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Quantum Entropy and Relative Quantum Entropy in a Proposition-State Structure

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Abstract

An axiomatic approach to classical and quantum mechanics in the form of an abstract proposition-state structure (pss) is considered. The notion of normal state is introduced in the pss that mimics properties of the states on the standard Hilbertian logic. A formulation of quantum entropy and relative quantum entropy is given for normal states. The entropy functions have the basic properties that are canonically required. A reversible dynamics for an isolated physical system is defined by a suitable Heisenberg picture. A corresponding Schrödinger picture is defined. It is shown that the reversible Schrödinger dynamics, besides preserving superposition of states, does preserve both the quantum and the relative quantum entropy. The extension of analog considerations to compound physical systems are suggested.

Keywords: Proposition-State Structure; Normal States; Quantum Entropy; Relative Quantum Entropy; Reversible Dynamics; Invariance

1 Introduction.

As well known the so called logic approach to classical and quantum mechanics is an axiomatic approach originated by the pioneer paper by Birkhoff and von Neumann [4]. Accordingly to the physical system there is associateda logic L of propositions and a set S of states, namely a suitable set of maps $s: L \to [0,1]$. A proposition $a \in L$ is interpreted to represent a set of equivalent yes-no experiments on the physical system while a state $s \in S$ a class of equivalent preparation procedures of the system. The number $s(a)$, $a \in L$, $s \in S$ gives the probability of the outcome yes for a test of a when the system is in the state S. The logic L is generally assumed to be a complete orthomodular lattice. The set S of states is a set of additive (or of σ-additive) positive measures on L of total mass 1. The mutual conceptual dependence of states and propositions is established by a set of axioms that further fix the formal structure (e.g., one can require the states to be strongly order determining on L). There results a scheme generally called *proposition-state structure* (pss). From the very origin, the scheme was studied in order to reproduce and generalize at the level of a general pss properties and concepts of the conventional quantum mechanics. At present there is a wide class of results which you can report. To mention only some books see [11, 23, 19, 2] (for open problems see e.g. [7]). An argument that recently has received attention is the problem of the formulation of the notion of quantum entropy and relative quantum entropy in a general pss. Of course the problem can be solved in case the logic satisfies the conditions to be represented, by Piron's thorem [18], by the logic $L(H)$ of the closed subspaces of a separable Hilbert space H on a numerical field conventionally chosen to be the complex one [18, 23, 15]. In that case, by Gleason's theorem [5], there exists a canonical affine isomorphism $\rho \to s_\rho$ of the convex set $K(H)$ of the positive trace 1 operators in H (density operators) on S, such that $s_\rho(a) \equiv s(a) = TrP^a\rho$, $\forall\rho \in K(H)$, $\forall a \in L$, where P^a is the orthogonal projection in H of range a. Then one canonically defines the entropy and the relative entropy for the states s_ρ, s_σ respectively by $E(s_\rho) = -Tr\rho\log\rho$ and $E(s_\rho|s_\sigma) = Tr\rho(\log\rho - \log\sigma)$. The properties of those definitions have been widely studied (e.g., [24, 17, 16, 22]; for relations among superposition, entropy, dynamical maps in the Hilbert model see also e.g., [28, 29]). For what concerns more general situations, the notion of entropy of partitions has been studied on quantum logics in view of possible applications to quantum computing ([25] and references therein). Similarly, entropy and conditional entropy (of partitions) have been considered in quantum space (L, s) where L is a logic and s a Bayessian state on L [12].

In this paper a formulation of quantum entropy and relative quantum entropy is proposed within a sufficiently general pss. The pss scheme here adopted is the one previously proposed for different studies [6, 26]. Besides the notionof characteristic, pure and maximal state previously considered [3] the notion of normal state of a pss is defined and discussed. Such states are intended to mimic at the level of an abstract pss, the role the density operators (states) play in the Hilbert model. The entropy and the relative entropy function are then formulated on the normal states. The entropy functions are shown to possess the basic conventional properties of the Hilbert formulation. The behaviour of entropy under time evolution of the physical system is then considered. To that end a one parameter group of automorphisms of the logic is introduced to characterize the reversible dynamics of an isolated physical system (Heisenberg picture). It is then shown that the corresponding Schrödinger picture, besides preserving the superposition relation of the states, leaves the entropy and the relative entropy unchanged. Finally some open problems concerning the behaviour of entropy under irreversible dynamics and composition of systems.

2 Assumptions and preliminary results.

We associate to the physical system a proposition-state structure defined as follows (for definition and results on lattices theory we refer to [15].)

Definition 1. A proposition-state structure (pss) is a pair (L, S) where L is a complete orhtomodular lattice (with greatest and least element $\mathbf{1}$, $\mathbf{0}$) and S a family of maps $s : L \to [0,1]$ such that denoting $S_1(a) = \{s \in S : s(a) = 1\}$, $S_0(a) = \{s \in S : s(a) = 0\}$ for $a \in L$, it holds:

$$
\begin{array}{ll}
A1 & a, b \in L \; a \leq b \Leftrightarrow S_1(a) \subset S_1(b) \\
A2 & \mathbf{1} = \vee_{a \in L} a \Rightarrow S_1(\mathbf{1}) = S \\
A3 & S_1(\wedge_\alpha a_\alpha) = \cap_\alpha S_1(a_\alpha) \\
A4 & a_i \in L,\; a_i \perp a_k \; i \neq k,\; s \in S \;\; \Rightarrow s(\vee_i a_i) = \sum_i s(a_i)
\end{array}
$$

where a' is the orthogonal complement of $a \in L$; $a \perp b$ means $a \leq b'$. The symbols $\leq$, $\vee$, $\wedge$ denote the order relation, the join and the meet in L respectively. From the additivity of the states one has $S_1(a) = S_0(a')$. In the following we denote $\wedge A \equiv \wedge_{a \in A} a \; \forall A \subset L$. According to the considerations of the introduction the number $s(a)$, $a \in L$, $s \in S$ is interpreted as the probability of the outcome yes for a tes of the class a when the system is the state s. If $D \subset S$ we define $L(D) = \{a \in L : s(a) = 1 \, \forall s \in D\} \equiv \{a \in L : S_1(a) \supset D\}$ [6]. $L(D)$ is a dual princible ideal in L. One has $\wedge L(D) = \vee_{s \in D}(\wedge L(s))$ and the identity $a = \wedge L(S_1(a))$.

Definition 2. In a pss (L, S), a state $s \in S$ is a superposition of the states in $D \subset S$ if anyone of the following equivalent conditions is verified

$$
\begin{array}{ll}
i) & \sigma(a) = 0 \, \forall \sigma \in D \Rightarrow s(a) = 0 \\
ii) & L(s) \supset L(D) \\
iii) & \wedge L(s) \leq \vee_{\sigma \in D}(\wedge L(\sigma))
\end{array}
$$

The definition i) was given in [23]. Applied to the Hilbert model the relation $L(s_\rho) \supset L(D)$ $(\rho, \sigma \in K(H),\; s_\sigma \in D)$ means that the range $[\rho]$ of ρ as an operator in H is contained in the linear span of the ranges of the operators σ such that $s_\sigma \in D :$ $[\rho] \leq \vee_{s_\sigma \in D}[\sigma]$ [26, 28, 29].

Definition 3. A state $s \in S$ is said to be characteristic if $s' \in S$ and $L(s) = L(s') \Rightarrow s = s'$. The set of the characteristic states is denoted by S_c.

A characteristic state is a pure state, that is, if $s \in S_c$ and $s = \alpha s_1 + (1 - \alpha)s_2$, $(s_1, s_2 \in S)$ $\Rightarrow s = s_1 = s_2$ [3]. Moreover $L(s)$ is a maximal dual principle ideal. Therefore, a priori one has $S_c \subset S_p \cap S_m$, S_p, S_m denoting the set of pure and maximal states, respectively, of the pss [3, 26] . It may happen $S_c = S_p = S_m =$empty set. This is the case of a pss (L, S) such that the center $C(L)$ of L is continous [26]. In case the logic L of the pss is distributive, that is $C(L) = L$, or in case $(L, S) \equiv (M^p, N)$, M^p the projections and N the normal states of a W^*-algebra, it holds $S_c = S_p = S_m$ [26]. In the following we will assume that the pss satisfies

$$A5 \qquad S_c = S_p = S_m.$$

Therefore a state $s \in S_c$ has also the properties:

i) $L(s)$ is a maximal dual ideal

ii) $e = \wedge L(s) \in A(L)$, $A(L)$ the atoms of L

iii) $S_1(\wedge L(s)) = S_1(e) = \{s\}$

This implies the existence of a one to one correspondence between atoms of L and characteristic states. In view of the following Sections we introduce another kind of state of pss.

Definition 4. A normal state of a pss is an $s \in S$ that can be uniquely represented in the form

i) $s = \sum_i \alpha_i s_i, \alpha_i > 0, \ \sum_i \alpha_i = 1$

ii) $\wedge L(s_i) \in A(L), \ \wedge L(s_i) \perp \wedge L(s_k) \ i \neq k$

We denote by S_n the set of the normal states.

Suppose that the state s of Definition 4 even admits the representation $s = \sum_i \beta_i u_i$, $\beta_i > 0$, $\sum_i \beta_i = 1$, $u_i \in S_c$, $\wedge L(u_i) \perp \wedge L(u_k)$ $i \neq k$. Then, by A5 and possibly modulo a permutation of the $\beta_i u_i$'s, one has $\alpha_i = \beta_i$, $s_i = u_i \, \forall i$.

Notice that, according to Definition 4, a normal state does not have a decomposition in terms of "non orthogonal" characteristic states. In the Hilbert model the definition makes sense on account of the spectral decomposition of the density operators [21].

Lemma 1. If s is a normal state, $s = \sum_i \alpha_i s_i$, $(\alpha_i > 0, \sum_i \alpha_i = 1)$ then the following holds

i) $\{s_i\} = S_1(e_i)$, $e_i = \wedge L(s_i)$

ii) $s_i(e_k) = \delta_{ik}$, $\quad s(e_k) = \alpha_k$

iii) $s(\vee_i e_i) = 1$; $s(a) = 1 \, \forall a \geq \vee_i e_i$; $\quad s(b) = 0 \ \forall b \leq (\vee_i e_i)'$.

Proof. i) follows from $s_i \in S_c$. ii) One has $s_i(e_i) = 1$ by definition, and hence $s_i(e_k) \leq s_i((e_i)') = 0$ because $e_k \leq (e_i)'$ $i \neq k$. Therefore one has $s(e_k) = \sum_i \alpha_i \delta_{ik} = \alpha_k$.

iii) From $L(s) = \cap_i L(s_i)$ there follows $\wedge L(s) = \vee_i(\wedge L(s_i)) = \vee_i e_i$ and $s \in S_1(\wedge L(s)) = S_1(\vee_i e_i)$. Hence $s(a) \geq s(\vee_i e_i) = 1$ if $a \geq \vee_i e_i$ because of the orthomodularity of L and the additivity of the states. Similarly the other results can be obtained. ∎

3 Entropy and relative entropy in a pss.

Definition 5. Let $s = \sum_i \alpha_i s_i \in S_n,$ so that $s_i \in S_c$, $\sum_i \alpha_i = 1,\ \alpha_i > 0$ and denote $e_i = \wedge L(s_i)$. Then the quantum entropy $E(s)$ of the system in the normal state s is defined by

$$E(s) = -\sum_k s(e_k) \log s(e_k) = -\sum_i \alpha_i \log \alpha_i \tag{1}$$

On account of the interpretation of a pss scheme, the definition of entropy can be justified, on physical ground, by the same arguments that motivate the definition of entropy in the Hilbert model (se e.g., [24, 16]). The entropy is uniquely defined as a consequence of the definition of normal state. It easily results $E(s) \geq 0$ while $E(s) = 0$ iff $s \in S_c$. Moreover it has the concavity property.

Proposition 1. Let $s, p, q \in S_n,\ s = \sum_i \alpha_i s_i,\ p = \sum_k \beta_k u_k,\ q = \sum_h \gamma_h v_h$ with $\alpha_i, \beta_k, \gamma_h > 0,\ \sum_i \alpha_i = \sum_k \beta_k = \sum_h \gamma_h = 1$ and atoms $e_i = \wedge L(s_i)$, $f_k = \wedge L(u_k)$, $g_h = \wedge L(v_h)$: $e_i \perp e_{i'}$, $f_k \perp f_{k'}$, $g_h \perp g_{h'}$, $(i \neq i',\ k \neq k'\ h \neq h')$. Suppose $s = \alpha p + (1-\alpha) q$, $0 < \alpha < 1$. Then

$$\begin{aligned} E(s) \geq & - \alpha \sum_k \beta_k \log \beta_k - \alpha \sum_{ik} \beta_k u_k(e_i) \log \beta_k u_k(e_i) \\ & - (1-\alpha) \sum_k \gamma_h \log \gamma_h - (1-\alpha) \sum_{ih} \gamma_h v_h(e_i) \log \gamma_h v_h(e_i) \end{aligned} \tag{2}$$

Proof. One has first $L(s) = L(p) \cap L(q)$ so that

$$\wedge L(s) = (\wedge L(p)) \vee (\wedge L(q)) = \vee_i e_i = (\vee_k f_k) \vee (\vee_h g_h)$$

and $s(a) = s_i(a) = u_k(a) = v_h(a) = 1\ \forall i, k, h,\ a = \vee_j e_j$. By the concavity of the function $-x \log x$ one has further

$$\begin{aligned} E(s) &= -\sum_i s(e_i) \log s(e_i) \\ &= -\sum_i \Big[\alpha p(e_i) + (1-\alpha) q(e_i)\Big] \log \Big[\alpha p(e_i) + (1-\alpha) q((e_i)\Big] \\ &\geq -\alpha \sum_i p(e_i) \log p(e_i) - (1-\alpha) \sum_i q(e_i) \log q(e_i) \end{aligned} \tag{3}$$

Moreover the terms in eq. (3) are such that

$$\begin{aligned} -\sum_i p(e_i) \log p(e_i) &\geq -\sum_{ij} \beta_k u_k(e_i) \log (\beta_k u_k(e_i)) \\ &\geq -\sum_{ik} \Big[\beta_k u_k(e_i) \log u_k(e_i) + u_k(e_i) \beta_k \log \beta_k\Big] \\ &\geq -\sum_{ik} \beta_k u_k(e_i) \log u_k(e_i) - \sum_k u_k(\vee_i e_i) \beta_k \log \beta_k \\ &\geq -\beta_k \log \beta_k - \sum_{ik} \beta_k u_k(e_i) \log u_k(e_i) \end{aligned} \tag{4}$$

where again the concavity of the function $-x\log x$ and the property $xy\log xy = x(y\log y) + y(x\log x)$ have been used. Similarly

$$-\sum_i q(e_i)\log q(e_i) \geq -\gamma_h \log\gamma_h - \sum_{ih}\gamma_h v_h(e_i)\log v_h(e_i) \tag{5}$$

By combining eqs. (3), (4) and (5) one has the result (2). ∎

A fortiori, a consequence of the Proposition 1 is the concavity of the entropy function for normal states. From eq. (2) and Definition 5 it is immediate

$$E(\alpha s_1 + (1-\alpha)s_2) \geq \alpha E(s_1) + (1-\alpha)E(s_2)$$

The result generalizes to $E(\sum_i \lambda_i s_i) \geq \sum_i \lambda_i E(s_i)$, $\lambda_i > 0$, $\sum_i \lambda_i = 1$ and s_i, $\sum_i \lambda_i s_i \in S_n$. Note also that in eq. (2) there is a refinement of the concavity property that is not present in the Hilbert model. Indeed in the separable Hilbert case if ρ_1, ρ_2, $\rho \in K(H)$ (the density operators in H) with $\rho = \alpha\rho_1 + (1-\alpha)\rho_2$, $0 < \alpha < 1$, one has from the definition of entropy $E(\rho) \equiv -Tr(\rho\log\rho \geq -\alpha Tr(\rho_1\log\rho_1) - (1-\alpha)Tr(\rho_2\log\rho_2) = \alpha E(\rho_1) + (1-\alpha)E(\rho_2)$ directly from the properties of the trace operation.

Definition 6. Let $s = \sum_i \alpha_i s_i \in S_n$ with associated atoms $e_i = \wedge L(s_i)$, $e_i \perp e_k$, $i \neq k$ and similarly $v = \sum_i \beta_i v_i \in S_n$, $f_i = \wedge L(v_i)$, $f_i \perp f_k$, $i \neq k$. The relative quantum entropy $E(s|v)$ of s with respect to v is defined by

$$E(s|v) = \sum_i s(e_i)\log s(e_i) - \sum_i s(f_i)\log v(f_i) = \sum_i \alpha_i\log\alpha_i - \sum s(f_i)\log\beta_i$$

if $L(s) \supset L(v)$ (s is a superposition of v), and $E(s|v) = +\infty$ otherwise.

In the Hilbert model if $s, v \in K(H)$ the relative entropy is defined by $E(s|v) = Tr\, s(\log s - \log v)$. The condition $L(s) \supset L(v)$ correspond to $Ker\, s \leq Ker\, v$ [22, 28] while in the language of the normal states of a W^*-algebra means that $supp\, s \leq supp\, v$ [26]. The definition given satisfies the conventional basic properties.

Proposition 2. If s, v are normal states of a pss then i) $E(s|v) \geq 0$ and *ii)* $E(s|v) = 0$ iff $s = v$.

Proof. One has by the definition of s and $E(s|v)$

$$\begin{aligned} E(s|v) &= \sum_i \alpha_i\Big(\log\alpha_i - \sum_k s_i(f_k)\log\beta_k\Big) \\ &\geq \sum_i \alpha_i\Big[\log\alpha_i - \log\Big(\sum_k s_i(f_k)\beta_k\Big)\Big] \\ &\geq -\sum_i \alpha_i \log\frac{\sum_k s_i(f_k)\beta_k}{\alpha_i} \\ &\geq \sum_i \alpha_i\Big(1 - \frac{\sum_k s_i(f_k)\beta_k}{\alpha_i}\Big) \\ &\geq 1 - \sum_k s(f_k)\beta_k \geq 0 \end{aligned}$$

where the properties $\log x \leq x - 1$ and the concavity of $\log x$ have been used. On the other hand if $E(s|v) = 0$ then, by the last result, $1 = \sum_k s(f_k)\beta_k \Rightarrow s(f_k) = 1 \ \forall k$ that is not possible if $s \neq v$. ∎

4 Entropy invariance under reversible dynamics

To introduce a characterization of the reversible dynamics in a pss (L, S) it is usefull to recall that an automorphism μ of a complete orthocomplemted lattice is a bijection $\mu : L \to L$ such that

i) $a \leq b \Leftrightarrow \mu(a) \leq \mu(b), \ a, b \in L$
ii) $\mu(a') = (\mu(a))'$

There follows that μ^{-1} is also an automorphism and [11]:

iii) $\mu(\wedge_\alpha a_\alpha) = \wedge_\alpha \mu(a_\alpha)$
iv) $\mu(\vee_\alpha a_\alpha) = \vee_\alpha \mu(a_\alpha)$, $\forall \{a_\alpha\} \subset L$
v) $\mu(e) \in A(L) \Leftrightarrow e \in A(L)$

A possible characterization of a reversible dynamics for an isolated physical system in the context of a pss is given by the following Definition. (For motivations and discussion of the problem see e.g., [2, 6]).

Definition 7. A dynamical group of a pss (L, S) in the Heisenberg picture is a one parametyer group of $t \to \mu_t, \ \mu_{t+t'} = \mu_t \mu_{t'}$ of automorphisms of L.

The definition becomes more attractive on physical ground if the corresponding Schrödinger picture is considered. To that end consider the definition of α_t given by

$$(\alpha_t s)(a) = s(\mu_t(a)), \ s \in S, \ \forall a \in L$$

Then α_t is a permutation of S provided $\alpha_t s \in S$ and $t \to \alpha_t$ a one parameter group of permutations of S that represents the Schrödinger picture. It has the property to preserve superpositon, namely $L(s) \supset L(D)$, $D \subset S \Rightarrow L(\alpha_t s) \supset L(\alpha_t D)$ [6]. Moreover α_t is affine for every t. Indeed if $s = \sum_i \alpha_i s_i$ one has

$$(\alpha_t s)(a) = s(\mu_t(a)) = \sum_i \alpha_i s_i(\mu_t(a)) = \sum_i \alpha_i (\alpha_t s_i)(a) \, \forall a \in L$$

so that $\alpha_t(\sum_i \alpha_i s_i) = \sum_i \alpha_i \alpha_t s_i$. One has also $s \in S_1(a)$ iff $\alpha_{-t} s \in S_1(\mu_t(a))$ because $(\alpha_{-t} s)(\mu_t(a) = s(\mu_{-t}(\mu_t(a))) = s(a)$.

Proposition 3. A Schrödinger map α_t maps normal states to normal states.

Proof. Suppose $s = \sum_i \alpha_i s_i \in S_n$ with $s_i \in S_c$, $\wedge L(s_i) = e_i \in A(L)$. By the previous considerations one has $\alpha_t s = \sum_i \alpha_i \alpha_t s_i$. Since $s_i \in S_1(e_i)$ iff $\alpha_t s_i \in S_1(\mu_{-t}(e_i))$ one has $\alpha_t s_i \in S_c$ since $\mu_{-t}(e_i) \in A(L)$. Moreover $\mu_{-t}(e_i) \equiv$

$\wedge L(\alpha_t s_i) \perp \wedge L(\alpha_t s_k) \equiv \mu_{-t}(e_k)$, $i \neq k$ again because μ_{-t} is an automorphism of L. Hence $s \in S_n$. ∎

As a consequence of the last Proposition it is possible to show that both the quantum entropy and the relative quantum entropy remain unchanged under reversible time evolution. Indeed for s, v as in Definition 6 one has

$$\begin{aligned}
E(\alpha_t s) &= E(\sum_i \alpha_i \alpha_t s_i) = -\sum_i \alpha_i \log \alpha_i = E(s) \\
E(\alpha_t s|\alpha_t v) &= E(\sum_i \alpha_i \alpha_t s_i | \sum_k \beta_k \alpha_t v_k) \\
&= \sum_i \alpha_i \log \alpha_i - \sum_k (\alpha_t s)(\wedge L(\alpha_t v_k)) \log \beta_k \\
&= \sum_i \alpha_i \log \alpha_i - \sum_k s(f_k) \log \beta_k \\
&= E(s|v)
\end{aligned}$$

because $(\alpha_t s)(\wedge L(\alpha_t v_k)) = s(\mu_t(\wedge L(\alpha_t v_k)) = s(\mu_t(\mu_{-t}(f_k))) = s(f_k)$

5 Remarks and comments.

In the previous Sections a formulation of quantum entropy and relative quantum entropy in a Proposition-State Structure has been proposed. The formulation is based on the notion of normal state that has been introduced having in mind the (spectral) properties of the density operators of the Hilbert model. They have been directly requested to have the main properties that are infact needed for successive considerations. An interesting problem would be of showing, instead of that explicit request, the uniqueness of the form by which the normal states are decomposed in terms of "orthogonal" characteristic states. Even if the scheme here considered seems quite weak to enable such proof, a characterization of the pss's for which this is possible would be an interesting result.

Another problem of interest concerns the time evolution of the physical system when it can no more be considered as an isolated system. This happens when the system interacts with its surrounding or it is subjected to projective measurements [8, 13, 14, 10, 29]. In those cases the dynamical evolution can no more be considered reversible. The time evolution is better described in those cases by the so called Irreversible Dynamical Semigroup for which there are wide studies and results (e.g., [10, 17] and references therein). The consideration of irreversible dynamics in the pss scheme would require the description of the compound system in terms of the component systems. The description of such situation already exists in terms of product of logics and of pss's according to the studies of different authors (e.g., [9, 1, 20, 27] and references therein). There are however problems that, as far as the author knows, have not yet been studied. Some of them, that are currently under investigation, are, e.g., the notion of state of the subsystems as derived from the state of the compound system; the characterization of a reduced dynamics from an assumed (reversible) global dynamics of the compound isolated system; the reduced entropy and the reduced relative entropy of the subsystems from the entropy of the compound system. Finally also the behavoiur of the entropy functions under irreversible time evolution should be studied.

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