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Plane 5D Worlds and Simple Compactification

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Abstract

The existence of a fifth dimension is investigated in the ground of a extended General Relativity Theory. The Minkowsian space-time is a solution to the 5-D exterior Einstein's field ecuations. Although, here is shown that an non-euclidean 5-D geometry is physically possible if the boundary conditions of the equations are suitable. The metric functions of this 5-D spacetime depend of an arbitary physical quantity. We speculate about the nature of the fifth coordinate, which could be associated to non linear field theories, non-Abelian Gauge fields and quantum fluctuations.

Keywords: exact, solution, einstein, ecuations, 5d, plane

1 Introduction

The use of a fifth dimension in Theoretical Physics has been a common way to accomplish the unification of the fundamental interactions.

Nordström [9] and, independently Kaluza [6] were the first to try unifying gravity with electromagnetism in a theory of five dimensions. In the other half, Kaluza showed that Four-dimensional Einstein-Maxwell equations $G^{\alpha\beta} = T^{\alpha\beta}_{EM}$ are equivalent to the field equations of a 5D theory in the vacuum $G^{AB} = 0$, where the $\alpha, \beta = 0, 1, 2, 3$ and $A, B = 0, 1, 2, 3, 4$.

Klein [7] was the first to introduce the concept of compactification, which explains how the additional coordinates are length-like and could be detected. The idea of extra-dimension compactification is necessary, there are also works that suggest extra-coordinates are not physically real (*Projective theories*) [15] and the so-called *Non-compactified theories*, in which they are not necessarily length-like or compact [10].

In the Kaluza-Klein theory, the metric tensor has the form:

$$g_{AB} = \begin{pmatrix} g_{\alpha\beta} + k\phi^2 A_\alpha A_\beta & k\phi^2 A_\alpha \\ k\phi^2 A_\beta & k\phi^2 \end{pmatrix}$$

where A_α is 4-vector electromagnetic potential, ϕ is a scalar field and k is a constant in order to get the right units.

One possible argument against this idea is based on the arbitrariness of the gauge transformation of the electromagnetic potential

$$A^\alpha \to A'^\alpha = A^\alpha + \frac{\partial f(x^\beta)}{\partial x^a}$$

that Classical Maxwell's Theory allows, hence, the metric tensor could not be expressed in a unique way for a certain coordinate system. Although, the experimental evidence in Moellenstedt and his collaborators work (1953) probed that the gauge arbitrariness is not possible and that the EM vector potential has a definite form. In Modern Kaluza-Klein theory, the cylinder condition – that states that all the derivatives of the fifth dimension are null – is necessary for a right consistency with the Einstein-Maxwell equations.

This point can be shown by some theoretical principles of physical statistics. For example, Alvarado [2] showed that the potential can be measurable on an ideal relativistic charged gas and that any consequence of the gauge arbitrariness could be detected experimentally.

Since the exact solution found by Davidson and Owen in the mid-eighties [4]. The dynamics of 5D-physics has been widely study since the late eighties by Wesson, Ponce de Leon and others and the name they gave to their formulation was *Theory of Induced Matter or TIM*[1] [18]. The main idea in this framework is to unify the concepts of space-time and associate a fifth with the matter. Their applications go from black holes, solitons and cosmological models .

One cosmological application of 5-D General Relativity worth quote is that of Vladimirov and Kokarev [17], who worked on a cosmological model based on the 5-dimensional vacuum solutions of the Einstein's equations. They associate the metric component g_{44} with the matter in the commoving frame and obtained matter equations of state as function of g_{44}, in a similar fashion to the ideas of Wesson and Ponce de Leon.

In the present work, the existence of solutions to the Einstein's field equations when the metric tensor dependents only of a new extra-coordinate in

[1] In the excellent review in [5], there is a large list of the literature in 5D-physics.

planar symmetry are discussed. That fifth coordinate could be any length-like parameter that can be related to the electromagnetic field as in Kaluza-Klein Theory or another physical entity (like mass in the DTG). The fifth element must vanish when the physical entity is null or when it disappears, based on the Klein principle of compactification. Alvarado [1] has showed that compactification to appears – 3 space dimensions + time transforms into 2 space dimensions + time – when Einstein's Field equations are solved (with $T_{\alpha\beta} = 0$) for a general static cylindrical metric that only is function of one coordinate and asymthotically planar solutions are looked for. The General Theory of Relativity (extended by hypothesis to a 5-manifold) and the Alvarado's compactification theorem is used in this work to investigate the existence of a fifth degree of freedom, with signature (+, - , - , - , -).

2 Metric Tensor, Christoffel symbols and Ricci Tensor

Let the following line element be, in gaussian units and using a static planar symmetry:

$$ds^2 = e^{\gamma(\eta)}dt^2 - e^{\tau(\eta)}dx^2 - e^{\mu(\eta)}dy^2 - e^{\nu(\eta)}dz^2 - e^{\rho(\eta)}d\eta^2 \tag{1}$$

where the new coordinate " η " is introduced. Here, all the metric functions $\gamma, \tau, \mu, \nu, \rho$ are only η-dependent.

Let the symbol $()' = \frac{d}{d\eta}$ .Then, the corresponding non-zero Christoffel symbols for this problem are :

$$\begin{aligned}
\Gamma^t_{t\eta} &= \frac{1}{2}\gamma' \quad \Gamma^x_{x\eta} = \frac{1}{2}\tau' \quad \Gamma^y_{y\eta} = \frac{1}{2}\mu' \\
\Gamma^z_{z\eta} &= \frac{1}{2}\nu' \quad \Gamma^\eta_{tt} = \frac{1}{2}e^{\gamma-\rho}\gamma' \quad \Gamma^\eta_{xx} = -\frac{1}{2}e^{\tau-\rho}\tau' \\
\Gamma^\eta_{yy} &= -\frac{1}{2}e^{\mu-\rho}\mu' \quad \Gamma^\eta_{zz} = -\frac{1}{2}e^{\nu-\rho}\nu' \quad \Gamma^\eta_{\eta\eta} = \frac{1}{2}\rho'
\end{aligned} \tag{2}$$

And now, using the expression for the Ricci tensor:

$$R_{AB} = \partial_I\Gamma^I_{AB} - \partial_B\Gamma^I_{AI} + \Gamma^I_{AB}\Gamma^L_{LI} - \Gamma^I_{AL}\Gamma^L_{BI} \tag{3}$$

It is found that only the diagonal elements from the Ricci tensor are non-zero, due to simple dependence of the metric functions. Those five components are:

$$\begin{aligned} R_{tt} &= \frac{1}{4}e^{\gamma-\rho}[\gamma'' + \gamma'^2 - \gamma'\rho' + \gamma'(\tau' + \mu' + \nu')] \\ R_{xx} &= -\frac{1}{4}e^{\tau-\rho}[\tau'' + \tau'^2 - \tau'\rho' + \tau'(\gamma' + \mu' + \nu')] \\ R_{yy} &= -\frac{1}{4}e^{\mu-\rho}[\mu'' + \mu'^2 - \mu'\rho' + \mu'(\gamma' + \tau' + \nu')] \\ R_{zz} &= -\frac{1}{4}e^{\nu-\rho}[\nu'' + \nu'^2 - \nu'\rho' + \nu'(\gamma' + \tau' + \mu')] \\ R_{\eta\eta} &= -\frac{1}{4}[2(\gamma'' + \tau'' + \mu'' + \nu'') - \rho'(\gamma' + \tau' + \mu' + \nu') \\ & +\gamma'^2 + \tau'^2 + \mu'^2 + \nu'^2] \end{aligned} \tag{4}$$

3 Field Equations and their solutions

It is well-known that the exterior Einstein's field equations for the metric functions are just:

$$R_{AB} = 0$$

Therefore, the five field equations are:

$$\gamma'' + \gamma'^2 - \gamma'\rho' + \gamma'(\tau' + \mu' + \nu') = 0 \tag{5}$$
$$\tau'' + \tau'^2 - \tau'\rho' + \tau'(\gamma' + \mu' + \nu') = 0 \tag{6}$$
$$\mu'' + \mu'^2 - \mu'\rho' + \mu'(\gamma' + \tau' + \nu') = 0 \tag{7}$$
$$\nu'' + \nu'^2 - \nu'\rho' + \nu'(\gamma' + \tau' + \mu') = 0 \tag{8}$$
$$2(\gamma''+\tau''+\mu''+\nu'') - \rho'(\gamma'+\tau'+\mu'+\nu') + \gamma'^2+\tau'^2+\mu'^2+\nu'^2 = 0 \tag{9}$$

Let $\chi = \gamma + \tau + \mu + \nu$, an arbitrary function of η and naturally the its square is:

$$\chi'^2 = \gamma'^2 + \tau'^2 + \mu'^2 + \nu'^2 + 2\gamma'(\tau' + \mu' + \nu') + 2\tau'(\mu' + \nu') + 2\mu'\nu'$$

And now by summing (5), (6), (7) and (8) we get

$$2\chi'' + \chi'^2 - \rho'\chi' = 0 \tag{10}$$

By direct integration, equation (10) transforms into:

$$\rho = \xi + 2\ln(\chi') + \chi$$

where ξ is a arbitrary constant. The value of ξ can be perfectly chosen as zero, because in $g_{\eta\eta} = e^{\rho} = \chi'^2 e^{\chi} e^{\xi}$ the constant e^{ξ} is just amplification parameter that only depends of the unit system.

According to $\rho' = 2\chi''/\chi' + \chi'$, and solving ρ' in equation (5) it is easy to see that:

$$\gamma = A_0\chi + B_0 \tag{11}$$

Applying this procedure to equations (6), (7) and (8), we can obtain the integrals

$$\tau = A_1\chi + B_1 \tag{12}$$

$$\mu = A_2\chi + B_2 \tag{13}$$

$$\nu = A_3\chi + B_3 \tag{14}$$

The arbitrary parameters $B_\alpha, \alpha = 0, 1, 2, 3,$ are chosen as zero because they are amplification factors in the corresponding metric functions.

Finally, by using the relation in (10) in (9) we obtain the important relation:

$$\chi'(F - 1) = 0 \tag{15}$$

where $F = A_0^2 + A_1^2 + A_2^2 + A_3^2$ is a real non-negative constant. Therefore, from (15) we must analyze the two separate cases:

Case: $F \neq 1$ This implies that necessary χ =constant. Hence, χ is only a scaling parameter, the 4-D Minkowskian space-time is recovered as we expected if $\chi = 0$:

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) \tag{16}$$

Case: $F = 1$ According to equation (15), the F parameter must be 1, or

$$A_0^2 + A_1^2 + A_2^2 + A_3^2 = 1 \tag{17}$$

Since $\chi = \gamma + \tau + \mu + \nu$ by definition, it is clearly seen from equations (11) to (14) that

$$A_0 + A_1 + A_2 + A_3 = 1 \tag{18}$$

Mathematically, one can find a map $Q : \mathbb{R}^2 \to \mathbb{R}^4$ that describes the solution to the algebraic equations (17) and (18). For simplicity, let define $A_0 = u$ and $A_1 = v$ and then, the Q map that relates (u, v) with (A_0, A_1, A_2, A_3) is given by the parametrization

$$A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} u \\ v \\ -\frac{1}{2}\left(u + v - 1 + \sqrt{\Delta}\right) \\ \frac{1}{2}\left(u + v + 1 + \sqrt{\Delta}\right) \end{pmatrix} \tag{19}$$

where $\Delta = -3(u^2+v^2)+2(u+v-uv)+1$. The necessity of real integration constants restricts the domain of Q to the region of the uv-plane described by the ellipse: $\Delta\,(u, v) = -3(u^2 + v^2) + 2(u + v - uv) + 1 = 0$ and all its interior points (Figure 1).

One particularly simple solution is, for example: $A_0 = -\frac{1}{2}$, $A_1 = A_2 = A_3 = \frac{1}{2}$. Then, the line element under this conditions would be

$$ds^2 = e^{-\frac{1}{2}\chi}dt^2 - e^{\frac{1}{2}\chi}dx^2 - e^{\frac{1}{2}\chi}dy^2 - e^{\frac{1}{2}\chi}dz^2 - (\chi')^2 e^{\chi} d\eta^2 \tag{20}$$

It is easy to see that for a $\chi(\eta) \to 0$, for instance, when $\eta \to \infty$ and that has a first derivative that behave identically, the fifth coordinate η vanishes and then the 4D flat space-time is gotten again. The result (20) is an asymptotically flat solution.

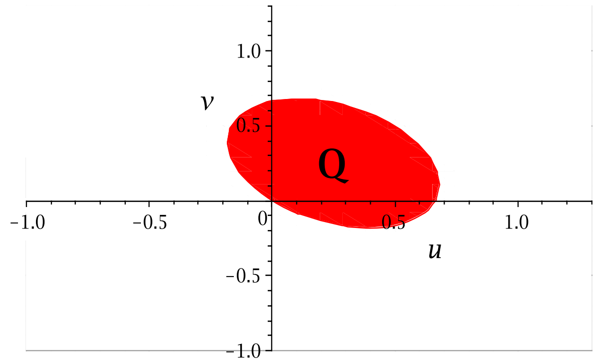

Figure 1: The domain of Q: All the (u, v) points on the ellipse and inside it define the parametrization of the integration constants.

4 Conclussions and some speculations

It is easy to see in equation (20) that the existence of the fifth length-like degree of freedom produce a natural bend of the space-time. The perception of 3-dimensions + time that is conceived by Lorentz transformation is recovered when the physical source (like EM field in Kaluza-Klein theory) is weak or null. As a physical condition, it must be posed that the χ-function in (20) is asymptotically zero at large distances from the field source.

We have shown the possibility of existence of a fifth element in the line due to physical entity that new theoretical formulation could describe. For instance, that fifth variation in the local interval could be associated with an non-Linear electromagnetic theory or a matter waves theory of arbitrary spin. This idea has it foundations in the fact that the of the Einstein's conception of the non-linear coupling of the gravitational field with it and the energy-matter – or in the words of Gary Au in his interview with Edward Witten "*The gravitational field gravitates*" [3]– could have a direct relation with any self-interacting field. In this framework, the fifth parameter also could be related to any other non-Abelian Gauge Theory [5].

Pursuing a more skeptic and conventional stand, the result of this calculation must carefully analyzed and therefore interpreted. This prediction just alters the traditional local invariant line element and must not be interpreted as proof of the manifestation of a new space independent direction. Although some geometrical analysis of the quantities in the transcendent dimension would promote them into physical observables depending of the value of proyection over the traditional space-time manifold or over the observer's world line [17], the experimental evidence points to confirm the 4-dimensional nature of the physical dynamical arena.

Although the solution (20) is not a 4-D Minkowskian, we could also argue that the $d\eta$ has very small dimensions, such as Planck length

$$d\eta \sim l_p \equiv \sqrt{\frac{\hbar G}{c^3}}$$

hence, an adequate definition (maybe based on variational principles) of the χ-function the metric functions $g_{\mu\nu}$ would be approximately 1. The nearly vanishing nature of the $g_{\eta\eta}$ element point that χ must be a slow variating function so that χ' is almost zero.

Under this conjecture, the tiny nature of the extra-dimension would ensure certain unobservable character at the experimental accessible range of energy. With this interpretation, the additional coordinate could be associated to quantum fluctuations of the local –and measurable– line element, which represent the actual and –almost– unique observable in General Relativity[2]. The idea of local quantum fluctuations is very old. For example, in Schwinger's variational formulation of Quantum Field Theory [16] the derivation of the generator of infinitesimal transformations is a direct consequence of the local variations of the Lagrangian function. Therefore, in the context of this random and unpredictable quantum fluctuations the χ-function could not be defined explicitly by deterministic manners. Then, the quantization procedure would be a little conservative and mechanic: i) elevate to quantum operator the metric tensor and therefore the χ-function, ii) construct a Hilbert space of physical states and iii) find a representation of the χ-operator [12]. By doing all this, we give a *geometric character* to quantum fluctuations that exists commonly the Nature. This concept geometric nature of the quantum fluctuation could be compatible with the Penrose's idea about the gravitational nature of the collapse of the wave function. All this new framework could be a alternative in the search of the Quantum Gravity, according to the words of Bill Unruh that "*...(QG) could not be obtained by 'tinkering with mathematics'; we need to feed in new conceptual ideas...* " [3].

References

[1] R. Alvarado. Ciencia y Tecnología **20** (1996) 145.

[2] R. Alvarado. *The relativistic temperature and Spinor Field equations exact solutions in the Gravity Theory.* PAIMS, Moscu. (In Russian)

[2] For example, what we measure in gravitational waves experiments are the displacements in masses positions [14] caused by the local perturbations of the geometry. Hence, we are measuring a quantity related closely to the line element between two events. This is an example of *partial observables*, a term coined by C Rovelli [16].

[3] G.K. Au. *The Quest for Quantum Gravity.* `Arxiv: gr-qc/9506001`

[4] A. Davidson, D.A. Owen. Phys Lett **155B** (1985) 247.

[5] B.S. DeWitt. Phys Rev Lett **12** (1964) 742.

[6] T. Kaluza. Sitz Preuss Akad Wiss Phys Math **K1** (1921) 966.

[7] O. Klein. Zeits Phys **37** (1926) 895.

[8] D.F. Lawden. *An Introduction to Tensor Calculus and Relativity.* Methuen. London. 1967.

[9] G. Nordström. Phys Zeits **15** (1914) 504.

[10] J.M. Overduin, PS Wesson. Phys Rep **283** (1997) 303.

[11] C. Rovelli. Living Rev Relativity. **1** (1998) 1. Online article founded on April 2005 at: `www.livingreviews.org/Articles/Volume1/1998-1rovelli.`

[12] C. Rovelli. *Quantum Gravity*. Cambridge. 2004.

[13] S. Rowan, J Hough. Living Rev Relativity. **3** (2000) 3. Online article founded in April 2005 at: `www.livingreviews.org/Articles/Volume3/2000-3hough.`

[15] E. Schmutzer. *A New 5-Dimensional Projective Unified Field Theory for Gravitation, Electromagnetism and Scalarism*. In: *Unified Field Theories of more than 4 dimensions*, ed. V De Sabbata, E Schmutzer. World Scientific. Singapure. 1982.

[16] J. Schwinger. Phys Rev **82** (1951) 914.

[17] Y.S. Vladimirov, S.S. Kokarev. *4-D homogeneous isotropic cosmological models generated by the 5-D vacuum.* `ArXiv: gr-qc/0210067`

[18] P.S. Wesson et al. Int J Mod Phys. **A11** (1996) 3247.

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