Solution of Nonlinear Oscillators Using Global Error Minimization Method

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Abstract
A modified variational approach called Global Error Minimization (GEM) method is developed for obtaining an approximate closed-form analytical solution for nonlinear oscillator differential equations. The proposed method converts the nonlinear differential equation to an equivalent minimization problem. A trial solution is selected with unknown parameters. Next, the GEM method is used to solve the minimization problem and to obtain the unknown parameters. This will yield the approximate analytical solution of the nonlinear ordinary differential equations (ODEs). This approach is simple, accurate and straightforward to use in identifying the solution.

Keywords: Nonlinear Oscillators; Analytical approximate solutions; Global Error Minimization method.

1- Introduction
Most phenomena in our world are essentially nonlinear and are described by nonlinear ordinary differential equations. Solving nonlinear ODEs is thus of great importance for gaining insight into real-world or engineering problems. However,
generally speaking, it is difficult to obtain accurate solutions of nonlinear problems. Consequently, solutions are approximated using numerical techniques, analytical techniques and a combination of these.

There are several methods used to find approximate solutions to nonlinear problems such as the parameter-expanding method [11], variational iteration method [2, 5, 7], homotopy perturbation method [1] and energy balance method [3, 6] were used to handle strongly nonlinear systems.

Our concern in this paper is the derivation of an approximate analytical solution for a nonlinear oscillatory differential equation. To do this we modify the variational approach proposed by He [4] and develop a method called GEM (Global Error Minimization) [10]. In the proposed method, the nonlinear differential equation is converted to an equivalent minimization problem. We combine the general idea of global error minimization in the AVK method [9] and He's variational approach for solving the nonlinear ODEs. The idea of error minimization is a natural process. Therefore, we believe that GEM provides a natural way to obtain a solution.

In the first part of the GEM, a simple sine or cosine term with unknown parameters is selected as the trial solution.

The unknown parameters are identified via the minimization of the global error. Next, more sine or cosine terms are added to increase the desired accuracy of the approximated solution. We will demonstrate that by using a few terms a solution with high accuracy is obtained.

2- Basic idea

In this section the Global Error Minimization (GEM) method is introduced and developed. The method is systematically described and will result in an approximate analytic solution for the strongly nonlinear oscillator ODEs. Consider a general second-order nonlinear oscillator differential equation:

$$u'' + F(u', u') = 0, \quad u(0) = A, \quad u'(0) = 0$$

(1)

With initial conditions:

$$u(0) = A, \quad u'(0) = B$$

(2)

**Definition:** Consider the nonlinear system (1); we define the following functional for the oscillator equation, called the global error functional [8]

$$E(u'', u', u) = \int_0^T \left( u'' + F(u', u') \right)^2 dt$$

(3)

$$T = \frac{2\pi}{\omega}, \quad \omega$$ is the primary natural frequency where $E$ is a continuous functional.

The solution of Eq. (1) can be expressed in the form of Fourier series:
Solution of nonlinear oscillators

\[ u(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos(n\omega t) + b_n \sin(n\omega t) \right) \]  

(4)

Where \( a_0, a_n, b_n \) are constants. These unknown constants could not be determined for the case of infinite Fourier series. However, we can approximate Eq. (4) by a finite series [10]:

\[ u(t) = a_0 + \sum_{n=1}^{m} \left( a_n \cos(n\omega t) + b_n \sin(n\omega t) \right) \]  

(5)

Various methods have been developed for determining the unknown constants used in Eq. (5) [8, 10]. In this paper, a natural and efficient method will be developed for determining these unknowns. The nonlinear problem (1) is first converted to the minimization problem (3). We directly substitute the trial solution (5) in the minimization problem. The solutions of the minimization problem are the unknown constants of Eq. (5).

3- Application

3-1 Example 1

First we consider the motion of a ball bearing oscillation in a glass tube that is bent into a curve such that the restoring force depends upon the cube of the displacement \( u \), the governing equation, ignoring frictional losses, is [7]:

\[ u'' + \varepsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0 \]  

(6)

We begin the procedure with the simplest trial solution:

\[ u_i(t) = A \cos(\omega t), \quad u_i(0) = A, \quad u_i'(0) = 0 \]  

(7)

Next, we convert Eq. (9) to the minimization problem (3):

\[ E(u^*, u', u) = \int_0^T \left( u_i^* + \varepsilon u_i^3 \right) dt, \quad T = \frac{2\pi}{\omega} \]  

(8)

By replacing \( u_i(t) = A \cos(\omega t) \) in Eq. (8) and performing the integration we get:

\[ E = \frac{1}{24} \left( \frac{A^2 \left( 24\omega^4\pi - 36\varepsilon^2 \omega^2 \pi + 15A^4 \varepsilon^2 \pi \right)}{\varepsilon} \right) \]  

(9)

The solution could be found through the condition \( \frac{\partial E}{\partial \omega} = 0 \):

\[ \omega = \frac{1}{6} \sqrt{9\varepsilon + 3\varepsilon \sqrt{39} A} \]  

(10)

Its period can be written in the form:

\[ T = 7.1587 A^{-1} \varepsilon^{-1} \]  

(11)
The exact period [7] is \( T = 7.4163A^{-1} \). Therefore, it can be easily proved that the maximal relative error is less than 3.61%.

If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly.

**Fig. 1.** Comparison of the GEM Method with the exact solution \((A = 1, \varepsilon = 1)\).

### 3.2- Example 2

We consider the quadratic nonlinear oscillator [7]:

\[
\dddot{u} + u + u^5 = 0
\]

With initial condition of:

\[
u(0) = A, \quad u'(0) = 0
\]

We convert Eq. (9) to the minimization problem (3):

\[
E(u, u', u) = \int_0^T \left( u'^2 + u + u^5 \right) dt, \quad T = \frac{2\pi}{\omega}
\]

By replacing \( u_i(t) = A \cos(\omega t) \) in Eq. (14) and performing the integration we get:

\[
E = \frac{1}{1920} \left( A^2 \left( -2400A^2\omega^2\pi + 945A^8\pi + 1920\pi + 2400A^4\pi + 1920\omega^2\pi - 3840\omega^2\pi \right) \right)
\]

The solution could be found through the condition \( \frac{\partial E}{\partial \omega} = 0 \):

\[
\omega = \frac{1}{12} \sqrt{48 + 30A^4 + 3\sqrt{1024 + 1280A^4 + 478A^8}}
\]
To show the remarkable accuracy of the obtained result, the approximate period with the He’s Energy Balance Method [3] and HPM [7] are compared in table 1.

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3.3- Example 3

Consider a relative harmonic oscillator [7]:

\[ x'' + (1+x'^2)^{3/2} x = 0 \]  

(17)

We make a transformation:

\[ x' = \frac{u}{\sqrt{1+u^2}} \]  

(18)

It follows:

\[ x'' = \frac{u'}{(1+u^2)^{3/2}} \]  

(19)

Substituting the expression of Eq. (18) for \( x' \) into Eq. (17):

\[ x'' = -(1-x'^2)^{3/2} x = \left(1 - \frac{u^2}{1+u^2}\right)^{3/2} x = -\frac{x}{(1+u^2)^{3/2}} \]  

(20)

Comparing Eq. (19) and Eq. (20), we have:

\[ u' = -x \]  

(21)

So Eq. (17) becomes:

\[ u'' + \frac{u}{\sqrt{1+u^2}} = 0 \]  

(22)

We rewrite:
\[ u^{n_2}(1 + u^2) - u^2 = 0 \]  
(23)

With initial condition of:

\[ u(0) = A, \quad u'(0) = 0 \]  
(24)

We convert Eq. (23) to the minimization problem (3):

\[ E(u', u, u) = \int_0^T \left( u'^2 (1 + u^2) - u^2 \right) dt, \quad T = \frac{2\pi}{\omega} \]  
(25)

By replacing \( u_i(t) = A \cos(\omega t) \) in Eq. (25) and performing the integration we get:

\[ E = \frac{1}{192} \frac{1}{\omega} A^4 \left( 144\pi + 240A^2\omega^2\pi + 105A^4\omega^4\pi + 144\omega^6\pi - 288\omega^4\pi - 240\omega^4A^2\pi \right) \]  
(26)

The solution could be found through the condition \( \frac{\partial E}{\partial \omega} = 0 \):

\[ \omega = \frac{1}{7} \frac{1}{80A^2 + 35A^4 + 48} \]

\[ \left( 1372 \frac{1}{2} \left( 30A^2 + 36 + \sqrt{1635A^4 + 3840A^2 + 2304} \right) \left( 80A^2 + 35A^4 + 48 \right)^{\frac{1}{2}} \right) \]  
(27)

### Table 2 Comparison of the GEM Method with the He’s frequency-amplitude formulation and HPM and Numerical.

<table>
<thead>
<tr>
<th>A</th>
<th>He’s frequency-amplitude formulation</th>
<th>Homotopy Perturbation Method [24]</th>
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<th>Numerical frequency</th>
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</table>

### 4- Conclusions

The GEM method was successfully applied to some strong nonlinear equations. The solution procedure of GEM method is of deceptive simplicity and the insightful
solutions obtained are of high accuracy even for the one-order approximation. The method is useful to obtain analytical solution for all oscillators and vibration problems, such as in the fields of civil structures, fluid mechanics, electromagnetics and waves, etc. Oscillator problems are very frequently encountered in all of the mentioned major fields of science and engineering. The GEM method provides an easy and direct procedure for determining approximations to the periodic solutions. The GEM method is a well-established method for analysing nonlinear systems which can be easily extended to any nonlinear equation. The accuracy and efficiency of the method was demonstrated by presenting some examples.

References


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