Adv. Studies Theor. Phys., Vol. 5, 2011, no. 14, 665 - 701

On the Mathematical Structure for Discrete and Continuous Metric Point Sets

John F. Moxnes

Department for Protection
Norwegian Defence Research Establishment
P.O. Box 25, N-2027 Kjeller, Norway
john-f.moxnes@ffi.no

Kjell Hausken

Faculty of Social Sciences
University of Stavanger
N-4036 Stavanger, Norway
kjell.hausken@uis.no

Abstract

We have studied fundamental properties of continuous and discrete metric point sets. Our focus is pure geometrical objects. We show how geodesic lines and angles can be constructed from an imposed metric even in discrete spaces. Lines in discrete and continuous metric point sets are constructed and compared with Euclid's five axioms. The angle between two lines is defined. Euclid's axioms E1 and E2 are sufficient to achieve local angles and to define an infinite space. Axiom E3 is sufficient to define a space with more than one dimension. Axiom E4 is sufficient to define a homogenous space. Axiom E5 is sufficient to define a flat space. We study how the concepts of vector spaces could appear from the metric point set. We have constructed arrows from each point in the metric point set. These arrows can be conceived as lines with a direction. The sum of arrows from each point is constructed algebraically without parallel transport. A method is presented for constructing coordinates. We have constructed coordinates in a metric point space by assuming that the arrow from a specific point o in the metric point space defines a vector space at each point p. We comment on the force concept. Different parallel transports are constructed geometrically. The concepts of tensors and tensor fields are briefly addressed.

Keyworlds: Point set, Discrete Point Space, Continuous Point spaces, Euclidean Point Space, Metric space, Vector space, Geodesic lines, Parallel transport, Auto parallels

1 Introduction

During years there has been increasingly interest in getting away from the classical concept of differential manifold as the arena in which physics takes place (Ambarzumian and Iwanenko 1930, Rurark 1931, Snyder 1947, Finkelstien 1969, Feynman 1982, Minski 1982,Yamamota 1984, Bombelli et al. 1987, t'Hooft 1990). This is in particular motivated by considerations about space-time structures at very small length scales and quantum gravity. A suspicion is that a fully satisfactory cure for the ultraviolet divergences of relativistic quantum field theory will eventually require some form of discrete space-time (Weingarten 1977, Jourjine 1987). Replacing a continuum field theory by a lattice theory is thus considered to be the most important non perturbative regularization of the quantum field theory. One may even speculate about the possibility to relate the deformation parameters of differential calculus to the Planck length (Dimakis et al. 1993). Pseudo- Riemannian geometry on finite and discrete sets has been developed (Regge 1961, Williams and Tuckey 1992).

Noncommutative geometry is the geometry of quantum space, which are generalized discrete spaces replacing classical quantum spaces (Woronowicz 1987, Connes 1994). The basic structure underlying noncommuative geometry is a differential calculus on an associative but not commutative algebra (Connes 1994 and references therein). Within the framework of noncommutative geometry, an analog of (pseudo-) Riemannian geometry on finite and discrete sets has been developed (Dimakis et al. (1992, 1993,1994,1995,1999), David 1992, Ambjørn 1994). In one formulation functions and differentials satisfy noncommutative relations depending on the lattice spacing. For vanishing lattice spacing they commute and one recovers the ordinary differential calculus. In another formulation Dimakis and Müller-Hoissen established a transformation of the noncommutative calculus of stochastic differential (Ito) into a noncommutative differential calculus[1]. The new differential operator d* does not have the properties of an exterior derivative (like the d in the calculus of differential forms on a manifold). Notice that essential in stochastic theories is how randomness is accounted for (Moxnes and Hausken 2010). In a noise formulation, noise could be Gaussian or more general, and it could be uncorrelated (white noise) or correlated (colored noise). A simple realization could be to let the state value at each time to be a random noise or alternatively a deterministic function of time added a random noise as typical in Markov (1906) process in discrete time and space. For Markov (1906) processes, the state value of $X_{t+\Delta t}$ at time t+Δt is given by the state value at time t, plus a state value of a "random variable" at time t. When time step approaches zero Ito calculus can be used. However, a system with uncorrelated noise is usually just conceived a coarse-grained version of a more fundamental microscopic model with correlation. Thus in the Stratonovich (1966) method noise is incorporated in a deterministic equation by adding a deterministic noise term that can be integrated in the normal Riemann sense[2]. The Stratonovich integral simply occurs as the limit of time correlated noise when the correlation time of the noise term approaches zero. Depending on the problem in consideration, the Stratonovich (deterministic noise) or the Ito model (stochastic noise) could be appropriate approximations. Indeed, for additive noise the Ito model gives the same answer as the Stratonovich model if the Stratonovich model uses a Gaussian distribution for the noise. But for multiplicative noise the results are different. (For a recent treatment of different interpretations of stochastic differential equations see for example Lau and Lubensky (2007).) Roughly, the reason for this difference is that the Stratonovich integral, which is a Riemann integral, applies on functions with bounded variations. The Ito integral applies on functions without bounded variations (i.e. white noise). However, if discrete time and space is true, time discrete Markov models on lattices would be an interesting approach, where the continues limit (Ito calculus) is only an approximation.

One can take the point in view that some form of differential calculus is the very basic structure necessary to formulate physical theories, and in particular dynamics of fields on space. It allows that introduction of further geometric notions like linear connections and the formulation of field theories and dynamics on finite sets (Bresser et al. 1996, Cho and Park 1997). Thus, the hypothesis of a physical theory (i.e. lattice theory) must be expressed in terms of the most primitive concepts, e.g. those of space and time. The notation of the distance between two points of a space is at the origin of geometry and physics. However, the classical Euclidean distance between two points p and q is defined by the infinum of length paths from p to q on a Riemannian manifold. The infinum of

[1] See Baehr et al for a systematic investigation of differential of associative algebraes.

[2] Allowing randomness in the initial values e.g. for prices and for a deterministic noise commonly implies more realistic models of physical situations than e.g. the Liouville process where a realization of the stochastic process is constructed deterministically allowing randomness only in the initial conditions of the prices.

length paths is calculated as extreme of the square root of $g_{\mu\nu}dx^{\mu}dx^{\nu}$ along paths from p to q. However this concept has to be reformulated (Connes 1994).

In the discrete theory the metric function is based on a correspondence between first order differential calculi and digraphs (the vertices of the latter are given by the elements of the finite set). Arrows originating from a vertex span its tangent space. Dimakis and Müller-Hoissen (1999) defined a metric as an element of the left-linear tensor product of the space of 1-forms with itself. In general, it is found that on a finite set, there is a counterpart of the continuum metric tensor with a geometric interpretation. In particular, in the case of the differential calculus on a finite set, the Euclidean geometry of polyhedra is recovered from conditions of metric compatibility and vanishing torsion. In the geometry on finite and discrete sets the Ricci tensor or a curvature scalar does not exists. But one can make use of the parallel transport associated with a connection.

Our main objective in this article is to make available a short but sufficiently precise article for an in depth exposure to discrete and continuous spaces, to the vectors, forces and parallel transport when focusing on the geometrical aspects. Traditionally these concepts are in mathematical physics either emphasized by problem solving techniques or by much more formal mathematical structure theories (Freedman et al. 1990, Hirsch 1997, Sklar 1977, Jürgen 2008). Special attention for reference is on the discrete and on the (continuum) Euclidean point spaces (typically tangent spaces). (See Vaillanta et al. for statistics on diffeomorphisms via tangent space representations.)

For historical interest we compare with the familiar Euclidean postulates for Euclidean point space. We construct vectors from a purely geometrical point of view. Although the Euclidean point space concerns old and well developed mathematical objects, an appropriate discrete geometry requires a much more precise formulation of geometrical concepts than what is currently available in the literature. This paper develops such a formulation. A first step was provided by Dimakis et al. (1993) who formulated a common mathematical framework for coordinates that includes both continuum and lattice theory. As we will see in this article, this is not so much a surprise since most of the fundamental objects in a geometrical continuum theory can be easily translated or applied to the lattice theory.

2 The metric point set (space) and the Euclidean postulates

We start with the most basic concept, a point set (Lee 2003). This point set is simply defined as a non-empty set X of elements which we call points. Points in the set are called p, i.e. $X \overset{def}{=} \{p\} \overset{mod}{\neq} \emptyset$, where "def" means definition and "mod" means model assumption. The association between the mathematical concept (points) and the real world (places) is assumed to be

$$p \rightarrow \textit{place in the real world} \tag{2.1}$$

Using this connection a point in the mathematical theory is associated to a place in the real world; our place of living. The set of all places in the world is the associated point set X. Note that we have not defined any coordinate system. However, we assume that places in the world can be identified. Places in the real world are identified by the positions of physical point objects which we name. For instance the sun, the moon, etc. could be the name of places. Thereby a frame of reference is given. Relation (2.1) applies both for continuous and discrete point sets.

Next the point set is "equipped" with a so-called metric function that takes two arbitrary points and gives a real number as output (Royden 1968). This function we name D for distance, and we write mathematically: $D: XxX \to R$. The measuring device could be a length stick, a strong line, a laser and a clock etc. To read: D is a function that applies two points in a point set and the output is a real number. This real number is called the distance "between" the two points. Note that D is a defined to be a function. Thus between two points we have only one distance. The association to the real world is given by

$$D \to the\ measured\ distance\ between\ two\ arbitray\ points(places) \tag{2.2}$$

A metric function (a metric for short) has some familiar generic properties that can be precisely stated (Royden 1968):

$$D(p,q) \geq 0, (a), D(p,q) = 0 \Leftrightarrow p = q, (b), D(p,q) = D(q,p), (c), D(p,q) \leq D(p,r) + D(r,q), (d)$$
$$\forall p,q,r \in X \tag{2.3}$$

a) states that the distance between two arbitrary points p and q is larger than or equal to zero. b) states that the distance between two arbitrary point p and q is zero if and only if the two points are the same. c) states that the distance between two arbitrary points p and q is equal to the distance between q and p. d) states that the distance between two arbitrary points p and q is smaller than or equal to the distance between p and r plus the distance between r and q. The point set X with its distance function D is now called a metric point set (or space) (we call it M for short). The metric point set could be discrete or continuous.

From now on we apply constructions. The objective is to apply constructions without any new physical "devices". Our only device is the distance function D that in the physical world is a physical object that can be used to measure distance.

We define a geodesic line (line for short). Two points p and q are assumed to be sufficient to describe a line. The line function l takes two points p and q in M and the output is a subset of M (the line “between” p and q). $l: MxM \rightarrow P(M)$.[3] The function is defined as

$$l(p,q) \overset{def}{=} \{r \in M \,|\, D(p,r) + D(r,q) = D(p,q)\}, \forall p,q,r \in M, \tag{2.4}$$

Thus the line between p and q is a set of points ($r \in M$). These points have the property that the distance from p to r, plus the distance from r to q is equal to the distance from p to q. It is easily observed that l(p,q)=l(q,p). For an arbitrary point r on the line l(p,q) it follows that $D(p,r) + D(r,q) = D(p,q) \Rightarrow D(q,p) = D(q,r) + D(r,p)$. Thus r is also on the line l(q,p) and vice versa. It also follows that the line between two points p and q is the shortest possible line. Assume that a point r is “between” p and q. Hence $D(p,r) + D(r,q) \geq D(p,q)$ according to (2.3d). A line has no direction. The line definition applies for continuous and discrete point sets. Thus lines are feasible even for a discrete metric point sets.

So far all the concepts apply for discrete and continuous sets. However, in what follows it is of interest to compare directly discrete sets with the Euclidean axioms of (continuous) space (geometry). Euclid’s first axiom says that (Lindsay and Margenau 1957): *E1: It is possible to draw a geodesic line joining any two points.* The important word here is “joining”. Our interpretation of this prosaic statement is that the metric point set is dense, i.e., “there are no “holes” between arbitrary points p and q. That means a continuum. To define this we will first introduce a concept called a shortening of a line. To shorten a line means to remove points from the points of a line. However, since l(p,q)= l(q,p) this has to be defined properly. During shortening we let $l(p,r) = \lambda \overset{p}{\square} l(p,q)$ be the line from p to r

---

[3] P(M) means power of M and is the set that consists of all subsets of M

achieved by starting with the larger line l(p,q) and cutting it off at point r. This gives one line l(p,r) that is preserved, and one line l(r,q) that is removed. l(p,r) is obtained by multiplying l(p,q) with the real number (factor) λ. We will apply the nomenclature $\lambda \overset{p}{\square} l(p,q)$, where λ is less than or equal to 1 but larger than or equal to zero. We define

$$0 \le \lambda \le 1 \in R, \forall p,q \in M,$$
$$\exists r \in M \square D(p,r) + D(r,q) \overset{mod}{=} D(p,q) \wedge D(p,r) \overset{mod}{=} \lambda D(p,q), l(p,r) = \lambda \overset{p}{\square} l(p,q), (a) \quad (2.5)$$

The line is thus shortened from l(p,q) to l(r,q) by cutting at point r, achieved by multiplying with λ. Thus we lose the point q in the new line l(p,r), and we lose all points in the removed line l(r,q) except point r. Notice that $0 \overset{p}{\square} l(p,q) = p$. Notice that even for a discrete point sets is line shortening possible, although not every λ can be used for shortening. However, the argument of E1 is that all such shortening lines, assuming $0 \le \lambda \le 1$, exist as lines, to read

$$\forall\ 0 \le \lambda \le 1 \in R, \forall p,q \in M,$$
$$\exists r \in M \square D(p,r) \overset{mod}{=} \lambda D(p,q), \exists \lambda \overset{p}{\square} l(p,q) = l(p,r), \qquad (E1: Continuous) \quad (2.6)$$

which is a mathematical statement of the Euclidean postulate E1, applicable only for continuous metric points sets. E1 does not apply for discrete metric points sets since not every λ is possible in a discrete metric point set. Thus a line in a discrete metric point set can only be cut for numerable specific λ values, that means at discrete points r.

Euclid's second axiom says that*: E2: A terminated geodesic line may be extended without limit in either direction.* Our interpretation is that the metric point set is infinite. We first define a new concept, the extension of a line, quite analogous to the shortening concept. We define the line given by the extension of l(p,q) from q to the point r with the factor $\lambda \geq 1$ as $l(p,r) = \lambda \overset{p}{\square} l(p,q)$. Thus we use the same nomenclature as when shortening. The mathematical definition is

$$\lambda \geq 1 \in R, \forall p,q \in M,$$
$$\exists r \in M \square D(p,q) + D(q,r) \overset{mod}{=} D(p,r) \wedge D(p,r) \overset{mod}{=} \lambda D(p,q), l(p,r) = \lambda \overset{p}{\square} l(p,q) \ , \qquad (2.7)$$

Euclid's second axiom states that all such extensions in (2.7) should exist. However, a discrete metric point set could be infinite also. To model a discrete metric point set we define that for every $\lambda \geq 1$ there is a point r on the extension of l(p,q) to l(p,r) that has a distance D(p,r) from p to r which is λ times larger than the distance D(p,q) from p to q, to read both for continuous and discrete spaces

$$\forall \lambda \geq 1 \in R, \forall p,q \in M,$$
$$\exists r \in M \square D(p,r) = \lambda' D(p,q), \exists \lambda' \overset{p}{\square} l(p,q) \ = l(p,r), \quad \lambda' \in R, \lambda' \geq \lambda \qquad (E2) \qquad (2.8)$$

For the continuous metric point set we state that $\lambda = \lambda'$ as E2. The equation $\lambda = \lambda'$ does not hold for the discrete space since not all extensions are possible. The distance function is one-valued by definition. That is, given two points of the metric space there exists only one distance between the points. Thus using the familiar spherical surface as our metric point set (without the south pole) (2.8) will not hold, since for every surface of a sphere we can find a numerical λ value, without being able to find two points in the metric point space (i.e. the sphere surface) which have a larger distance than λ. To illustrate, all great circle arcs between antipodal points on a sphere with radius r have the same length, i.e. half the circumference of the circle, or πr, which is the maximum possible distance between two points on the sphere. Since λ is easily chosen to yield a larger distance than πr, (2.8) does not hold. (We are due to the uniqueness of the distance function not allowed to "move around the sphere" when measuring the distance.)

We can also define multiplication of the line l(p,q) with a negative scalar or zero, to read

$$\forall \lambda \leq 0 \in R, \forall p, q \in M, \\ \lambda \overset{p}{\cdot} l(p,q) \overset{def}{=} \left(|\lambda|+1\right) \overset{q}{\cdot} l(q,p) - l(q,p) \cup p, \tag{2.9}$$

By "-l(q,p)" we mean that the points in l(q,p) is not in the set. Equation (2.9) applies for all $\lambda \in R$ if we use E1 and E2.

Euclid's third axiom says that: *E3: It is possible to draw a circle with a given center and through a given point.* This axiom is tricky since a circle is not defined. However, a sphere could be defined as a set of points with the same distance to a point p. Our minimal interpretation of this postulate is that the metric point space has more than one dimension[4]. We write that

$$\forall p, q \in M, and\ \forall \lambda \in R, \exists\, r \in M \cdot D(p,r) \overset{mod}{=} \lambda \wedge r \overset{mod}{\notin} l(p,q) \quad (E3) \tag{2.10}$$

Euclid's fourth axiom says that*: E4: All right angles are equal. Our interpretation is that the metric point set is homogeneous and isotropic.* We first define the angle between lines and finally right angles. The angle between lines is indeed not straightforward to define for discrete metric point sets. We start with continuous metric point set. Let l(p,q) and l(p,r) be two lines with a common point p. First we

[4] There are many definitions that can be used to define the dimension of the set ξ, e.g. covering

define the ratio F between two lines. Let l(p,r') be the line $\lambda \overset{p}{\cdot} l(p,r), 0 < \lambda \le 1$. For each λ, let z' be the point on the line l(p,q) or on the shortening of l(p,q) or on the extension l(q,p) with the shortest distance D(z',r'). If z' is on the shortening or on the extension of l(p,q) the ratio F is defined by

$$\forall l(p,q), l(p,r) \in P(M), D(r',z') = minimum, F_{l(p,q),\lambda \overset{p}{\cdot} l(p,r)} \overset{def}{=} D(p,z')/D(p,r') \tag{2.11}$$

The angle is then defined by

$$\forall l(p,q), l(p,r) \in P(M),$$
$$\theta_{l(p,q),l(p,r)} = ArcCos\left(Lim_{\lambda \to 0} F_{l(p,q),\lambda \overset{p}{\cdot} l(p,r)} \right) \tag{2.12a}$$

Thus the angle is the inverse Cosine of the limit of the ratio when the shortening factor λ approaches zero. However, if z' is on the extension l(q,p), the ratio and the angle becomes

$$\forall l(p,q), l(p,r) \in P(M),$$
$$F_{\theta_{l(p,q),l(p,r)}} \overset{def}{=} -D(p,z')/D(p,r'), \tag{2.12b}$$
$$\theta_{l(p,q),l(p,r)} = ArcCos\left(Lim_{\lambda \to 0} F_{l(p,q),\lambda \overset{p}{\cdot} l(p,r)} \right)$$

For some cases of metric point sets, referred to as flat metric point sets, the ration F becomes the same for all finite λ values. However, the same ratio as a definition of flat space will not be used in this article.

---

dimension and inductive dimension (Kuratowski 1961, Engelking 1978).

For a discrete metric points set the definitions in (2.12a) and (2.12b) do not apply since the limiting procedure $\lambda \to 0$ is not possible. However, two lines that cross in discrete metric point set have a common point (crossing point). We choose the nearest discretely located neighboring point to the crossing point along each line and define the angle as a purely algebraic relation following from the distance between the three points in the triangle. Assume that the lines A, B and C constitute the triangle with the length A, B and C. A and B cross, and the angle between A and B is α. The angle between A and B is then simply defined by

$$Cos(\alpha) \overset{def}{=} \left(A^2 + B^2 - C^2\right)/\left(2AB\right). \tag{2.12c}$$

Next we define right angles. Choose a point r on the line l(p,q). We define H_{pq} as the set of points which all have the same distance to the two endpoints p and q of the line l(p,q), i.e.

$$H_{pq} \overset{def}{=} \left\{r' \in M \,\square\, D(r',p) = D(r,p), D(r',q) = D(r,q)\right\} \tag{2.13}$$

Define l(r,r') as the line from r on the line l(p,q), to the point r' in H_{pq} not on the line l(p,q), i.e. $r' \in H_{pq}, r' \neq r, p, q$. The definition is that the angle between l(r,q) and l(r,r') is the same and is a right angle (and equals $\pi/2$). Hence according to Euclid's fourth axiom we get

$$\begin{aligned} &r \in l(p,q), r \in H_{pq}, \\ &\forall r' \in H_{pq}, r' \neq r, p, q, \theta_{l(r,q),l(r,r')} \overset{mod}{=} const. = \pi/2, \qquad (E4) \end{aligned} \tag{2.14}$$

For discrete metric point sets we can also apply (2.14) when using the definition of angle in (2.12c) that is applicable for discrete metric point sets.

Finally, Euclid’s fifth axiom says that: *E5: If two geodesic lines in a plane meet another geodesic line in the plane so that the sum of the interior angles on the same side of the latter geodesic line is less than two right angles, the two geodesic lines will meet on that side of the latter geodesic line.* There are quite many different versions of this axiom. Euclid intends to state that the space is flat. We stated after (2.12b) that a metric point set may be defined as flat if all fractions F between two lines in (2.11) and (2.12ab) are equal when shortening. This definition applies for discrete metric point sets also. However, another suggestion for continuous metric point sets is as follows. Assume the line l(p,q). We define the set B_{pq} around the point r on l(p,q), where r is defined in (2.15a), i.e.

$$\begin{aligned} &r \in l(p,q), r \in H_{pq}, (a) \\ &B_{pq} = \{r' \in M \,\square\, D(r',r) = D(r,p)\}, (b) \end{aligned} \tag{2.15}$$

Then let $\theta_{l(r',p),l(r',q)}$ be the angle between the line l(r’,p) and l(r’,q). We thus postulate that all such angles are $\pi/2$, to read

$$\forall p,q \in M, \forall r' \in B_{pq}, \theta_{l(r',p),l(r',q)} \overset{mod}{=} \pi/2, \qquad (E5) \tag{2.16}$$

Equation (2.16) the becomes the mathematical statement of Euclid’s 5 th axiom. Finally we will define a special subset of the metric point set. Assume that two lines l(p,q) and l(p,r) are given. Define next the line l(q,r) and its extension in both directions. Construct then all the lines through p and through a point l(q,r) and its of the extensiona. The union of all such lines is a subset of M which we call $M_{l(p,q),l(p,r)}$, where $M_{l(p,q),l(p,r)} \subset M$.

Summarizing: We have constructed lines in discrete and continuous metric point sets and compared with Euclid’s five axioms. Of importance is the ability to define the angle between two lines. We only need Euclid's axioms E1 and E2 to achieve local angles and to define an infinite space. We need axiom E3 to define a space with more than one dimension. We need axiom E4 to define a homogenous space. We need axiom E5 to define a flat space.

3 The arrows

In section 2 we defined angles between lines and we defined a shortening and extension process of lines. In this section we apply that lines can be extended and shortened.

We define arrows from a point p. The arrow given by p and q is defined as $\vec{a}: MxM \rightarrow PP(M)$

$$\vec{a}(p,q) \overset{def}{=} \{\{p\}, l(p,q)\} \tag{3.1}$$

An arrow from p to q consists in some sense of a line with a direction. We set that $\vec{a}(p,p) = \{\{p\}, l(p,p)\} = \{\{p\}, p\} \overset{def}{=} \vec{\omega}_p$. Observe that an arrow $\vec{a}(p,q) \notin P(M)$. The arrow set consists of all arrows $A \overset{def}{=} \{\vec{a}\}$. Notice that the definition applies both for discrete and continuous metric point sets.

The arrow set $A = \{\vec{a}\}$ is equipped with a function that takes any arrow and multiplies it with a scalar. Call this function $mu(\cdot\ for\ short): RxA \rightarrow A$. The definition is

$$\lambda \in R \square \lambda \cdot \vec{a}(p,q) \overset{mod}{=} \{\{p\}, \lambda \overset{p}{\cdot} l(p,q)\} \tag{3.2}$$

Thus we closely follow the procedure outlined during shortening and extension of a line. All $\lambda \in R$ are feasible for continuous metric point sets, while for discrete metric points sets all $\lambda \in R$ are not feasible. More specifically, for discrete metric point sets we could have $\lambda \in Q_0$, the set of all rational numbers including zero.

The arrow set is also equipped with a product function that takes two arrows starting at the same point and multiplies them, where $pr(\text{"}\cdot\text{"}\ \mathit{for\ short}) : AxA \to R$. The construction is as follows both for discrete and continuous point sets ("inner product").

$$
\begin{aligned}
&\vec{a}(p,q)\cdot\vec{a}(p,r) \overset{def}{=} D(p,q)D(p,r)Cos\left(\theta_{\vec{a}(p,q)\vec{a}(p,r)}\right), \theta_{\vec{a}(p,q)\vec{a}(p,r)} \overset{def}{=} \theta_{l(p,q)l(p,r)}, (a)\\
&\vec{\omega}_p \cdot \vec{a}(p,q) \overset{def}{=} \vec{\omega}_p, \vec{\omega}_p \overset{def}{=} \vec{a}(p,p), (b)\\
&L\left(\vec{a}(p,q)\right) \overset{def}{=} \left(\vec{a}(p,q)\cdot\vec{a}(p,q)\right)^{1/2} = D\left(p,q\right), (c)\\
&L\left(\vec{a}(p,p)\right) = L\left(\vec{\omega}_p\right) \overset{def}{=} D\left(p,p\right) = 0, (d)
\end{aligned}
\tag{3.3}
$$

Notice that the angle between arrows is defined to be equal to the angle between lines defined in (2.10). Also observe that $\vec{a}(p,q)\cdot\vec{a}(p,q) = D\left(p,q\right)^2$. The length of an arrow is defined by $L\left(\vec{a}(p,q)\right) \overset{def}{=} \left(\vec{a}(p,q)\cdot\vec{a}(p,q)\right)^{1/2} = D\left(p,q\right)$.

Finally, we define the sum of two arrows from the same point p. To define the sum of two arrows is challenging. Notice that the sum of two lines was not defined. We assume that the sum of two arrows from a point is again an arrow from the same point. Let us define $\theta_{\vec{a}(p,q),\vec{a}(p,r)}$ as the angle between the arrows $\vec{a}(p,q)$ and $\vec{a}(p,r)$. The sum (+ for short) : AxA-> A of $\vec{a}(p,q)$ and $\vec{a}(p,r)$ is then defined purely algebraically as

$$\begin{aligned}
&\vec{a}(p,q)+\vec{a}(p,r)=\vec{a}(p,r'),r'\in M_{l(p,q)l(p,r)},(a)\\
&D(p,r')\overset{def}{=}D(p,q)^2+D(p,r)^2+2D(p,q)D(p,r)Cos\left(\theta_{\vec{a}(p,q),\vec{a}(p,r)}\right),(b)\\
&Cos\left(\theta_{\vec{a}(p,r'),\vec{a}(p,r)}\right)\overset{def}{=}\frac{D(p,q)^2+D(p,r')^2-D(p,r)^2}{2D(p,r')D(p,r)},(c)\\
&\vec{a}(p,q)+\vec{\omega}_p\overset{def}{=}\vec{\omega}_p+\vec{a}(p,q)\overset{def}{=}\vec{a}(p,q),(d)
\end{aligned} \tag{3.4}$$

Equation (3.4a) says that the endpoint of a summed arrow is in the subset spanned by the two lines l(p,q) and (p,r). Equation (3.4b) gives the length of the summed arrow. Equation (3.4c) gives the direction of the summed arrow. Observe that the construction of the sum does not need any parallel transport. However, the length and direction of the summed arrow is as if the summed arrows should have been constructed geometrically in the familiar way learned at school. However, this is not performed, or even possible to perform in general since we are not yet allowed to "move" arrows as in the parallel transport. Indeed, the parallel transport is not defined yet. In general arrows are not vectors, but they can be, as shown and defined in Appendix B.

Summarizing: We have constructed arrows from each point in the metric point set. These arrows can be conceived as lines with a direction. The sum of arrows from each point is constructed algebraically without parallel transport.

4 The coordinates

In this section our focus is the coordinates of the metric point set M. We confine attention to those metric point sets where arrows from a point p have a property that is not general. Indeed, we assume that the metric point set has some very special property, i.e. an arrow is a vector as defined in appendix B. Hence

$V_p = \{\vec{a}(p,q)\}, \forall q \in M$ at the point p[5].

Let us choose an origin of M, i.e. a specific point p of M called o. We assume that all the arrows from an arbitrary point p define a vector space at o. For the vector space V_o (called tangential space at V_o) we assume that an orthonormal basis $\{e_i(o)\}$ exists, i.e. every arrow in V_o can be written

$$\begin{aligned} &\vec{a}(o,q) = \sum_{i=1}^{n} \alpha_i(o,q)\,\vec{e}_i(o) \\ &\forall \vec{a}(o,q) \in V_o, \alpha_i(o,q) \in R, \vec{e}_i(o) \in V_o, \Rightarrow \vec{e}_i(o)\cdot\vec{e}_j(o) = \delta_{ij}, i = 1,..n, j = 1,..n \end{aligned} \tag{4.1}$$

Observe that for a specifically chosen origin o, the functions $\alpha_i(o,q)$ are functions of the type $M \to R$. Let $\vec{a}_1$ and $\vec{a}_2$ denote two arbitrary arrows at the point o with components α_{1i} and α_{2i} relative to the given basis set. The length of the arrow (vector) is $L(\vec{a}_1) \overset{def}{=} (\vec{a}_1 \cdot \vec{a}_1)^{1/2}$, to read

$$\begin{aligned} &L(\vec{a}_1) \overset{def}{=} (\vec{a}_1\cdot\vec{a}_1)^{1/2} = \sum_{i=1}^{n} \alpha_{i1}{}^2, L(\vec{a}_2) \overset{def}{=} (\vec{a}_2\cdot\vec{a}_2)^{1/2} = \sum_{i=1}^{n} \alpha_{i2}{}^2 \\ &d(\vec{a}_1,\vec{a}_2) \overset{def}{=} L(\vec{a}_1 + (-1)\cdot\vec{a}_2) = \sum_{i=1}^{n} (\alpha_{i1} - \alpha_{i2})^2 \end{aligned} \tag{4.2}$$

In order to construct the coordinates the following procedure is used. Define functions (coordinates) $K_i : M \to R, i = 1,..n$ by

$$K_i(q) \overset{def}{=} \alpha_i(o,q), i = 1..n, \forall q \in M \tag{4.3}$$

where α_i is the component of the arrow $\vec{a}(o,q)$ according to an orthonormal

[5] Since the vector space is equipped with and inner product, i.e., we have an inner product space.

basis. We call $K_i(q)$ the coordinates of M relatively to the basis and the reference point o. Observe that the coordinates depend on the basis and the point o.

Theorem 1:

$$D(o,q) = \sum_i K_i(q)^2, \forall q \in M \tag{4.4}$$

Proof: It follows from the definition of the arrow and the definition of length that

$$D(o,q) = L(\vec{a}(o,q)) = \left(\vec{a}(o,q) \cdot \vec{a}(o,q)\right)^{1/2} = (\sum_{i=1} (\alpha_i(q)^2)^{1/2} = (\sum_{i=1} (K_i(q)^2)^{1/2}$$

Qed.

Summarizing: We have constructed coordinates in a metric point space by assuming that the arrow from a specific point o in the metric point space defines a vector space at each point p.

5 The force

The literature reports no measuring device that measures force in all physical applications. We have accordingly conceived only one experimental device, i.e. a length measuring device. However, the lack of identification of a measuring device that measures force does not mean that such a device is impossible to find in the future. Hence in this section we hypothetically assume an additional measuring device that measures force. The force we conceive has a direction in the metric point space M, and has a numerical value read off from the experimental device. We use $\vec{f}(p,q,n)$, to read: the force at the point p in the "direction of q" with numerical value n. So what could be meant by a direction read off from a force instrument? We simply assume that the force $\vec{f}$ measurement in a point p delivers a real number n and a point q in the point space

M. The two points p and q can then be used to construct an arrow $\vec{a}(p,q)$. And a second arrow $\vec{a}(p,r)=\lambda\cdot\vec{a}(p,q), D(p,r)=n$ can be constructed. The forces $\vec{f}(p,q,n)$ could indeed be a vector, i.e. an element in a vector space. However, assume as a special case that the set of all forces at p is isomorphic to the set of all arrows at p. We set that $\vec{f}(p,q,n) \square\ \vec{a}(p,q), n=L\left(\vec{a}(p,q)\right)$. Thus every force can be associated with a specific arrow. However, this implies that the force is a vector only if the arrow is a vector.

6 Different points; the parallel transport (PT)

Section 3 constructed a set of arrows starting at each point p of M. In physics one often compares forces and arrows from different points of the metric point set. The comparison entails developing a mapping that maps forces and arrows from one point to another point such that forces and arrows can be compared at the same point. Different mappings are indeed possible. No unique mapping exists a priori.

We now construct a specific mapping called parallel transport (PT) of an arrow $\vec{a}(p,q)$ to the point $\vec{a}(p',q')$. We simply write

$$P_t\left(\vec{a}(p,q), p'\right)_{l(p,p')} = \vec{a}(p',q') \qquad (6.1)$$

This we should read: $\vec{a}(p',q')$: the arrow starting at p’ and ending at q’ , is the parallel transported (mapping) of the arrow $\vec{a}(p,q)$ to the point p’ along the (geodesic) line l(p,p’). We apply the following rules to define the familiar Levi-Cevita (LC) PT or the Cartan (C) PT

$$\begin{aligned} &a)\ L\left(\vec{a}(p',q')\right)=L\left(\vec{a}(p,q)\right),(same\ length),LC,C \\ &b)\ \theta_{l(p,q),l(p,p')}=\pi-\theta_{l(p',q'),l(p',p)},(same\ angle\ with\ the\ line),LC,C \\ &c)\, l(p',q')\in M_{l(p,p'),l(p,q)},(same\ \ plane;no\ torsion),LC \end{aligned} \tag{6.2}$$

(6.2a) says that the arrow keeps its length when PT. (6.2b) says that the arrow keeps its angle with the line when PT. (6.2c) says that the arrow is in the same plane when PT. The rules a) and b) apply both for the LC PT and the C PT. For the C PT torsion is allowed. Thus (6.2c) does not apply.

We define a curve as segments of lines. The PT along any curve between any two points can be found by using that the curve consists of segments of lines. Thus we simply write that

$$P_t\left(\vec{a}(p,q),p'\right)_{C(p,p')}=\vec{a}(p',q') \tag{6.3}$$

To read: $\vec{a}(p',q')$ is the arrow found by PT of the arrow $\vec{a}(p,q)$ to p' along the curve C(p,p').

Section 2 commented on the fifth postulate of Euclid which gives the flatness. The modern mathematical definition of flatness is different, and is connected to the PT. The metric point set is called flat if an arrow PT from a point to another point gives the same arrow, independently of the chosen curve that one parallel transports along. The metric point set is called Euclidean if every arrow parallel transported from a point to another point according to the LC PT is independent of the chosen curve that one parallel transports along. Thus in this modern definition the flatness is not connected to the metric point set, but to the chosen PT. Thus in principle the metric point set could be flat according to the old definition, but curved according to the modern definition.

We now use the LC PT. According to the modern definition we could write for flatness

$$P_t\left(\vec{a}(p,q),p'\right)_{C(p,p')}=\vec{a}(p',q'),\forall C(p,p');Flatness \tag{6.4}$$

This property means that it is possible to construct an object $\vec{A}(p,q)$ by collecting an arbitrary $\vec{a}(p,q)$ together with it's PT arrows as one object; the set $\vec{A}(p,q)$. This is indeed the familiar concept learned in school.

However, consider three arbitrary arrows $\vec{a}(p,q)$, $\vec{a}(o,p)$ and $\vec{a}(o,q)$. $\vec{a}(o,p)$ and $\vec{a}(o,q)$ can be subtracted locally and algebraically; to read $\vec{a}(o,r)=\vec{a}(o,q)-\vec{a}(o,p)$. We PT $\vec{a}(p,q)$ to o, to read

$$P_t(\vec{a}(p,q),o)_{C(o,p)}=\vec{a}(o,r') \tag{6.5}$$

Assume that $\vec{a}(o,r')=\vec{a}(o,r)$. It is easily proven that this is fulfilled if we assume the condition (6.4). It can also be proven that $\vec{a}$ becomes an element in a vector space. We have that

Theorem 2:

$$D(p,q)=\sum_i (K_i(p)-K_i(q))^2,\forall p,q\in M \tag{6.6}$$

Proof: Let $\vec{a}(p,q)$ be the arrow from p to q. Let $\vec{a}(o,p)$ and $\vec{a}(o,q)$ have the components α_{ip} and α_{iq} respectively. Then by using $\vec{a}(o,r')=\vec{a}(o,r)$ it follows that

$$D(p,q)=L(\vec{a}(p,q))\overset{LC,C}{=}L\left(P_t(\vec{a}(p,q),o)\right)\overset{LC}{=}L\left(\vec{a}(o,r')\right)\overset{Flatness}{=}L\left(\vec{a}(o,r)\right)=L\left(\vec{a}(o,q)-\vec{a}(o,p)\right)$$

$$=\left(\sum_{i=1}^{n}(\alpha_{iq}-\alpha_{ip})^2)\right)^{1/2}=\left(\sum_{i=1}^{n}(K_i(q)-K_i(p))^2)\right)^{1/2}$$

Qed.

Thus assume that the points p and q are given coordinates when choosing o as the reference point. If the distance between the points is $D(p,q)=\left(\sum_{i=1}^{n}(K_i(q)-K_i(p))^2)\right)^{1/2}$, it is easily proven that $\vec{a}(o,r')=\vec{a}(o,r)$.

Although we have defined PT for arrows in this section, we could also apply the PT for forces in the same way.

Finally, we comment what the literature refers to as auto parallel lines. These are curves where the PT of an arrow, that is tangential to the curve, stays tangential also after the PT. Auto parallel lines do not in general equal the geodesic lines defined according to the metric properties in section 2.

Summarizing: We have constructed different parallel transports geometrically.

7 Conclusions

We have studied fundamental properties of continuous and discrete metric point sets. Our focus is pure geometrical objects. We show how geodesic lines and angles can be constructed from an imposed metric even in discrete spaces. Lines in discrete and continuous metric point sets are constructed and compared with Euclid's five axioms. The angle between two lines is defined. Euclid's axioms E1 and E2 are sufficient to achieve local angles and to define an infinite space. Axiom

E3 is sufficient to define a space with more than one dimension. Axiom E4 is sufficient to define a homogenous space. Axiom E5 is sufficient to define a flat space. We study how the concepts of vector spaces could appear from the metric point set. We have constructed arrows from each point in the metric point set. These arrows can be conceived as lines with a direction. The sum of arrows from each point is constructed algebraically without parallel transport. A method is presented for constructing coordinates. We have constructed coordinates in a metric point space by assuming that the arrow from a specific point o in the metric point space defines a vector space at each point p. We comment on the force concept. Different parallel transports are constructed geometrically. The concepts of tensors and tensor fields are briefly addressed.

Appendix A: The Euclidean point space

The literature provides the following definition of an Euclidean point space: A manifold (a set) M is usually called a *Euclidean Point Space (EPS)* and denoted as ξ if:

$$\textit{There exist a function } a: MxM \to V, \textit{and a Su function}, \textit{written} +, \textit{such that}:$$

$$\begin{aligned} &i)\; a(p,q) = a(p,r) + a(r,q), \forall p,q,r \in M, \\ &ii)\; \forall p \in M, \textit{and } \forall\, v \in V, \exists q \in M \textit{ such that } a(p,q) = v \end{aligned} \qquad \text{(A1)}$$

V is an inner product set (space) with the traditional properties (Appendix A, [1]):

The elements of ξ are called points, and the inner product space V is called the translation space of ξ. The function $a(p,q)$ is the called the vector determined by the end point q (NB) and the initial point p. The condition ii) is equivalent to requiring the function $a_p(q): \xi \to V$ *defined by* $a_p(q) = a(p,q)$ to be one-to-one for each q. The dimension of ξ, written as *dim* ξ is defined to be the dimension of the translation space V. If the vector space V does not have an inner product, the set ξ is called an affine space. Observe that the inner product space V is made an Euclidean point space if one define a function $a(p,q)=(-1p)+q,\forall p,q\in V$. So an inner product set (space) can be an Euclidean point space.

Appendix B: The vector space.

The literature provides the following definition of a vector space and inner product space. There exists a sum function: $Sum:VxV\to V$ (+ for short) and a multiplication function $Mu:RxV\to V$ (. for short) with the following properties

$$\begin{aligned}
&a),\vec{u}+\vec{v}=\vec{v}+\vec{u},b),(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w}),c),\exists\theta\in V\,\Box\,\vec{v}+\theta=\vec{v},\forall\vec{v}\in V,\\
&d),\lambda\cdot(\vec{u}+\vec{v})=\lambda\cdot\vec{u}+\lambda\cdot\vec{v},e),(\lambda+\mu)\cdot\vec{v}=\lambda\cdot\vec{u}+\mu\cdot\vec{v}\\
&f),\lambda\cdot(\mu\cdot\vec{v})=(\lambda\,\mu)\cdot\vec{v},g),1\cdot\vec{v}=\vec{v},h),0\cdot\vec{v}=\theta\\
&where,\lambda,\mu\in R,\vec{u},\vec{v},\vec{w}\in V.
\end{aligned} \tag{B1}$$

There exists a product $P:VxV\to R$ (also called the dot or the inner product $\sqcup$ for short) with the following properties

$$\begin{aligned}
&a)\,\vec{u}\Box\vec{v}=\vec{v}\Box\vec{u},b)\,(\vec{u}+\vec{v})\Box\vec{w}=\vec{u}\Box\vec{w}+\vec{v}\Box\vec{w},\\
&c)\lambda\cdot\left(\vec{u}\Box\vec{v}\right)=(\lambda\cdot\vec{u})\Box\vec{v},d)\vec{u}\Box\vec{v}\geq 0,e)\,\vec{u}\Box\vec{u}=0\Leftrightarrow\vec{u}=\theta
\end{aligned} \tag{B2}$$

The length of a vector is defined as $L(\vec{v})\overset{\text{def}}{=}\left(\vec{v}\Box\vec{v}\right)^{1/2}$. We can check whether

$\vec{a}(p,q)=\{\{p\},l(p,q)\},\vec{\omega}_p\overset{def}{=}\vec{a}(p,p)$ is an element in a vector space. Some of the axioms are simply fulfilled by construction.

$$
\begin{aligned}
&\textit{Vector space}:\\
&a),\vec{a}(p,q)+\vec{a}(p,r)=\vec{a}(p,r)+\vec{a}(p,q), \textit{By construction}\\
&b),(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})?,\\
&c),\vec{a}(p,q)+\vec{\omega}_p=\vec{a}(p,q), \textit{By construction}\\
&d),\lambda\cdot(\vec{u}+\vec{v})=\lambda\cdot\vec{u}+\lambda\cdot\vec{v}?,\\
&e),(\lambda+\mu)\cdot\vec{v}=\lambda\cdot\vec{u}+\mu\cdot\vec{v},?\\
&f),\lambda\cdot(\mu\cdot\vec{a}(p,q))=\lambda\cdot(\{\{p\},\mu\overset{p}{\cdot}l(p,q)\})=\{\{p\},(\lambda\,\mu)\overset{p}{\cdot}l(p,q)\}=(\lambda\,\mu)\cdot\vec{a}(p,q), \textit{By construction}\\
&g),1\cdot\vec{a}(p,q)=\vec{a}(p,q), \textit{By construction},\\
&h),0\cdot\vec{a}(p,q)=\vec{\omega}_p, \textit{By construction}
\end{aligned}
\tag{B3}
$$

Further for the inner product

$$
\begin{aligned}
&a),\vec{a}(p,q)\square\vec{a}(p,r)=D(p,q)D(p,r)Cos\left(\theta_{\vec{a}(p,q)\vec{a}(p,r)}\right)=\vec{a}(p,r)\square\vec{a}(p,q), \textit{By construction}\\
&b)\ (\vec{u}+\vec{v})\square\vec{w}=\vec{u}\square\vec{w}+\vec{v}\square\vec{w},?\\
&c),\lambda\cdot\left(\vec{a}(p,q)\square\vec{a}(p,r)\right)=(\lambda\cdot\vec{a}(p,q))\square\vec{a}(p,r), \textit{By construction}\\
&d),\vec{a}(p,q)\square\vec{a}(p,r)\geq 0, \textit{By construction}\\
&e)\ \vec{a}(p,q)\square\vec{a}(p,q)=0\Leftrightarrow\vec{a}(p,q)=\vec{\omega}_p, \textit{By construction}
\end{aligned}
\tag{B2}
$$

Appendix C

Theorem C1 (Schwarz inequality):

$$u\cdot v\leq L(u)L(v),\forall u,v\in V \tag{C1}$$

Proof: Define the vector $L(u)^2\cdot v-(v\cdot u)\cdot u$. Then it follows that

$$
\begin{aligned}
&L\left(L(u)^2\cdot v-(v\cdot u)\cdot u\right)=L(u)^4L(v)^2-(u\cdot v)^2L(u)^2\geq 0,\\
&\Rightarrow(u\cdot v)^2\leq L(u)^2L(v)^2\Rightarrow(u\cdot v)\leq L(u)L(v)
\end{aligned}
$$

Qed.

Theorem C1 (Triangle inequality): $L(u+v) \le L(u) + L(v)$

Proof:

$$L(u+v)^2 = (u+v)\cdot(u+v) = L(u)^2 + L(v)^2 + 2(u\cdot v) \le L(u)^2 + L(v)^2 + 2L(u)L(v)$$
$$= (L(u)+L(v))^2 \Rightarrow L(u+v) \le L(u) + L(v) \qquad \text{(C2)}$$

Qed.

Theorem C2 (Specific length relation):

$L(u-v) \ge L(u) - L(v)$ (Not valid for a general norm) (C3)

Proof: It follows that

$$L(u-v)^2 = (u-v)\cdot(u-v) = L(u)^2 - 2(u\cdot v) + L(v)^2 \ge L(u)^2 - 2L(u)L(v) + L(v)^2$$ Q
$$= (L(u)-L(v))^2 \Rightarrow L(u-v) \ge L(u) - L(v)$$

ed

Theorem C3 (The length is a norm):

Proof: The norm on a vector set (space) has the following properties

$$a)\|v\| \ge 0, b)\|v\| = 0 \Leftrightarrow v = \theta, c)\|\lambda\cdot v\| = Abs(\lambda)\|v\|,$$
$$d)\|u+v\| \le \|u\| + \|v\| \qquad \text{(C4)}$$

Proof: a) $a)L(v) \ge 0, b)L(v) = 0 \Leftrightarrow v = \theta, c)L(\lambda\cdot v) = Abs(\lambda)L(v),$ $d)L(u+v) \le L(u) + L(v)$

Qed.

Theorem C4 (The distance function d is a metric)$: d: VxV \to R, d(u,v) = L(u-v)$

Proof:

$$a)d(u,v) \geq 0, b)d(u,v) = 0 \Leftrightarrow u = v, c)d(u,v) = L(u-v) \\ = L((u-w)+(w-v)) \leq L(u-w) + L(w-v) = d(u,w) + d(w,v) \tag{C5}$$

Qed.

Theorem C5: $W(v) = \Omega \ x \ v \Leftrightarrow W(v) = -W^t(v),$ *and* $\dim(V) = 3$

Proof: left to right. Let $\{e_i\}$ *i=1,.3, be a basis for V. Then*

$$P(u, W(v)) = P(u, \Omega \ x \ v) = (u_p e_p, \varepsilon_{ijk} \Omega_i v_j e_k) = \varepsilon_{ijk} \Omega_i u_k v_j = -\varepsilon_{ikj} \Omega_i u_k v_j = P(\varepsilon_{ikj} \Omega_i u_k e_j, v_p e_p)$$

Right to left: Define $\Omega_k = -(1/2)\varepsilon_{ijk} W_{ij}, i.e. \Omega_1 = -W_{23}, \Omega_2 = W_{13}, \Omega_3 = -W_{12}$,

where $W_{ij} = P(W(e_i), e_j)$

Qed.

Appendix D: Tensors and tensor fields

We are in the position to define tensors and tensors fields. Let V be a vector space with elements v. Let V* be the set of all linear transformations from V into the real numbers (called the dual space), with elements v*. Finally let V** be the set of all linear transformations from V* into R, with elements v**. All those spaces are easily defined to be vector spaces. As an example we use

$$v \in V, v = v^1\vec{e}_1 + v^2\vec{e}_2, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \vec{e}_1, \vec{e}_2 = base\ vectors,$$
$$v^* \in V^*, v^* = v_1 * \vec{e}^1 * + v_2 * \vec{e}^2 *, v^* = \begin{bmatrix} v^1 * & v^2 * \end{bmatrix}, \vec{e}^1 *, \vec{e}^2 * = base\ vectors, \quad (D1)$$
$$v^{**} \in V^{**}, v^{**} = v^1 ** \vec{e}_1 ** + v^2 ** \vec{e}_2 **, v^{**} = \begin{bmatrix} v_1 ** \\ v_2 ** \end{bmatrix}, \vec{e}_1 **, \vec{e}_2 ** = base\ vectors,$$

where two transformations are defined as

$$v^*(v) = \begin{bmatrix} v^1 * & v^2 * \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v^1 * v_1 + v^2 * v_2 \in R, v^* : V \to V^*$$
$$v^{**}(v^*) = \begin{bmatrix} v^1 * & v^2 * \end{bmatrix} \begin{bmatrix} v_1 ** \\ v_2 ** \end{bmatrix} = v^1 * v_1 ** + v^2 * v_2 ** \in R, v^{**} : V^* \to V^{**} \quad (D2)$$

Now we demand that

$$v^*(v) \overset{mod}{=} v^{**}(v^*), \forall v^1 *, v^2 * \Rightarrow v^1 * v_1 + v^2 * v_2 = v^1 * v_1 ** + v^2 * v_2 **, \forall v^1 *, v^2 *$$
$$\Rightarrow v_1 ** = v^1, v_2 ** = v^2 \quad (D3)$$

This enables us to establish an isomorphic relation between the space V and the space V**, which enables us to identify the space V with V**. Applying the isomorphic relation causes the number of variables to become much smaller. Define the multi linear functions T by:

$$T : \underbrace{V^* x V^* x V^* ... V^*}_{p\ times} x \underbrace{V^{**} x V^{**} x V^{**} ... V^{**}}_{q\ times} \to R \quad (D4)$$

T is called a tensor of contra variant order p and covariant order q. In the example above T= v** is a tensor of order p=1,q=0, called a pure contra variant tensor of order 1.

Consider the example

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$T_1^1(v^*,v^{**})=\begin{bmatrix} v^1* & v^2* \end{bmatrix} A \begin{bmatrix} v^1** \\ v^2** \end{bmatrix} = v^1*\left(a_{11}v^1**+a_{12}v^2**\right)+v^2*\left(a_{21}v^1**+a_{22}v^2**\right)$$
$$T_{11}(v^*,v'^*)=\begin{bmatrix} v^1* & v^2* \end{bmatrix}\begin{bmatrix} v^1{}'* \\ v^2{}'* \end{bmatrix} = v^1*v^1{}'*+v^2*v^2{}'*$$

(D5)

T_1^1 is a mixed tensor. T_{11} is a pure covariant tensor of order 2. We now define a special transformation called the tensor product TP ; V*xV**->R by the following rule

$$TP: V*xV** \to R,$$
$$TP_{v^*,v^{**}}\left(u^*,u^{**}\right) \overset{def}{=} \begin{bmatrix} u^1* & u^2* \end{bmatrix}\begin{bmatrix} v_1** \\ v_2** \end{bmatrix}\begin{bmatrix} v^1* & v^2* \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$
$$=\begin{bmatrix} u^1* & u^2* \end{bmatrix} A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, A \overset{def}{=} \begin{bmatrix} v_1**v^1* & v_1**v^2* \\ v_2**v^1* & v_2**v^2* \end{bmatrix} \overset{def}{=} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$e_{11}=\begin{bmatrix} u^1* & u^2* \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_{12}=\begin{bmatrix} u^1* & u^2* \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$e_{21}=\begin{bmatrix} u^1* & u^2* \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_{22}=\begin{bmatrix} u^1* & u^2* \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

(D6)

where e_i are the bases. Further we have that

$$TP : V^* x V^{**} \to R,$$
$$a_{11} = TP_{v^*,v^{**}}\left(\begin{bmatrix}1 & 0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right), a_{12} = TP_{v^*,v^{**}}\left(\begin{bmatrix}1 & 0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right),$$
$$a_{21} = TP_{v^*,v^{**}}\left(\begin{bmatrix}0 & 1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right), a_{22} = TP_{v^*,v^{**}}\left(\begin{bmatrix}0 & 1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right), \tag{D7}$$

Assuming generally that the dimensions of V* and V** are n, it follows that

$$TP : V^* x V^{**} \to R,$$
$$T^1_{1\,v^*,v^{**}}(u^*, u^{**}) = \begin{bmatrix}u^1 * & u^2 *\end{bmatrix}\begin{bmatrix}v_1 ** \\ v_2 **\end{bmatrix}\begin{bmatrix}v^1 * & v^2 *\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} \tag{D8}$$
$$Dim = N^{p+q}, p = 1 = q \Rightarrow Dim = 4,$$

Tensors of higher order are defined accordingly. If T^n_m and $S^{n'}_{m'}$ are two tensors of contra variant order n and covariant order m, and contra variant order n' and covariant order m' respectively, the components are given by $a\, {}^{\overbrace{i..j}^{n\,times}}_{\underbrace{p..q}_{m\,times}}$ and $b\, {}^{\overbrace{k..l}^{n'times}}_{\underbrace{q..s}_{m'times}}$.

The tensor product is written as $T \otimes S$. The components of this tensor is written as

$$c^{i..jk..l}_{p..q..s} = a^{i..j}_{p..q} b^{k..l}_{q..s} \tag{D9}$$

Tensors defined according to tensor products are called simple tensors. In general the tensor product can be seen as a mapping from V*xV** into a tensor T^1_1 for second order tensors. In general tensors of order 2 closely follow the familiar matrix rules outlined in elementary linear algebra.

Assume that the basis of V * and V** are changed. Thus the components of the matrix of the tensor from V*xV** also change. Assume that

$$\overline{e}_k* = \dot{H}_k^q \overline{e}_q*, \overline{e}_k** = \ddot{H}_k^q e_q**, \overline{e}_k; new \tag{D10}$$

Then it follows that the new components are given by

$$\overline{a}_{ij} = a_{kl} \dot{H}_i^k \ddot{H}_j^l \tag{D11}$$

Generally such a relation could be used as a definition of a tensor instead of the definition in (D4).

Assume that we have a family of tensors a_{kl} at different points of space. Our intention is to define a tensor field, which is quite different from the concept of a tensor. Assume that the coordinates are defined. Introduce the mappings

$$\begin{aligned} &a_{kl} : M \to T_p, K_i : M \to R^n \\ &A_{kl} = a_{kl}(K_i^{-1}(R^n)) : R^n \to T \end{aligned} \tag{D12}$$

Thus to each point in the point space we have a tensor.

Assume that we change basis for V**. The coordinates are related to the components. What happens then with the components of the tensor at each given point? They change according to the rule

$$\begin{aligned} &a_{kl} : M \to T_p, K_i : M \to R^n \\ &\overline{A}_{ij} = A_{kl} \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial x^l}{\partial \overline{x}^j}, x^k = x^k(\overline{x}_1, \overline{x}_2, \overline{x}_3), x_i : old, \overline{x}_i; new \end{aligned} \tag{D13}$$

Appendix E: A construction of a vector space without any specific parallel transport

Define a new set called ap$\subset A$. The set is constructed by collecting an arbitrary $\vec{a}(p,q)$ together with it's PT arrows in one set $\vec{A}(p,q)$. This familiar concept is learned in school. The arrow together with all its PT arrows from p are conceived as simply one arrow $\vec{A}(p,q)$.

$$\vec{A}(p,q) \overset{def}{=} \{\vec{a}(r,s) \,|\, P_t\left(\vec{a}(r,s), p\right) = \vec{a}(p,q)\}, r,s,p,q \in M \quad \text{(E1)}$$

References

[1] V. Ambarzumian and D. Iwanenko, Zur Frage nach Vermeidung der unendlichen Selbstruckwirkung des Electrons. Z. Phys. 64 (1930), 563-567.

[2] J. Ambjørn, Quantization of geometry. Lectures given at Les Houches Session LXII, Fluctuation Geometries in Statistical Mechanics and Field Theory, 1994.

[3] H.C. Baehr, A. Dimakis and F. Müller-Hoissen, Differential calculi on commutative algebras. J.Phys. A: Math Gen , 2 (1995), 3197-3222.

[4] L. Bombelli, J. Lee, D. Meyer and R.D. Sorkin, Space-time and causal set. Phys. Rev. Lett. 59 (1987), 521-524.

[5] K. Bresser, Müller-Hoissen F., Dimakis A., Sitarz A., Non-commutive geometry of finite groups. J. Phys A: Math. Gen., 29 (1996), 2705-2735.

[6] S. Cho, and K.S. Park, Linear connections on graphs. J. Math. Phys., 38 (1997), 11, 5889-5904.

[7] A. Connes, Noncommutative Geometry, Academic Press, Inc, San Diego, New York Boston, ISBN 0-12-185860-X, 1994.

[8] A. Connes and J. Lott, Particle models and non-commutative geometry. Nucl. Phys. B, (Proc Suppl), 18B (1990), 29-47.

[9] F. David, Simplicial quantum gravity and random lattices. Lectures given at Les Houches Session LVII, Gravitation and Quantization, 1992.

[10] A. Dimakis and F. Müller-Hoissen, Connes' distance function on one-dimensional lattices. Int. J. Theort. Phys., 37, 3 (1994), 907-913.

[11] A. Dimakis and F. Müller-Hoissen, Differential calculus and gauge theory on finite sets. J. Phys A: Math. Gen., 27 (1994), 3159-3178.

[12] A. Dimakis and F. Müller-Hoissen, Discrete Riemannian geometry. J. Math. Phys. 40 (1999), 1518, doi:10.1063/1.532819.

[13] A. Dimakis and F. Müller-Hoissen, Quantum mechanics on a lattice and q-deformations. Phys. Lett., B, 295 (1992), 242-248.

[14] A. Dimakis and F. Müller-Hoissen, Stochastic Differential Calculus, the Moyal * product, and noncommutative geometry. Lett. Math. Physics, 28 (1993), 123-137.

[15] A. Dimakis, F. Müller-Hoissen, and T. Striker, From continuum to lattice theory via deformation of the differential calculus. Phys. Lett. B, 300 (1993), 141-144.

[16] A. Dimakis and F. Müller-Hoissen and F. Vanderseypen, Discrete differential manifolds and dynamics on networks. J. Math. Phys. 36 (1995), 3771-3791.

[17] R. Engelking, Dimension Theory, North Holland, 1978.

[18] R.P. Feynman, Simulating physics with computers. Int. J. Theor. Phys., 21 (1982), 467-488.

[19] D. Finkelstein, Space-time code. Phys. Rev. 184 (1969), 1261-1271.

[20] M H. Freedman and F. Quinn, Topology of 4-Manifolds. Princeton University Press, 1990.

[21] M. Hirsch, Differential Topology. Springer Verlag, 1997.

[22] A.N. Joourjine, Discrete gravity without coordinates. Physical Rev., D, 35 (1987), 10, 2983-2986

[23] K. Ito, On stochastic differential equations. Mem. Amer. Math. Soc., No 4, 1951.

[24] J. Jürgen, *Riemannian Geometry and Geometric Analysis* (5th ed.), Berlin, New York: Springer-Verlag, ISBN 978-3540773405, 2008.

[25] C. Kuratowski, Topologie II PWN, 1961.

[26] A.W.C. Lau and T.C. Lubensky, State-dependent diffusion: Thermodynamic consistency and its path integral formulation. Phys. Rev. E, 76, 011123 (17 pages), (2007).

[27] J.M. Lee, Introduction to Smooth Manifolds.Springer-Verlag, 2003.

[28] R.B. Lindsay and H. Margenau, Foundations of physics, Library of Congress Catalog Card Number: 57-14416, Dover edition, USA, 1957.

[29] A.A. Markov, Extension de la loi de grands nombres aux e`ne`venements dependants les uns de autres, Bull Soc Phys-Math, Kasan 15 (1906), 135-156.

[30] M. Minski, Cellular vacuum. Int. J. Theor. Phys. 21 (1982), 537-551.

[31] J.F. Moxnes and K. Hausken, Introducing randomness into first-order and second-order deterministic differential equations. Adv. Math. Physics., (2010), doi:10.1155/2010/509326.

[32] T. Regge, Gereral relativity without cordinates. Nuovo Cimento, 19 (1961), 558-571

[33] H.L. Royden, Real Analysis, second Edition Macmillan Publishing Co., Inc, NewYork, 1968.

[34] A. Ruark, The roles of discrete and continuous theories in physics. Phys. Rev., 37 (1931), 315-326.

[35] L. Sklar, Space, time, and space-time, University of California Press, California, 1977.

[36] R.L. Stratonovich, A new interpretation for stochastic integrals and equations. J. Siam. Control, 4 (1966), 362-371.

[37] H.S. Snyder, Quantized spacetime. Phys. Rev. 71 (1947), 38-41.

[38] G. T'Hooft, Quantization of discrete deterministic theories by Hilbert space extension. Nucl. Phys. B, 342 (1990), 471-485.

[39] M. Vaillanta, M.I. Millera, L.C. Younesa and A. Trouvéd, Statistics on diffeomorphisms via tangent space representations. NeuroImage Volume 23 (2004), Supplement 1, S161-S169.

[40] D. Weingarten, Geometric formulation of electrodynamics and general relativity in discrete space-time. J. Math. Phys., 18, 1 (1977), 165-170.

[41] R. Williams and P.A. Tuckey, Regge calculus: a brief review and bibliography. Class. Qunatum Grav., 9 (1992), 1409-1422.

[42] S.L. Woronowicz, Compact matrix pseudo groups. Commun. Math. Phys., 111 (1987), 613-665.

[43] H. Yamamoto, Quantum field theory on discrete spacetime. Phys. Rev. 30 (1984), 1727-1732.

Received: May, 2011