Exact Solutions and Light Bullet Soliton Solutions to the Generalized (3+1)-Dimensional Nonlinear Schrödinger Equation

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Abstract

A generalized $G'/G$-expansion method is applied to solve the generalized (3+1)-dimensional nonlinear Schrödinger equation. Hyperbolic function solution, trigonometric function solution and rational exact solution with parameters are obtained. Selecting parameters and parameter functions properly, novel light bullet soliton solutions with or without the chirp are presented.

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1 Introduction

The nonlinear Schrödinger equation (NLSE) is one of the most useful generic mathematical models \cite{1} that naturally arises in many fields of physics. Major interest in the NLSE was piqued by the discovery of solitary wave solutions.\cite{2,3} Stable exact soliton solutions the NLSE are known only in (1+1) dimensional [(1+1)D], for the simple reason than the inverse scattering method, responsible for the existence and stability of 1D solitons, works only in (1+1)D. There are no known exact stable solitons in (2+1)D or (3+1)D. Recently, great interest has been generated when it was suggested that the (2+1)D generalized NLSE with distributed coefficients may lead
to stable 2D solitons.\[4\] The stabilizing mechanism was the sign-alternating Kerr nonlinearity in a layered medium. A vigorous search for the stabilized localized solutions of the (2+1)D NLSE has been launched;\[5–7\] however, out of necessity, it has been numerical. We present here analytical travelling wave and solitons to the NLSE in (3+1)D. Our interest is focus on the generalized NLSE in (3+1)D with distributed coefficients\[8,9\];

\[
i \frac{\partial u}{\partial z} + \frac{\beta(z)}{2} (\partial_{\perp} u + \frac{\partial^2 u}{\partial t^2}) + \chi(z)|u|^2 u = i\gamma(z)u, \tag{1}
\]

which describes the evolution of a lowly varying wave packet envelope \(u(z, x, y, t)\) in a diffractive nonlinear Kerr medium with anomalous dispersion, in the paraxial approximation. Here \(z\) is the propagation coordinate, \(\Delta_{\perp} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\) represents the transverse Laplacian, and \(t\) is the reduced time, i.e., time in the frame of reference moving with the wave packet. All coordinates are made dimensionless by the choice of coefficients. The functions \(\beta, \gamma\) and \(\chi\) stand for the diffraction or dispersion, nonlinearity and gain coefficients, respectively. The generalized NLSE is of considerable importance, as it describes the full spatiotemporal optical solitons, or light bullets, in (3+1) dimension. In this paper, we extend the so called \(z^\epsilon\)-expansion method\[10–12\] which is present very recently to consider the generalized (3+1)D NLSE, many types of exact solutions including light bullet soliton solutions are obtained.

## 2 Exact solutions for the generalized (3+1)D NLSE

For the generalized (3+1)D NLSE (1), we define the complex \(u\) field as

\[
u(z, x, y, t) = A(z, x, y, t)e^{iB(z, x, y, t)}, \tag{2}
\]

where \(A(z, x, y, t)\) and \(B(z, x, y, t)\) are real functions. Substituting \(u(z, x, y, t)\) into Eq.(1), we find the following coupled equations for the phase \(B(z, x, y, t)\) and the amplitude \(A(z, x, y, t)\):

\[
\frac{\partial A}{\partial z} + \frac{\beta(z)}{2} \left(2 \frac{\partial A}{\partial x} \frac{\partial B}{\partial x} + 2 \frac{\partial A}{\partial y} \frac{\partial B}{\partial y} + 2 \frac{\partial A}{\partial t} \frac{\partial B}{\partial t} + A \frac{\partial^2 B}{\partial x^2} + A \frac{\partial^2 B}{\partial y^2} + A \frac{\partial^2 B}{\partial t^2}\right) - \gamma(z)A = 0, \tag{3}
\]

\[
-A \frac{\partial B}{\partial z} + \frac{\beta(z)}{2} \left[\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial t^2} - A \left(\frac{\partial B}{\partial x}\right)^2 - A \left(\frac{\partial B}{\partial y}\right)^2 - A \left(\frac{\partial B}{\partial t}\right)^2\right] + \chi(z)A^3 = 0. \tag{4}
\]

According to the balance principle and the \(z^\epsilon\)-expansion method, the solution of Eqs.(3) and (4) can be expressed in the following form:

\[
A(z, x, y, t) = f_0(z) + f_1(z) \frac{G(\xi)'}{G(\xi)}, \quad \xi = k(z)x + l(z)y + m(z)t + \omega(z), \tag{5}
\]

\[
B(z, x, y, t) = a(z)(x^2 + y^2 + t^2) + b(z)(x + y + t) + e(z) \tag{6}
\]
where \( f_0, f_1, k, l, m, \omega, a, b \) and \( e \) are the parameters to be determined. The function \( G = G(\xi) \) satisfied the a second order linear ordinary differential equation:

\[
G'' + \lambda G' + \mu G = 0,
\]

where \( G'' = \frac{\partial^2 G(\xi)}{\partial \xi^2}, G' = \frac{\partial G(\xi)}{\partial \xi}, \) and \( \lambda, \mu \) are real constants. Substituting Eqs.(5) and (6) into Eqs.(3) and (4) and collecting all the terms with the same order of \( G \) together, the left sides of Eqs.(3) and (4) are converted into two polynomials of \( x^{i_1}y^{j_1}t^{k_1}(\frac{G'}{G})^{h_1} (i_1 = 0, 1; j_1 = 0, 1; k_1 = 0, 1; h_1 = 0, 1, 2) \) and \( x^{i_2}y^{j_2}t^{k_2}(\frac{G'}{G})^{h_2} (i_2 = 0, 1, 2; j_2 = 0, 1, 2; k_2 = 0, 1, 2; h_2 = 0, 1, 2, 3) \) respectively. Setting each coefficient of the two polynomials to zero, after simplified one can finally derive a system of algebraic or first-order ordinary differential equations for \( f_0, f_1, k, l, m, \omega, a, b \) and \( e \) as follows:

\[
-f_1 \mu \frac{\partial \omega}{\partial z} + 3 \beta f_0 a - \gamma f_0 + \frac{\partial f_0}{\partial z} - \beta f_1 k b \mu - \beta f_1 l b \mu - \beta f_1 m b \mu = 0,
\]

\[
-f_1 p(\frac{\partial k}{\partial z} + 2 \beta k a) = 0,
\]

\[
-f_1 p(\frac{\partial l}{\partial z} + 2 \beta l a) = 0,
\]

\[
-f_1 p(\frac{\partial m}{\partial z} + 2 \beta m a) = 0,
\]

\[
-f_1 \left( \frac{\partial \omega}{\partial z} + \beta k b + \beta l b + \beta m b \right) = 0,
\]

\[
-f_1 \left( \frac{\partial a}{\partial z} + 2 \beta a^2 \right) = 0,
\]

\[
-f_1 \left( \frac{\partial b}{\partial z} + 2 \beta a b \right) = 0,
\]

\[
\frac{\partial f_1}{\partial z} + 3 \beta f_1 a - \gamma f_1 - f_1 \frac{\partial \omega}{\partial z} - \beta f_1 k b \lambda - \beta f_1 l b \lambda - \beta f_1 m b \lambda = 0,
\]

\[
-\frac{3}{2} \beta f_0 b^2 + \chi f_0^3 - f_0 \frac{\partial e}{\partial z} + \frac{1}{2} \beta f_1 k^2 \lambda \mu + \frac{1}{2} \beta f_1 l^2 \lambda \mu + \frac{1}{2} \beta f_1 m^2 \lambda \mu = 0,
\]

\[
\frac{1}{2} f_1 (6 \chi f_0^2 - 2 \frac{\partial e}{\partial z} + 2 \beta k^2 \mu + 2 \beta l^2 \mu + 2 \beta m^2 \mu - 3 \beta b^2 + \beta k^2 \lambda^2 + \beta l^2 \lambda^2 + \beta m^2 \lambda^2) = 0,
\]

\[
\frac{3}{2} f_1 (2 \chi f_0 f_1 + \beta k^2 \lambda + \beta l^2 \lambda + \beta m^2 \lambda) = 0,
\]

\[
f_1 (\chi f_1^2 + \beta k^2 + \beta l^2 + \beta m^2) = 0,
\]

where \( p = 1, \lambda, \mu \) and \( j = 0, 1. \)

Solving Eqs.(8)-(19) self-consistently with the help of Maple or Mathematica, we have a set of conditions on the coefficients and parameters as follow:

\[
f_0 = f_{10} \exp(\int_0^z \gamma - 3 \beta a_0 \alpha dz), \quad f_1 = \frac{2 f_{10}}{\lambda} \exp(\int_0^z \gamma - 3 \beta a_0 \alpha dz),
\]
\[ a = a_0\alpha, \ b = b_0\alpha, \ k = k_0\alpha, \ l = l_0\alpha, \ m = m_0\alpha, \]  
\[ \omega = \omega_0 - \alpha(k_0 + l_0 + m_0)b_0 \int_0^z \beta dz, \]  
\[ e = e_0 + [(k_0^2 + l_0^2 + m_0^2)(\mu - \frac{\lambda^2}{4}) - \frac{3}{2}b_0^2] \int_0^z \beta \alpha^2 dz, \]  
where \( f_{10}, a_0, b_0, k_0, l_0, m_0, \omega_0, e_0 \) are real constants, and \( \alpha = (1 + 2a_0 \int_0^z \beta dz)^{-1} \) is the normalized chirp function which is related to the wave front curvature and presents a measure of the phase chirp imposed on the wave. One should note the universal influence of the chirp function \( \alpha \) on the solutions. The chirp function is related only to the diffraction or dispersion coefficients; however, it affects all of the parameters. In the case when there is no chirp, \( a_0 = 0 \) and \( \alpha = 1 \), the parameters \( k,l,m \) and \( b \) are all constants. In the presence of chirp, they all acquire the prescribed \( z \) dependence. The chirp also influence the form of the amplitude \( A \) through the dependence of \( f_0, f_1 \) and \( \xi \) on \( \alpha \). It should also be noted that \( \chi \) is not arbitrary but depends on \( \alpha, \beta \) and \( \gamma \):  
\[ \chi = -\frac{\beta \lambda^2 \alpha^2}{4f_{10}^2}(k_0^2 + l_0^2 + m_0^2)e^{\exp(-2 \int_0^z \gamma - 3\beta a_0 \alpha dz)}. \]  
Hence, the solutions found can exist only under certain conditions and the system parameter functions \( \beta(z), \gamma(z) \) and \( \chi(z) \) cannot be all chosen independently, i.e., the functions \( \beta(z), \gamma(z) \) are chosen as independently while the nonlinearity coefficient \( \chi(z) \) is determined by Eq.\( (24) \). And to obtain exact solutions in a lossy medium, the nonlinearity coefficient \( \chi \) must grow exponentially.

Substituting Eqs.\( (5) \) and \( (6) \) with Eqs.\( (20)-(23) \) into Eq.\( (2) \), we have the fundamental solution of Eq.\( (1) \):  
\[ u = f_{10} \exp(\int_0^z \gamma - 3\beta a_0 \alpha dz)[1 + \frac{2}{\lambda}(\frac{G'}{G})] \exp\{i[a_0\alpha(x^2 + y^2 + t^2) + b_0\alpha(x + y + t) + ((l_0^2 + m_0^2)(\mu - \frac{1}{4}\lambda^2) - \frac{3}{2}b_0^2)] \int_0^z \beta \alpha^2 dz + e_0\}, \]  
where \( G = G(\xi), \xi = k_0\alpha x + l_0\alpha y + m_0\alpha t + \omega_0 - \alpha(k_0 + l_0 + m_0)b_0 \int_0^z \beta dz, \) and \( \alpha = [1 + 2a_0 \int_0^z \beta dz]^{-1} \).

Substituting the general solutions of Eq.\( (7) \) into Eq.\( (25) \), we have three types of exact solutions of Eq.\( (1) \) as follows:

When \( \lambda^2 - 4\mu > 0 \), we obtain hyperbolic function solution
\[ u_1 = f_{10} \exp(\int_0^z \gamma - 3\beta a_0 \alpha dz)[1 + \frac{2}{\lambda}(\frac{C_1 \sinh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi) + C_2 \cosh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi)}{C_1 \cosh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi) + C_2 \sinh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi)})] \exp\{i[a_0\alpha(x^2 + y^2 + t^2) + b_0\alpha(x + y + t) + ((l_0^2 + m_0^2)(\mu - \frac{1}{4}\lambda^2) - \frac{3}{2}b_0^2)] \int_0^z \beta \alpha^2 dz + e_0\}, \]  
where \( \xi = k_0\alpha x + l_0\alpha y + m_0\alpha t + \omega_0 - \alpha(k_0 + l_0 + m_0)b_0 \int_0^z \beta dz, \) and \( \alpha = [1 + 2a_0 \int_0^z \beta dz]^{-1} \), and \( C_1, C_2 \) are arbitrary constants.
When $\lambda^2 - 4\mu < 0$, we have trigonometric function solution

$$u_2 = f_{10} \exp\left(\int_0^z \gamma - 3\beta a_0 \alpha dz\right) \left[1 + \frac{2}{\lambda} \left(-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)\right)\right] \exp\{ia_0 \alpha (x^2 + y^2 + t^2) + b_0 \alpha (x + y + t) + ((k_0^2 + l_0^2 + m_0^2)(\mu - \frac{1}{4} \lambda^2) - \frac{3}{2} b_0^2)\int_0^z \beta \alpha^2 dz\} + e_0\}, \quad (27)$$

where $\xi = k_0 \alpha x + l_0 \alpha y + m_0 \alpha t + \omega_0 - \alpha (k_0 + l_0 + m_0) b_0 \int_0^z \beta dz$, $\alpha = [1 + 2a_0 \int_0^z \beta dz]^{-1}$, and $C_1, C_2$ are arbitrary constants.

When $\lambda^2 - 4\mu = 0$, we get rational solution

$$u_3 = f_{10} \exp\left(\int_0^z \gamma - 3\beta a_0 \alpha dz\right) \left[1 + \frac{2}{\lambda} \left(\frac{C_2}{C_1 + C_2 \xi}\right)\right] \exp\{ia_0 \alpha (x^2 + y^2 + t^2) + b_0 \alpha (x + y + t) + ((k_0^2 + l_0^2 + m_0^2)(\mu - \frac{1}{4} \lambda^2) - \frac{3}{2} b_0^2)\int_0^z \beta \alpha^2 dz\} + e_0\}, \quad (28)$$

where $\xi = k_0 \alpha x + l_0 \alpha y + m_0 \alpha t + \omega_0 - \alpha (k_0 + l_0 + m_0) b_0 \int_0^z \beta dz$, $\alpha = [1 + 2a_0 \int_0^z \beta dz]^{-1}$, and $C_1, C_2$ are arbitrary constants.

For the solutions of Eq.(1) above, due to the arbitrary functions $\gamma(z)$ and $\beta(z)$ ($\chi(z)$ depends on $\gamma(z)$ and $\beta(z)$ by Eq.(24)), we have freedom in selecting these functions appropriately according to some actual physical requirements. Thus we can construct abundant periodic wave solutions and solitary solutions. Here, for example, we only consider a simple special case. For the solution $u_1$ expressed in Eq.(26), when $C_2 = 0$, and $\mu = \frac{\lambda^2 - 4R}{4}$, we have the solution

$$u_4 = f_{10} \exp\left(\int_0^z \gamma - 3\beta a_0 \alpha dz\right) \left[1 + \frac{2}{\lambda} \left(\frac{\tan h(\sqrt{R} \xi)}{\tan h(\sqrt{R} \xi)}\right)\right] \exp\{ia_0 \alpha (x^2 + y^2 + t^2) + b_0 \alpha (x + y + t) + ((k_0^2 + l_0^2 + m_0^2)(\mu - \frac{1}{4} \lambda^2) - \frac{3}{2} b_0^2)\int_0^z \beta \alpha^2 dz\} + e_0\}, \quad (29)$$

where $\xi = k_0 \alpha x + l_0 \alpha y + m_0 \alpha t + \omega_0 - \alpha (k_0 + l_0 + m_0) b_0 \int_0^z \beta dz$, $\alpha = [1 + 2a_0 \int_0^z \beta dz]^{-1}$, $R > 0$ and $\chi$ satisfies Eq.(26). Taking the diffraction (or dispersion) coefficient $\beta$ to be of the form $\beta = \beta_0 \cos(k_0 z)$ and the fain (loss) coefficient $\gamma$ to be a small constant, we obtain light bullet soliton solutions of the $(3+1)$D NLSE. Fig1.(a) and (b) present the light bullet soliton solutions, with and without the chirp. In Fig1 (a)-(b), we plot the intensity $U = |u_4|^2$ determined by Eq.(29), and the functions of the propagation distance are presented as functions of $X = k_0 x + l_0 y + m_0 t$ and $z$. Obviously, the effect of the particular periodic chirp function determined by the dispersion coefficient $\beta = \beta_0 \cos(k_0 z)$ is to produce a periodic variation along the propagation direction ($z$ direction) and a monotonic asymmetric change in the transverse direction ($X$ direction).
Figure 1: Intensity plots of the light bullet solitary wave solutions $u_4$ expressed by Eq.(29). (a) chirped wave, $a_0 = 0.1$, (b) unchirped wave, $a_0 = 0$, and the parameter selections: $\gamma = \gamma_0 = 0$, $\beta = \frac{4}{3} \cos(z)$, $f_{10} = k_0 = l_0 = m_0 = b_0 = R = 1$, $\lambda = 0.2$, $\omega_0 = 0$.

3 Summary and discussion

To summarize, applying the $G'/G$-expansion method, three types of new exact solutions including hyperbolic function solution, trigonometric function solution and rational solution for the generalized (3+1)-dimensional nonlinear Schrödinger equation with distributed diffraction, dispersion, nonlinearity, and gain are obtained. Selecting the parameters and the arbitrary functions in the solutions freely, more abundant exact solutions including soliton solutions can be derived. We present novel exact light bullet soliton solutions with or without the chirp. The influence of the spatiotemporal chirp function on the phase and the amplitude of solutions is displayed. The more about the propagation characteristics and its stability for the spatial bullet solution will be investigated somewhere else. The $G'/G$-expansion method is generalized and extended to solve variable coefficients high dimension nonlinear physical models and shown effective and powerful. More explore about the solutions for the (3+1)D NLSE and seeking exact solutions for other higher-dimensional nonlinear models by the generalized $G'/G$-expansion method is further worth studying.

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