q(b_2 + \frac{q}{4})} + q \sqrt{-b_3 + b \sqrt{\nu_0} + C_0 b_2}}} \times P_n \left[ s^{\frac{2}{q} \sqrt{\nu_0 - b_3 + b \sqrt{\nu_0} + \frac{q}{4}}} \left( 1 - qs \right)^{\frac{2}{q} \sqrt{\nu \sqrt{\nu_0(1-\nu)^2 + q(b_2 + \frac{q}{4})} + q \sqrt{-b_3 + b \sqrt{\nu_0} + C_0 b_2}}} \right] (1 - 2qs), \tag{61}
\]
where \(B_n\) is the normalized constant and it was determined by the condition
\[
\int_{-\infty}^{\infty} F_{nk}(s) ds = 1, \tag{62}
\]
and the lower component of Dirac spinor can be calculated by applying (44) as
Dirac-Hua problem

\[ G_{nk}(r) = \frac{1}{M + E_{nk} - \Delta(r)} \left( \frac{d}{dr} + \frac{k}{r} - U(r) \right) F_{nk}(r), \]  

\[ b_1 = (E_{nk} + M - C_s), \quad b_2 = (k + H)(k + H + 1), \]

\[ b_3 = \frac{E_{nk} - M}{4\alpha^2}, \quad \vec{V}_0 = \frac{V_0}{4\alpha^2}, \]

3.2 Pseudo-Spin symmetry case

Substituting (1) and (2) into (47) and considering pseudo-spin symmetry (the condition of pseudo-spin symmetry \( d\Sigma(r)/dr = 0 \) or \( \Sigma(r) = \text{const} = C_{ps} \)) [49, 50], we have

\[ \left\{ \begin{array}{l} \frac{d}{dr} \left[ \frac{1}{r^2} \frac{d}{dr} \right] \left( (k + H)(k + H - 1) \right) \\
+ (E_{nk} - M - C_{ps})(M + E_{nk}) - V_0(E_{nk} - M - C_{ps}) \left[ \frac{1 - e^{-2\alpha r}}{1 - q e^{-2\alpha r}} \right]^2 \end{array} \right\} G_{nk}(r) = 0, \]

By using Eq. (49) and \( s = e^{-2\alpha r}, \; 0 < s < 1 \), we obtain

\[ \left\{ \begin{array}{l} \frac{d^2}{ds^2} + \frac{1}{s} \frac{d}{ds} \left[ s^2 (1-q s)^2 \right] \left[ -(-b_s b_s q_s^2 + b_s \vec{V}_0 + C_0 b_s q_s^2) \right] \\
+ (-2 b_s b_s q_s + 2 b_s \vec{V}_0 - 2 b_s C_0 q_s - b_s) \left[ (b_s + b_s \vec{V}_0 + C_0 b_s) \right] \end{array} \right\} G_{nk}(s) = 0, \]

and using the NU method and similar procedures presented in subsect. 3.1., the eigenvalues for the symmetry are obtained as by using (22) to be

\[ n^2 q + \left( q + 2 \sqrt{\vec{V}_0 b_s (1-q)^2} + q (b_s + \frac{q}{4}) + 2 q \sqrt{b_s C_0 - b_s b_s + \vec{V}_0 b_s} \right) n \]

\[ + \left( 1 + 2 q \sqrt{b_s C_0 - b_s b_s + \vec{V}_0 b_s} \right) \left( \sqrt{\vec{V}_0 b_s (1-q)^2} + q (b_s + \frac{q}{4}) + q \sqrt{b_s C_0 - b_s b_s + \vec{V}_0 b_s} \right) \]

\[ -2 b_s C_0 q_s + b_s + 2 b_s b_s q_s - 2 \vec{V}_0 b_s + \frac{q}{2} = 0, \]

The lower component of the Dirac spinor can be calculated by using (31) as

\[ G_{nk}(s) = a_n s^{\sqrt{b_s C_0 - b_s b_s + \vec{V}_0 b_s}} \left( 1 - q s \right)^{\frac{1}{2} - \frac{1}{2} \sqrt{\vec{V}_0 b_s (1-q)^2 + q (b_s + \frac{q}{4})}} \]

\[ \times P_n^{\left( \frac{1}{2} - \frac{1}{2} \sqrt{\vec{V}_0 b_s (1-q)^2 + q (b_s + \frac{q}{4})} \right)} \left( 1 - 2 q s \right), \]

where \( a_n \) is a normalization constant and it was determined by the condition

\[ \int_{-\infty}^{\infty} G_{nk}(s) ds = 1, \text{ and} \]
\[ b_4 = (k + H)(k + H - 1), \quad b_5 = E_{nk} - M - C_{ps}, \quad b_6 = \frac{E_{nk} + M}{4\alpha^2}. \] (68)

4 Conclusion

In this work, we have been obtained analytically the approximate energy equation and the corresponding wave functions of the Dirac equation for the Hua potential coupled including a Coulomb-like tensor under the condition of the spin and pseudo-spin symmetries using the parametric generalization of the NU method. The energy eigenvalues equation and the corresponding unnormalized wave functions have been obtained in terms of the Jacobi polynomials.

Acknowledgment: The author would like to thank the kind my wife Mrs. Seyedeh Soghra Moosavi for their positive suggestions which have improved the present work.

References


Received: April, 2011