A Covariant Relativistic Formalism
for the New Dirac Equation

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Abstract

In the present work we construct a covariant relativistic lagrangian to describe a nonzero rest mass particle, with nonzero spin, which has a special internal structure. From the minimum action principle we obtain the new relativistic equation similar to one proposed by Dirac in 1971. By this method we may study Noether’s currents, energy-impulse tensor and field moment. Interaction with other fields remains an unsolved question.

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1 Introduction

It has been known that the two major theories of twentieth century physics are relativity and quantum theory were obtained quite differently. The essentials
of relativity theory were formulated in the first decade of last century. But the discovery of quantum theory spanned three decades. The first formulation of a relativistic wave equation with a standard probability interpretation was made by Dirac in 1928. But when the physical interpretation of the of the Dirac equation was finally worked out, it turned out that the wave function was not a Schrödinger wave function. It is only in the interaction-free limit that it could be interpreted as a wave function. Dirac assiduously searched for a relativistic Schrödinger equation describing a composite particle, which has a special internal structure. However, he found this description was inconsistent in the present of electromagnetic interaction, and Dirac seems to have abandoned this programme some time ago. N. Mukunda and his collaborators, including E Sudarshan found a way out of this difficulty [1]. A new relativistic wave equation for particles of non-zero rest-mass is proposed by Dirac [2], allowing only positive values for the energy. There is great formal similarity between it and the usual relativistic wave equation for the electron, but the physical significance is very different, in particular the new equation gives integral values for the spin. The internal degrees of freedom involve two harmonic oscillators.

Dirac’s equation of 1928 was the first spectacularly successful combination of the principles of quantum mechanics and of special relativity. There had been a previous attempt to combine these principles, and it had led to the equation of Klein and Gordon. Even Niels Bohr had felt that that equation was quite satisfactory. Dirac however was convinced that it was not, for two reasons: it was in conflict with the probabilistic interpretation of quantum mechanics, and it did not conform to the principles of the transformation theory of quantum mechanics which Dirac himself had set up.

The new relativistic Dirac equation has the following form

$$\left\{ \frac{\partial}{\partial x_0} + \alpha_r \frac{\partial}{\partial x_r} + m\beta \right\} \psi = 0,$$

(1)

where $\alpha_r (r = 1, 2, 3)$ are real matrices $4 \times 4$ and $\beta$ is an antisymmetric matrix. Note that $\beta^2 = -I$. The symbol $\tilde{q}$ will denote a $4 \times 1$ matrix (vector) $(q_1, q_2, q_3, q_4)$ where $q_1, q_3 = p_1$ and $q_2, q_4 = p_2$ stand for the dynamic variables associated to two harmonic oscillators describing particle’s internal structure. The quantities $q_a (a = 1, 2, 3, 4)$ satisfy following commuting law

$$[q_a, q_b] = q_a q_b - q_b q_a = i\beta_{a,b}$$

(2)

In a similar way, matrices $\alpha_r (r = 1, 2, 3)$ and $\beta$ satisfy the Dirac algebra

$$\alpha_r \alpha_s + \alpha_s \alpha_r = 2\delta_{r,s}.$$  

(3)
Let $\partial^\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\alpha_0 = I$.

In view of previous considerations, equation (1) takes the form

$$(\alpha_\mu \partial^\mu + m\beta)q\psi = 0,$$  

(5)

where $\mu = 0, 1, 2, 3$

2 Main Results

The new relativistic Dirac equation (1) (equivalently, equation (5)) and the old one may be obtained starting from the pair of field functions $q_a \psi$, $\bar{\psi} q_b$, where $a, b = 1, 2, 3, 4$.

Let $L_D$ be the Lagrange density function; it is obvious that this function must be a relativistic invariant. With the aid of the field functions $q_a \psi$, $\bar{\psi} q_b$, $\partial^\mu \bar{\psi} q_b$ we construct the relativistic bilinear form given by

$$L_D = -\frac{1}{4}(\bar{\psi} \tilde{q} \alpha_\mu \beta \partial^\mu q\psi - \partial^\mu \bar{\psi} \tilde{q} \alpha_\mu \beta q\psi) + \frac{1}{2} \bar{\psi} \tilde{q} m q\psi.$$  

(6)

We now introduce following expressions

$$q_a \psi = \varphi_a, \bar{\psi} q_b = \bar{\varphi}_b, \partial^\mu q_a \psi = \varphi_a^\mu, \partial^\mu \bar{\psi} q_b = \bar{\varphi}_b^\mu.$$  

(7)

Taking into account equations (6)-(7) we have

$$L_D = L_D(\varphi_a, \bar{\varphi}_b, \varphi_a^\mu, \bar{\varphi}_b^\mu).$$  

(8)

Let us define the action by

$$I = \int L_D(x) dx.$$  

(9)

Action variation is

$$\delta I = I' - I = \int L_D' dx' - \int L_D dx.$$  

(10)

More explicitly,

$$\delta I = \int L_D(\varphi_a + \delta \varphi_a, \bar{\varphi}_b + \delta \bar{\varphi}_b, \varphi_a^\mu + \delta \varphi_a^\mu, \bar{\varphi}_b^\mu + \delta \bar{\varphi}_b^\mu) dx' - \int L_D dx.$$  

(11)
or more generally,
\[ \delta I = \int \left[ L_D(\varphi_a, \tilde{\varphi}_b, \varphi_a^\mu, \tilde{\varphi}_b^\mu) + \frac{\partial L_D}{\partial \varphi_a} \delta \varphi_a + \frac{\partial L_D}{\partial \tilde{\varphi}_b} \delta \tilde{\varphi}_b + \frac{\partial L_D}{\varphi_a^\mu} \delta \varphi_a^\mu + \frac{\partial L_D}{\tilde{\varphi}_b^\mu} \delta \tilde{\varphi}_b^\mu \right] dx' - I. \]  
(12)

In view of the fact that
\[ dx' = (1 + \frac{\partial \delta x_k}{\partial x_k}) dx, \]  
(13)
from equation (12) we obtain
\[ \delta I = \int L_D \delta x_k - \int \frac{\partial L_D}{\partial x_k} \delta x_k dx + \int \delta L_D dx. \]  
(14)

Integrating by parts the first term of (14) gives
\[ \delta I = L_D \delta x_k - \int \frac{\partial L_D}{\partial x_k} \delta x_k dx + \int \delta L_D dx. \]  
(15)

Since coordinates in the integration limits miss, \( \delta x^k = 0 \) and the lagrangian does not depend explicitly on the coordinates, we may write
\[ \delta I = \int \left[ \frac{\partial L_D}{\partial \varphi_a} \delta \varphi_a + \frac{\partial L_D}{\partial \tilde{\varphi}_b} \delta \tilde{\varphi}_b + \frac{\partial L_D}{\varphi_a^\mu} \delta \varphi_a^\mu + \frac{\partial L_D}{\tilde{\varphi}_b^\mu} \delta \tilde{\varphi}_b^\mu \right] dx \]  
(16)

After integrating by parts the two last terms in last equation, it converts to
\[ \delta I = \int \left( \frac{\partial L_D}{\partial \varphi_a} \delta \varphi_a - \frac{\partial}{\partial x_\mu} \frac{\partial L_D}{\partial \varphi_a^\mu} \delta \varphi_a + \frac{\partial L_D}{\partial \tilde{\varphi}_b} \delta \tilde{\varphi}_b - \frac{\partial}{\partial x_\mu} \frac{\partial L_D}{\partial \tilde{\varphi}_b^\mu} \delta \tilde{\varphi}_b^\mu \right) dx + f(\delta \varphi_a, \delta \varphi_a), \]  
(17)

where \( f(\delta \varphi_a, \delta \varphi_a) \) is given by
\[ f(\delta \varphi_a, \delta \varphi_a) = \frac{\partial L_D}{\varphi_a^\mu} \delta \varphi_a + \frac{\partial L_D}{\tilde{\varphi}_b^\mu} \delta \tilde{\varphi}_b. \]  
(18)

According to principle of least action, \( \delta I = 0 \) and taking into account that the variation of field functions \( \delta \varphi_a \) y \( \delta \varphi_b \) equals zero at the limits of integration, we obtain
\[ \delta I = \int \left[ \left( \frac{\partial L_D}{\partial \varphi_a} - \frac{\partial}{\partial x_\mu} \frac{\partial L_D}{\partial \varphi_a^\mu} \right) \delta \varphi_a + \left( \frac{\partial L_D}{\partial \tilde{\varphi}_b} - \frac{\partial}{\partial x_\mu} \frac{\partial L_D}{\partial \tilde{\varphi}_b^\mu} \right) \delta \tilde{\varphi}_b \right] dx = 0. \]  
(19)
The two variations $\delta \varphi_a$ and $\delta \tilde{\varphi}_b$ are independent and then

$$\frac{\partial L_D}{\partial \varphi_a} - \frac{\partial}{\partial x_\mu} \frac{\partial L_D}{\partial \dot{\varphi}^\mu_a} = 0. \tag{20}$$

$$\frac{\partial L_D}{\partial \tilde{\varphi}_b} - \frac{\partial}{\partial x_\mu} \frac{\partial L_D}{\partial \dot{\tilde{\varphi}}^\mu_b} = 0. \tag{21}$$

These equations may be expressed in terms of the field functions $q \psi$ and $\bar{\psi} \bar{q}$ as follows

$$\frac{\partial}{\partial x_\mu} \left( \frac{\partial L_D}{\partial (\partial^\mu q \psi)} \right) - \frac{\partial L_D}{\partial q \psi} = 0. \tag{22}$$

$$\frac{\partial}{\partial x_\mu} \left( \frac{\partial L_D}{\partial (\partial^\mu \bar{\psi} \bar{q})} \right) - \frac{\partial L_D}{\partial \bar{\psi} \bar{q}} = 0. \tag{23}$$

From equations (6) and (23) we conclude that

$$\frac{\partial L_D}{\partial \bar{\psi} \bar{q}} = -\frac{1}{4} \alpha_\mu \beta \partial^\mu q \psi + \frac{1}{2} m q \psi \tag{24}$$

and

$$\frac{\partial}{\partial x_\mu} \left( \frac{\partial L_D}{\partial (\partial^\mu \bar{\psi} \bar{q})} \right) = \frac{1}{4} \alpha_\mu \beta \partial^\mu q \psi. \tag{25}$$

Finally,

$$\left\{ \frac{\partial}{\partial x_\alpha} + \alpha_r \frac{\partial}{\partial x_r} + m \beta \right\} q \psi = 0$$

which is the new Dirac equation.

3 Conclusions

In this work we showed how to obtain a new Dirac equation by relativistic lagrangian method. By this method we can obtain the Noether’s currents, energy-impulse tensor and field moment. Interaction with other fields remains an unsolved question. The problem with the electromagnetic interaction stay to be still to be done.
References


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