On Twisting Pure Radiation and Einstein-Maxwell Fields

William Davidson

Mathematics and Statistics Department, University of Otago
Dunedin, New Zealand
wdav295@btinternet.com

Abstract. Two new special solutions of the Einstein-Maxwell equations are presented, applicable to spacetimes having a multiple null eigenvector of the Weyl tensor forming a geodesic, shearfree and twisting congruence $k$, and where $k$ is also aligned as an eigenvector of the Maxwell field tensor. In one solution the spacetime has mass and is radiating it away, the other is non-radiating, of constant mass but possessing a field charge $Q$. In an appropriate complex null tetrad the physical and geometric properties of the spacetimes are calculated, including the Ricci tensor and the Maxwell tensor which both tend to zero as the parameter $r$ of the $k$ congruence tends to infinity. The metrics are both of Petrov type II and without singularity.

Keywords: Einstein-Maxwell, twisting rays

1 Permanent address: 21 Rowbank Way, Loughborough, UK, LE11 4AJ

1. INTRODUCTION

In this paper we present special solutions to the Einstein-Maxwell equations. We shall be concerned with Einstein-Maxwell fields in spacetimes where the Weyl tensor has a multiple null eigenvector $k$ forming a geodesic ($\kappa = 0$), shearfree ($\sigma = 0$) and twisting congruence, and where $k$ is also aligned as an eigenvector of the Maxwell field tensor $F_{ab}$. Thus we can write as properties of $k$ ($\Theta$ being the divergence and $\omega$ the twist):

$$\kappa = \sigma = 0, \quad \rho = - (\Theta + i\omega) \neq 0$$

(1.1)

in terms of the Newman-Penrose spin coefficients $\kappa, \sigma$ and $\rho$ and
in terms of the Maxwell field tensor.

We shall refer such spacetimes to a complex null tetrad \( \mathbf{m}, \mathbf{mbar}, \mathbf{n} \) and \( \mathbf{k} \) with labels 1,2,3 and 4, respectively. Since \( \mathbf{k} \) is an eigenvector of \( F_{ab} \) the field tensor scalar \( \phi_0 \) will vanish,

\[
\phi_0 = F_{ab} k^a m^b = 0,
\]

and in turn this implies that the Ricci tensor has, relative to the tetrad, the zero components

\[
R_{11} = R_{14} = R_{44} = 0.
\]

Note that the indices in (1.4) are tetrad, as specified in accordance with (1.22). Taken together the above conditions mean that these spacetimes are algebraically special so that of the Weyl complex coefficients we have

\[
\psi_0 = \psi_1 = 0.
\]

The metric and field equations for such twisting Einstein-Maxwell spacetimes were introduced by Robinson, Schild and Strauss [13], and further developed by Trim and Wainwright [19]. This followed the metric derivation for twisting vacuum fields by Kerr [8], Robinson and Robinson [14], Robinson, Robinson and Zund [15] and Debney, Kerr and Schild [1]. To display the metric and field equations we adopt coordinates \( x^i = x, y, r \) and \( u \) for \( i = 1, 2, 3 \) and 4, respectively. The spacelike coordinate \( x \) is complex, \( y \) its conjugate, \( r \) an affine parameter along the \( k \) lines and \( u \) a retarded time.

The metric signature will be taken to be +2 and we choose units so that the speed of light and the Einstein gravitational constant are unity. The spacetime metric, in terms of 1-forms relating the null tetrad to the \( x^i \) system, is [13,19]:

\[
d s^2 = 2 \omega^1 \omega^2 - 2 \omega^3 \omega^4,
\]

where

\[
\omega^1 = -dx/(P\bar{P}), \quad \omega^2 = -dy/(P\rho),
\]

\[
\omega^3 = du + Ldx + \bar{L}dy,
\]

\[
\omega^4 = dr + Wdx + \bar{W}dy + H\omega^3.
\]

The dual system is then, in the \( x^i \) system:

\[
m^i = (-P\bar{P}, 0, PW\bar{P}, PL\bar{P}),
\]

\[
\bar{m}^i = (0, -P\rho, P\bar{W}\rho, P\bar{L}\rho),
\]

\[
n^i = (0, 0, -H, 1),
\]

\[
k^i = (0, 0, 1, 0).
\]
Pure radiation and Einstein-Maxwell fields

\[
\rho = -\frac{1}{(r + r_0 + i\Sigma)}, \\
\Sigma = -iP^2 \left( \overline{\partial L} - \partial \overline{L} \right) / 2, \\
W = L_\alpha / \rho + \partial (r_0 + i\Sigma), \\
\overline{\partial} \equiv \partial_x - L\partial_u. \\
\] (1.8)

Thus the coefficient of \(du^2\) is \(-2H\) where

\[
H = K / 2 - (r + r_0)P^{-1}P_\alpha - \left( m (r + r_0) + M\Sigma - \phi_0^0 \phi_1^0 \right) / \left( (r + r_0)^2 + \Sigma^2 \right) + r_{\alpha u}, \\
K = 2P^2 \text{Re}\left( \partial \ln \left( P - \overline{L}_\mu \right) \right), \\
M = \Sigma K + P^2 \text{Re}\left( \partial \overline{\Sigma} - 2\overline{L}_\alpha \partial \Sigma - \Sigma \partial_u \overline{L} \right). \\
\] (1.9)

Here \(P, K, m, M\) and \(r_0\) (disposable) are real functions and \(L\) a complex function of \(x, y\) and \(u\). \(H\) is a real function, and \(W\) a complex function, of \(x, y, r\) and \(u\). A subscripted comma indicates partial differentiation. The Einstein-Maxwell field equations to be satisfied are then

\[
\overline{\partial} - 2L_\mu \phi_1^0 = 0, \\
\overline{\partial} - L_\mu \left( P^{-1} \phi_2^0 \right) + \left( P^{-3} \phi_1^0 \right)_\mu = 0, \\
P \left( 3L_\mu - \overline{\partial} \right) (m + iM) = 2\phi_1^0 \phi_2^0, \\
P \left( \overline{\partial} - 2L_\mu + 2P^{-1} \overline{\partial} P \right) \overline{\partial} C - P^3 \left( P^{-3} (m + iM) \right)_\mu = \phi_0^0 \phi_2^0, \\
\] (1.10) (1.11) (1.12) (1.13)

where

\[
C = \overline{\partial} \left( P^{-1} \overline{\partial} P - \overline{L}_\mu \right) + \left( P^{-1} \overline{\partial} P - \overline{L}_\mu \right)^2. \\
\] (1.14)

The items \(\phi_1^0\) and \(\phi_2^0\) are complex functions of \(x, y, u\) and are connected to the invariants of the Maxwell field

\[
\phi_1 = \frac{1}{2} F_{ab} \left( k^a n^b + \overline{m}^a m^b \right), \\
\phi_2 = F_{ab} \overline{m}^a n^b, \\
\] (1.15) (1.16)

by the relations

\[
\phi_1 = \rho^2 \phi_0^0, \\
\phi_2 = \rho \phi_0^0 + \rho^2 P \left( 2L_\mu - \overline{\partial} \right) \phi_1^0 + 2i \rho^3 P \left( \Sigma L_\mu - \overline{\partial} \Sigma \right) \phi_0^0. \\
\] (1.17) (1.18)

In the general case, as well as the geodesic curvature \(\kappa\) and the shear \(\sigma\), certain other spin coefficients for the congruence vanish:

\[
\tau = \pi = \lambda = \epsilon = 0. \\
\] (1.19)

On the other hand from (1.8) we obtain for \(\rho = -\left( \Theta + i\omega \right)\):
\[
\rho = -1\left[ r + r_0 + P^2 \left( L_y - L_x - L_{yu} + L_{uu} \right) / 2 \right].
\] (1.20)

As \( \rho \) depends on \( 1/(r + r_0) \) (entirely in the non-twisting case) we can infer from (1.17) that \( \phi_1^0 \) is associated with a charged field (electric/magnetic), while (1.18) indicates that \( \phi_2^0 \), multiplied by \( 1/(r + r_0) \) linearly, is associated with radiation.

The electromagnetic field energy tensor is
\[
T_{ab} = F_{ac} F^c_b - 1 / 4 g_{ab} F_{cd} F^{cd}
\] (1.21)

where, in the null tetrad,
\[
g_{ab} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\] (1.22)

Implicit in the field equations in the absence of field charge is the fact that the Maxwell tensor has to satisfy
\[
F^{*ab} = 0, \quad F^{*ab} = F^{ab} + i \tilde{F}^{ab}, \quad \tilde{F}^{ab} = \varepsilon_{abcd} F^{cd} / 2,
\] (1.23)

\( \varepsilon_{abcd} \) being the alternating symbol in the tetrad. In addition, the Ricci scalar will vanish:
\[
R = 0.
\] (1.24)

All tetrad components of \( R_{ab} \) in fact vanish except
\[
R_{12} = R_{34} = 2 \phi \bar{\phi}_1, \quad R_{13} = 2 \phi \bar{\phi}_2, \quad R_{34} = 2 \phi_2 \bar{\phi}_2
\] (1.25)

and \( R_{13} = \bar{R}_{13} \) (conjugate of \( R_{13} \)). The results (1.24) and (1.25) follow from the Einstein equations for an energy tensor given by (1.21).

Of the metric functions the quantity \( m(x, y, u) \) can be taken to represent mass. In particular, if we set \( m = \text{const} = m_0 \), \( L = \phi_1^0 = \phi_2^0 = r_0 = 0 \), and choose \( P = 1 + xy / 2 \) our equations lead to \( R_{ab} = 0 \), a vacuum. If we then make the transformation

\[
x = \sqrt{2} \tan \frac{\theta}{2} e^{i\xi} \quad \left( y = \sqrt{2} \tan \frac{\theta}{2} e^{-i\xi} \right)
\]

we obtain the metric
\[
ds^2 = r^2 \left( d\theta^2 + \sin^2 \theta d\xi^2 \right) - 2du dr - \left( 1 - 2m_0 / r \right) du^2,
\] (1.26)

which of course is the Schwarzschild solution. Because of the three famous tests of general relativity involving the Sun we know that the Schwarzschild \( m_0 \) does indeed represent the isolated mass producing the field.

Also, if we now again assume \( \phi_1^0 = 0 \) but \( \phi_2^0 \neq 0 \) and set \( m = m(u) \) we derive the metric for Vaidya’s ‘shining star’ radiating solution [20], where \( m(u) \) is the time dependent mass associated with the emission of pure radiation.

We may therefore infer that, at least when \( m \) is a constant or dependent on \( u \) only, that \( m \) represents the mass of the system (see also [17,9]).
We have here reproduced the metric and differential equations in their generality. To obtain particular spacetimes we shall seek to make simplifying assumptions about one or more of the independent functions in the metric. For previous solutions to, and discussions of, the twisting Einstein-Maxwell and pure radiation equations see under REFERENCES. Our motivation here is to obtain solutions in general relativity that contribute to the investigation of Einstein’s equations, particularly in the relation between gravitation and electromagnetism, in a chosen set of circumstances.

2. A NULL ELECTROMAGNETIC FIELD AND PURE RADIATION

We derive first a solution representing pure radiation that arises from a null electromagnetic field. We start with values for the following functions (taking $L, _u = 0$):

$$L = i \left( Ae^{ix} \cos(dy) + Be^{iy} \sin(dx) \right), \quad (A, B, c, d \text{ real constants}), \quad (2.1)$$

$$P = 1. \quad (2.2)$$

For a null field we have $\phi_0 = \phi_1 = 0$, and so from (1.7b), (1.25), (1.17) and (1.18):

$$T_{ab} = R_{xy} k_x k_y = 2\phi_x \phi_y k_x k_y = 2\phi_x^0 \phi_y^0 \rho \rho k_x k_y. \quad (2.3)$$

Evidently the equation (1.12) reduces to

$$0_0 = 0.$$  

which requires (with $M$ given by (1.9):

$$-\ddot{m} / \dot{u} + i \left\{ \left[ \ddot{m} / \dot{u} + c^2 d^2 / 2 \right] \left[ Ae^{ix} \cos(dy) + Be^{iy} \sin(dx) \right] \right\} = 0. \quad (2.4)$$

We may satisfy (2.4) by assuming that $m$ depends on $u$ only and setting

$$-\ddot{m} / \dot{u} = -c^2 d^2 :$$

$$m = m_0 - c^2 d^2 u, \quad m_0 = \text{const}, \quad (2.5)$$

and in addition

$$B = -Ad / c. \quad (2.6)$$

We now have

$$M = -Acd^2 \left[ e^{iy} \cos(dx) + e^{ix} \cos(dy) \right]. \quad (2.7)$$

Turning to equations (1.13), (1.14) we find that

$$\phi_0^0 \phi_2^0 = c^2 d^2, \quad (2.8)$$

so that we may take $\phi_0^0 = c d$, and hence from (1.18)

$$\phi_k = cd / \left[ -(r + r_0) + iAd \left( e^{ix} \sin(dy) + e^{iy} \sin(dx) \right) \right], \quad (2.9)$$

which $\to 0$ as $r \to \infty$. Equations (1.10)-(1.13) are now fulfilled and the associated metric follows from (1.7), (1.8). Since $P(x,y,u) = 1$, if we identify the complex coordinates $x, y$ by:
\[ x = X + iY, \quad y = X - iY, \] then the local 2-space \( X, Y \) is flat.

From \( \rho \) given in (1.8) the twist of the \( k \) lines is

\[
\omega = Ad \left[ e^{\nu y} \sin(dy) + e^{\nu y} \sin(dx) \right] / \left[ (r + r_0)^2 + A^2 d^2 \left( e^{\nu y} \sin(dy) + e^{\nu y} \sin(dx) \right)^2 \right].
\]

(2.10)

For the outstanding values of the Ricci tensor (1.25) yields

\[
R_{12} = R_{34} = R_{13} = R_{23} = 0,
R_{33} = 2c^2 d^2 \left[ (r + r_0)^2 + A^2 d^2 \left( e^{\nu y} \sin(dy) + e^{\nu y} \sin(dx) \right)^2 \right].
\]

(2.11)

In addition we have (1.24) and calculation also gives

\[
R_{ab} R^{ab} = 0.
\]

(2.12)

In order to obtain the field tensor \( F_{ab} \) for a null Maxwell field \((\phi_0 = \phi_1 = 0)\) we invoke the relation, as an expansion of a self-dual bivector,

\[
F^*_{ab} = 2 \varphi_2 \left( k_a m_b - k_b m_a \right),
\]

(2.13)

and so from (2.9), (2.23) and (1.18), we derive as the only non-zero tetrad components of \( F_{ab} \):

\[
F_{13} = -F_{31} = -\rho \phi_2^0 = -cd \left[ (r + r_0) + iAd \left( e^{\nu y} \sin(dy) + e^{\nu y} \sin(dx) \right) \right],
F_{23} = -F_{32} = \rho \phi_2^0 = -cd \left[ (r + r_0) - iAd \left( e^{\nu y} \sin(dy) + e^{\nu y} \sin(dx) \right) \right].
\]

(2.14)

We can now verify that

\[
F_{ab} F^{ab} = 0, \quad \bar{F}_{ab} F^{ab} = 0,
F_{ab} \rho = 0, \quad \bar{F}_{ab} \rho = 0,
\]

(2.15)

as required for an Einstein-Maxwell null field.

In the local frame, by reference to the values of the invariants \( \phi_0, \phi_1 \) and \( \phi_2 \) of the electric and magnetic fields, \( E \) and \( H \) lie in the \( X, Y \) plane at right angles and of equal magnitude \((E,H) = 0, \ E = H)\).

The spacetime is free of singularity and of Petrov type II.

**Pure radiation**

Pure radiation is characterised by an energy tensor of the form

\[
T_{ab} = \phi^2 k_a k_b,
\]

(2.16)

where \( k \) is a null vector. The radiation is propagated along the \( k \) lines and \( \phi^2 \) is the energy density of that radiation, which must be positive. Here we identify \( k \) with our geodesic, shearfree congruence, \( k \) being a multiple eigenvector of the Weyl tensor. Evaluation of the zero divergence, \( T_{ab} \rho = 0\), referring to (1.7b) and (1.1)

then leads to \( \phi^2 \) having the form

\[
\phi^2 = \psi^2 (x, y, u) \rho \bar{\rho}.
\]

(2.17)
This is precisely the form of the energy tensor in the spacetime of our null Einstein-Maxwell field, as given by (2.3). Hence we can identify this spacetime as host to pure radiation with $\varphi^2$ given as (cf. (2.8) or (2.11)):

$$\varphi^2 = 2c^2d^2 \left[ (r + r_0)^2 + A^2d^2 \left( e^{\sigma} \sin (dy) + e^{\sigma} \sin (dx) \right)^2 \right].$$  \hfill (2.18)

which, as required is $> 0$, and vanishes as $r \to \infty$.

We see from (2.5) that the radiation steadily reduces the mass $m$ of the field in $u$ time.

### 3. A CHARGED EINSTEIN-MAXWELL FIELD WITHOUT RADIATION

For this solution we shall again take $L$ and $P$ to be independent of $u$:

$$L = -icy/ (axy + b)^2, \quad P = axy + b,$$  \hfill (3.1)

$a$, $b$ and $c$ being positive constants. We also set

$$m = \text{const} = m_0, \quad r_0 = 0.$$  \hfill (3.3)

Referring to (1.9) we find that

$$M = 0,$$  \hfill (3.4)

$$K = 2ab,$$

so that the local 2-space $X, Y$ has positive constant curvature.

The spacetime being non-radiating we shall have $\phi_0^0 = 0$ and we shall suppose that

$$\phi_0^0 = Q(x, y, u).$$  \hfill (3.5)

However, because of (1.11) we can restrict $\phi_0^0$ to the form

$$\phi_0^0 = f(x, y)P^2 = f(x, y)(axy + b)^2.$$  \hfill (3.6)

Then (1.10) can be satisfied if we take

$$f(x, y) = g(y)/(axy + b)^2,$$  \hfill (3.7)

so that

$$\phi_0^0 = Q = g(y).$$  \hfill (3.8)

We can confirm that the four equations (1.10)-(1.13) are now fulfilled for a non-radiating Einstein-Maxwell field of charge function $Q$.

From (1.17) and (1.18) it now follows that

$$\phi_1 = g(y)(axy + b)^2/ \left[ r(axy + b) + ic(axy - b) \right]^2,$$

$$\phi_2 = \frac{(axy + b)^2 \left[ 4iabcxy(y) - (axy + b) \left[ r(axy + b) + ic(axy - b) \right] g'(y) \right]}{\left[ r(axy + b) + ic(axy - b) \right]^2}. \hfill (3.9)$$
Referring to (1.25) we may now obtain the non-zero values of the Ricci tensor. 

An illustrative example

Henceforth we shall adopt the simplest choice, namely $g(y) = \text{const}$, so that

$$\phi_i^0 = \text{const} = Q_0. \quad (3.10)$$

There follows

$$\phi_1 = Q_0 (axy + b)^2 / \left[ r(axy + b) + ic(axy - b) \right]^2,$$
$$\phi_2 = 4i Q_0 abcy (axy + b)^2 / \left[ r(axy + b) + ic(axy - b) \right]^3. \quad (3.11)$$

By (1.25) we find that

$$R_{12} = R_{44} = 2Q_0^2 (axy + b)^4 / \left[ r^2 (axy + b)^2 + c^2 (axy - b)^2 \right]^2,$$
$$R_{13} = R_{23} = -8i Q_0^2 abcy (axy + b)^4 / \left[ r^2 (axy + b)^2 + c^2 (axy - b)^2 \right]^2 \left[ r(axy + b) - ic(axy - b) \right], \quad (3.12)$$
$$R_{33} = \frac{32Q_0^2 a^2 b^2 c^2 xy(ax + b)^4}{\left[ r^2 (axy + b)^2 + c^2 (axy - b)^2 \right]^3},$$

all of which vanish as $r \to \infty$.

In addition we have the invariants

$$R_{ab} R^{ab} = 16 Q_0^4 (axy + b)^8 / \left[ r^2 (axy + b)^2 + c^2 (axy - b)^2 \right]^4, \quad (3.13)$$

$$\Theta = r(ax + b)^2 / \left[ r^2 (axy + b)^2 + c^2 (axy - b)^2 \right], \quad (3.14)$$
$$\omega = -c(a^2 x^2 y^2 - b^2) / \left[ r^2 (axy + b)^2 + c^2 (axy - b)^2 \right].$$

There are no singularities in the spacetime, which is of Petrov type II.

For this spacetime ($\phi_0 = 0$) the self-dual bivector $F^*_{ab}$ has the expansion

$$F^*_{ab} = 2\phi_1 \left( m_a \bar{m}_b - m_b \bar{m}_a - k_a n_b + k_b n_a \right) + 2\phi_2 \left( k_a m_b - k_b m_a \right). \quad (3.15)$$

Accordingly, for the non-zero tetrad components of the field tensor $F_{ab}$ we derive:
CONCLUSION

We have provided two solutions to the Einstein-Maxwell equations, each featuring a spacetime in which the Weyl tensor has a multiple null eigenvector $k$ forming a geodesic, shearfree and twisting congruence; $k$ is also aligned as an eigenvector of the Maxwell field. One of these solutions is radiating, the other non-radiating but possessing a field charge $Q$.

In each case the Ricci and Maxwell tensors, evaluated in the constructed complex null tetrad, vanish as the parameter $r$ of the $k$ congruence tends to infinity. The spacetimes, of Petrov type II, are well behaved without singularity.

In the first solution, we have seen that electromagnetic radiation steadily reduces the mass $m$, in accordance with Einstein’s fundamental relation between mass and energy. The second solution demonstrates, via (3.13), how a charged electromagnetic field is registered on the basic geometry of the spacetime, while (3.16) provides its field distribution with twisting rays.

REFERENCES


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