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Non-Linear Schrodinger Equation (NLSE) Solitons Related to Momentum and Hamiltonian QM Hermitian Operators

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Abstract

Propagation effects are analyzed for electromagnetic (EM) waves which satisfy the non-linear Schrodinger equation (NLSE) in a dispersive wave guide. The coupling between momentum and frequencies due to dispersion relation is treated by a coupled Hamiltonian-Momentum operator. One-Soliton solution of NLSE is analyzed with quantum-mechanical (QM) effects. The integrability condition for NLSE is related to Hamiltonian and Momentum Hermitian operators.

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1. Introduction

In the present article we would like to analyze some quantum mechanical (QM) effects for electromagnetic (EM) waves which satisfy the non-linear Schrodinger equation (NLS) in a dispersive wave guide using quantum optics methods. The common procedure for studying QM effects for the NLSE is by the use of quantum field theory [1-2]. In such approach one uses the field operators $\hat{\phi}(z,t)$ and $\hat{\phi}^{\dagger}(z,t)$ satisfying the boson commutation relations (CR)

$$\left[\hat{\phi}(z,t),\hat{\phi}^{\dagger}(z',t)\right]=\delta(z-z') \quad , \tag{1}$$

where z is the one dimensional propagation coordinate. For one dimensional Hamiltonians one uses multiplications of such operators and their derivatives integrated over the z coordinate. Usually QM solutions of NLSE by using quantum field theories turn to be quite complicated. In Quantum Optics one can use the one dimensional equal space CR for the annihilation and creation operators given as [3-5]

$$\left[\hat{a}(z),\hat{a}^{\dagger}(z')\right]=\delta(z-z') \quad . \tag{2}$$

In conventional QM analysis $\hat{a}^{\dagger}(t)\hat{a}(t)$ represents the photon number operator in the quantization volume while in our analysis $\hat{a}^{\dagger}(z,t)\hat{a}(z,t)$ represents the number of photons per unit length at coordinates z,t. The total photon number operator $\hat{n}$ in a soliton is given by $\int \hat{a}^{\dagger}(z,t)\hat{a}(z,t)dz=\hat{n}$. Soliton squeezing effects obtained by Kerr interaction have been treated by Haus [6] and Haus and Lai [7] using a linearized approach. Although some of the derivations in the present work are similar to those presented in [6] the approach for treating Kerr effects is different and more similar to other conventional quantum optics methods [8-12].

For various systems the momentum operator $\hat{G}$ has been used for evaluating QM propagation effects, including coupling between different modes [3,13-14]). In the present work we treat, however, one mode of the EM field which includes a coupling between momentum and frequencies due to dispersion relation. For this purpose it is useful to relate the integrability condition to Hamiltonian and Momentum Hermitian operators.

The article is arranged as follows. In Section 2 QM propagation effects for linear dispersive wave guide are treated. In Section 3 we analyze one-soliton solution of the NLSE in which non-linear Kerr interaction effects are taken into account. QM effects are discussed. We show the relation between the integrability condition for NLSE and Hamiltonian and Momentum matrices representing Lax pair [15]. We relate the present analysis for the 'compatibility condition' to the one-soliton solution of NLSE. In Section 5 we summarize our results and conclusion.

2. Propagation of EM waves in a linear dispersive wave guide

In treating EM waves in *linear dispersive* waveguide we encounter the problem that the frequency ω and the wavevector k are coupled by the dispersion relation [6] :

$$\omega(k)\approx\omega_0(k_0)+\left(\frac{d\omega}{dk}\right)_{k=k_o}\delta k+\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\delta k^2 \quad . \tag{3}$$

Here ω_0 is the resonant frequency corresponding to the wavevector k_0,

$$\left(\frac{d\omega}{dk}\right)_{k=k_o} = v_g \qquad (4)$$

is the group velocity, and $\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}$ is the group velocity dispersion. Higher order terms in the series expansion of ω as function of δk are neglected here as we assume that we have only a narrow distribution of wavevectors around the central wavevector k_0. We limit the analysis to the one dimensional case.

The coupled Hamiltonian-Momentum operator which includes the coupling between frequencies and wavevectors is given as

$$\hat{H} = \hbar \int \omega(k_0+\delta k)\hat{a}^\dagger(k_0+\delta k)\hat{a}(k_0+\delta k)d(\delta k) \quad . \qquad (5)$$

In Eq. (5) $\hat{a}^\dagger(k_0+\delta k)\,\hat{a}(k_0+\delta k)$ represents the number of photons which have momentum $k_0+\delta k$ with the energy $\hbar\omega(k_0+\delta k)$ summed over all modes, represented by the integration over δk.

By substituting Eq. (3) into Eq. (5) we get:

$$\hat{H} =$$

$$\hbar\int\left[\omega_0\left(k_0\right)+\left(\frac{d\omega}{dk}\right)_{k=k_o}\delta k+\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\delta k^2\right]\hat{a}^\dagger(k_0+\delta k)\hat{a}(k_0+\delta k)d(\delta k) \,. \qquad (6)$$

We will transform the dependence of the creation and annihilation operators in Eq. (6) on the momentum $k_0+\delta k$ to dependence on the coordinate z by using the following Fourier transforms:

$$\hat{a}(k_0+\delta k) = \frac{1}{\sqrt{2\pi}}\int \hat{a}(z)\exp\left[i\left(k_0+\delta k\right)z\right]dz \quad ;$$

$$\hat{a}^\dagger(k_0+\delta k) = \frac{1}{\sqrt{2\pi}}\int \hat{a}^\dagger(z')\exp\left[-i(k_0+\delta k)z'\right]dz' \quad ; \qquad ,(7)$$

$$\hat{a}^\dagger(k_0+\delta k)\hat{a}(k_0+\delta k) = \frac{1}{2\pi}\int dz\int dz'\hat{a}^\dagger(z')\hat{a}(z)\exp\left[i\left(k_0+\delta k\right)\left(z-z'\right)\right]$$

where the operators $\hat{a}(z)$ and $\hat{a}^\dagger(z)$ satisfy the CR given in Eq. (2).

We substitute Eq. (7) into Eq. (6) obtaining

$$\hat{H}/\hbar =$$

$$\int\left[\omega_0+\left(\frac{d\omega}{dk}\right)_{k=k_o}\delta k+\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\delta k^2\right]\iint dzdz'\,\hat{a}^\dagger(z')\hat{a}(z)\exp\left[i\left(k_0+\delta k\right)\left(z-z'\right)\right]$$

$$\hat{a}^\dagger(z')\hat{a}(z)\exp\left[i\left(k_0+\delta k\right)\left(z-z'\right)\right]d(\delta k)$$

$$(8)$$

and perform the following three integrals:

$$\frac{1}{2\pi}\int \exp\left[i\left(k_0+\delta k\right)\left(z-z'\right)\right]d(\delta k)=\delta\left(z-z'\right) \quad , \tag{9}$$

$$\begin{aligned}&\frac{1}{2\pi}\int\left\{\left(\frac{d\omega}{dk}\right)_{k=k_o}\delta k\right\}\exp\left[i\left(\delta k\right)\left(z-z'\right)\right]d(\delta k)=\\&-\frac{i}{2\pi}\left(\frac{d\omega}{dk}\right)_{k=k_o}\frac{\partial}{\partial z}\int\exp\left[i\left(\delta k\right)\left(z-z'\right)\right]d(\delta k)=-i\left(\frac{d\omega}{dk}\right)_{k=k_o}\frac{\partial}{\partial z}\delta\left(z-z'\right)\end{aligned} \quad , \tag{10}$$

$$\begin{aligned}&\frac{1}{2\pi}\int\left\{\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\delta k^2\right\}\exp\left[i\left(\delta k\right)\left(z-z'\right)\right]d(\delta k)=\\&-\frac{1}{2\pi}\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\frac{\partial^2}{\partial z^2}\int\exp\left[i\left(\delta k\right)\left(z-z'\right)\right]d(\delta k)=-\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\frac{\partial^2}{\partial z^2}\delta\left(z-z'\right)\end{aligned} \quad . \tag{11}$$

Substituting Eqs. (9-11) into Eq. (8) and transforming the derivatives by partial integration we get:

$$\begin{aligned}&\hat{H}=\\&\hbar\omega_0\int\hat{a}^\dagger(z')\hat{a}(z')dz'-i\hbar\left(\frac{d\omega}{dk}\right)_{k=k_o}\int\hat{a}^\dagger(z')\left\{\frac{\partial}{\partial z'}\hat{a}(z')\right\}dz'\\&-\hbar\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\int\hat{a}^\dagger(z')\left\{\frac{\partial^2}{\partial z'^2}\hat{a}(z'\right\}dz'\end{aligned} \quad . \tag{12}$$

In Eq. (12) we arranged the space derivative so that they operate to the right . By using partial integration Eq. (12) can be transformed into the other form

$$\begin{aligned}&\hat{H}=\\&\hbar\omega_0\int\hat{a}^\dagger(z')\hat{a}(z')dz'+i\hbar\left(\frac{d\omega}{dk}\right)_{k=k_o}\int\left\{\frac{\partial}{\partial z'}\hat{a}^\dagger(z')\right\}\hat{a}(z')dz'\\&-\hbar\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\int\left\{\frac{\partial^2}{\partial z'^2}\hat{a}^\dagger(z'\right\}\hat{a}(z')dz'\end{aligned} \quad . \tag{13}$$

The Hamiltonian of Eq. (12) or of Eq. (13) is the generator for the coupled space-time propagation. The equation of motion for the annihilation operator is given by

$$i\hbar\frac{\partial}{\partial t}\hat{a}(z,t)=\left[\hat{a}(z,t),\hat{H}\right] \quad , \tag{14}$$

where $\hat{H}$ is given by Eq. (12) and the CR of Eq. (2) can be used. By substituting Eq. (12) into Eq. (14) and performing the CR we get

$$\frac{\partial}{\partial t}\hat{a}(z,t)=-i\left(\omega_0\hat{a}(z,t)-i\left(\frac{d\omega}{dk}\right)_{k=k_o}\frac{\partial}{\partial z}\hat{a}(z,t)-\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\frac{\partial^2}{\partial z^2}\hat{a}(z,t)\right) \; . \tag{15}$$

The corresponding equation for $\frac{\partial}{\partial t}\hat{a}^{\dagger}(z)$ can be obtained by the dagger of Eq. (15) (or correspondingly from Eq. (13)).

We define:

$$\hat{a}(z) = \tilde{\hat{a}}(z)\exp(-i\omega_0 t) \quad , \tag{16}$$

and then Eq. (15) is transformed into

$$\frac{\partial}{\partial t}\tilde{\hat{a}}(z,t) = -v_g \frac{\partial}{\partial z}\tilde{\hat{a}}(z,t) + i\frac{1}{2}\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\frac{\partial^2}{\partial z^2}\tilde{\hat{a}}(z,t) \quad , \tag{17}$$

where v_g is the group velocity defined in Eq. (4).

We use the following additional transformation:

$$t' = t - \frac{z}{v_g} \quad , \tag{18}$$

and then Eq. (17) is transformed into

$$\frac{\partial}{\partial t'}\tilde{\hat{a}}(z,t) = i\frac{1}{2}\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\frac{\partial^2}{\partial z^2}\tilde{\hat{a}}(z,t) \quad . \tag{19}$$

The time $t' = t - \frac{z}{v_g}$ indicates that we removed the time delay $\frac{z}{v_g}$ from the ordinary time and the use of operator $\tilde{\hat{a}}(z)$ indicates that we removed the rapid carrier oscillation frequency $\exp(-i\omega_0 t)$ from $\hat{a}(z)$.

One should notice that the Hamiltonian of Eq. (12) or (13) includes integration over the z' coordinate where such integration is in analogy with quantum field Hamiltonians which include also such space integration.

For simplicity of notation, from now on, in using Eq. (19) we will remove the prime and the tilde from this equation but we need to take into account that the time in such equation represents a time delayed by $\frac{z}{v_g}$ and that from $\hat{a}(z)$ we removed the rapid variation $\exp(-i\omega_0 t)$.

The nonlinear Schrodinger Hamiltonian can be obtained by adding to the linear Hamiltonian the Kerr effect represented by nonlinear Hamiltonian

$$\hat{H}_K(t) = -\hbar\frac{K}{2}\int dz\hat{a}^{\dagger}(z,t)\hat{a}^{\dagger}(z,t)\hat{a}(z,t)\hat{a}(z,t), \tag{20}$$

where K is the Kerr constant .

By taking into account the non-linear momentum operator $\hat{H}_K$ the equation of motion for $\hat{a}(z,t)$ becomes

$$\frac{\partial}{\partial t}\hat{a}(z,t) =$$
$$i\frac{1}{2}\left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}\frac{\partial^2}{\partial z^2}\hat{a}(z,t) - \frac{i}{\hbar}\left[\hat{a}(z),\hat{H}_k(z,t)\right] = \qquad (21)$$
$$i\frac{1}{2}C\frac{\partial^2}{\partial z^2}\hat{a}(z,t) + iK\hat{a}^\dagger(z,t)\hat{a}(z,t)\hat{a}(z,t)$$

We have added here the CR with $\hat{H}_K$ and in a short notation the constant $C = \left(\frac{d^2\omega}{dk^2}\right)_{k=k_o}$ is representing the group velocity dispersion.

3. One-soliton solution of NLSE with QM effects

For obtaining a classical soliton solution of NLSE we exchange the operator $\hat{a}(z,t)$ into its classical representation $a_c(z,t)$ and then we get the classical equation

$$\frac{\partial}{\partial t}a_c(z,t) = i\frac{1}{2}C\frac{\partial^2}{\partial z^2}a_c(z,t) + iK\hat{a}_c^*(z,t)a_c(z,t)a_c(z,t) \quad . \qquad (22)$$

A soliton solution of Eq. (22) depends on 4 constants: a constant x_0 representing the pulse center, the carrier frequency of the soliton, an arbitrary phase constant θ_0 of the soliton and the total intensity of the soliton. The solution of Eq. (22) is simplified by eliminating these constants: The constant x_0 representing the soliton center is eliminated by choosing a coordinate system whose origin is at the pulse center. The arbitrary constant phase θ_0 of $a_c(z,t)$ is chosen to be equal to zero. The carrier frequency of the soliton is assumed to coincide with the frequency ω_0. The total number of photons in the soliton remains as an important parameter . Under these assumptions the one-soliton classical solution of Eq. (22) is given as [6]

$$a_c(z,t) = A\exp\left[i\left(\frac{KA^2}{2}t\right)\right]\sec h\left(\frac{z}{\xi}\right) \quad , \qquad (23)$$

with the constraint

$$KA^2 = \frac{C}{\xi^2} \quad . \qquad (24)$$

The normalization constant A is assumed to be real. The complex classical amplitude $a_c(z,t)$ is normalized so that by integrating its magnitude squared over z we get the number n of photons in the soliton:

$$\int dz\,|a_c(z,t)|^2 = \int A^2\sec h^2\left(\frac{z}{\xi}\right)dz = 2A^2\xi = n \quad . \qquad (25)$$

We notice according to Eq. (25) that the number of photons in the soliton is proportional to A^2 representing the normalization constant squared. The parameter ξ is related to the pulse shape. For larger values of ξ the soliton pulse becomes narrower with a larger amplitude at the pulse center and larger number of photons. The one-soliton solution of the NLSE represents a balance between linear dispersion, which tends to break up the soliton wave packet, and a focusing effect of the cubic nonlinearity, produced by Kerr effect.

We can change the space z dependence of Eq. (24) into the wavevector k dependence by using the Fourier transform

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ikz) \sec h\left(\frac{z}{\xi}\right) dz = \xi \sqrt{\frac{\pi}{2}} \sec h\left(\frac{\pi k \xi}{2}\right) \qquad (26)$$

Then the one-soliton solution can be transformed into

$$a_c(k,t) = \exp\left[i\left(\frac{KA^2}{2} t\right)\right] F(k) \quad . \qquad (27)$$

Eq. (27) has a quite simple explanation showing that in the soliton pulse we have a certain distribution of wavevectors producing its wavepacket . One needs, however, to take into account that the rapid frequency dependence $\exp(-i\omega_0 t)$ has been eliminated by using Eq. (16) (omitting for simplicity of notation the tilde on operator $\hat{a}$). Also the relative simple forms of Eq. (23) or (27) is obtained under the above simplifying conditions..

We can use a certain integration over the z dependence of the soliton exchanging the classical solution of Eq. (23) into

$$a_c(t) = \sqrt{2\xi}\ A \exp\left[i \frac{KA^2}{2} t\right] \quad , \qquad (28)$$

so that in agreement with Eq. (25) we will get

$$|a_c(t)|^2 = n = 2A^2\xi \quad . \qquad (29)$$

A possible quantum analog of the classical amplitude $a_c(t)$ of Eq. (28) might be obtained by assuming that the soliton is produced as a coherent state $|\alpha(t)\rangle$ where

$$|\alpha(t)|^2 = n = 2A^2\xi \quad , \qquad (30)$$

and the phase term of $\alpha(t)$ is given by $\exp\left[i \frac{KA^2}{2} t\right]$. The photon number distribution of the one-soliton state becomes then :

$$|\alpha(t)\rangle = \frac{\left\{\sqrt{2\xi}\ A \exp\left[ikA^2 t/2\right]\right\}^n}{\sqrt{n!}} \exp\left(-\xi A^2\right)|n\rangle \quad . \qquad (31)$$

For each number state $|n\rangle$ in the coherent photon distribution an additional phase given by $\exp\left[ikA^2 tn/2\right]$ is introduced. The phase $KA^2 tn/2$ is proportional to the photon number and to A^2 (see Eq.(25)) and increases linearly with time. Similar phases are known to be obtained by the ordinary Kerr interactions[8-10]. Poissonian photon number uncertainty has been obtained also in [7].

4. The integrability condition for NLSE related to Hermitian operators

Our interest in the present Section is to show that the integrability condition for NLSE can be related to Hamiltonian and Momentum operators representing Lax pair both described by Hermitian matrices. Let us assume that we have a two dimensional wavefunction depending both on z and t

$$\vec{\psi}(z,t) = \begin{pmatrix} \psi_1(z,t) \\ \psi_2(z,t) \end{pmatrix} \quad . \tag{32}$$

Suppose that $|\vec{\psi}(z,t)\rangle$ satisfy the Schrodinger equation :

$$\frac{\partial}{\partial t}\begin{pmatrix} \psi_1(z,t) \\ \psi_2(z,t) \end{pmatrix} = \frac{\mathrm{H}}{i}\begin{pmatrix} \psi_1(z,t) \\ \psi_2(z,t) \end{pmatrix} = \tilde{H}\begin{pmatrix} \psi_1(z,t) \\ \psi_2(z,t) \end{pmatrix} \quad , \tag{33}$$

where H is a two dimensional Hermitian matrix which can include a function of $u(z,t)$ and its derivatives. Suppose also that $|\vec{\psi}(z,t)\rangle$ satisfy the momentum equation [3-5, 13-14] equation :

$$\frac{\partial}{\partial z}\begin{pmatrix} \psi_1(z,t) \\ \psi_2(z,t) \end{pmatrix} = i\mathrm{M}\begin{pmatrix} \psi_1(z,t) \\ \psi_2(z,t) \end{pmatrix} = \tilde{M}\begin{pmatrix} \psi_1(z,t) \\ \psi_2(z,t) \end{pmatrix} \quad , \tag{34}$$

where M is an Hermitian two dimensional matrix which also can include a function of $u(z,t)$ and its derivatives. We find it convenient to represent the equations of motion with $\tilde{H}$ and $\tilde{M}$ which are not Hermitian but are related to the Schrodinger and Momentum Hermitian matrices by Eqs. (33-34).

We assume the *Compatibility Condition*

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}\vec{\psi}(z,t)\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}\vec{\psi}(z,t)\right) \quad . \tag{35}$$

where on the left side of this equation we perform first the derivative of $\vec{\psi}(z,t)$ according to t and afterwards according to z while on the right side we inverse the order of these derivatives. Using Eqs, (33-34) we get

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}\vec{\psi}(z,t)\right) = \frac{\partial}{\partial z}\left(\tilde{\mathrm{H}}\vec{\psi}(z,t)\right) = \left(\frac{\partial}{\partial z}\tilde{\mathrm{H}} + \tilde{\mathrm{H}}\tilde{\mathrm{M}}\right)\vec{\psi}(z,t) \quad , \tag{36}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}\vec{\psi}(z,t)\right) = \frac{\partial}{\partial t}\left(\tilde{\mathrm{M}}\vec{\psi}(z,t)\right) = \left(\frac{\partial}{\partial t}\tilde{M} + \tilde{\mathrm{M}}\tilde{\mathrm{H}}\right)\vec{\psi}(z,t) \quad . \tag{37}$$

Substituting Eqs. (36-37) into Eq. (35) we get the *Compatibility Condition*

$$\tilde{\mathrm{H}}\tilde{\mathrm{M}} - \tilde{\mathrm{M}}\tilde{\mathrm{H}} - \frac{\partial}{\partial t}\tilde{\mathrm{M}} + \frac{\partial}{\partial z}\tilde{\mathrm{H}} = 0 \tag{38}$$

The idea is that by using special forms for the matrices $\tilde{\mathrm{M}}$ and $\tilde{\mathrm{H}}$ which satisfy the compatibility equation (38) they lead to *integrable* nonlinear equation. We demonstrate here such an approach for the NLSE.

In the present analysis we define:

$$\tilde{\mathrm{M}} = \begin{pmatrix} 0 & -u(z,t) \\ u^*(z,t) & 0 \end{pmatrix} \quad , \tag{39}$$

$$\tilde{\mathrm{H}} = \begin{pmatrix} -i\,|u(z,t)|^2 & i\dfrac{du(z,t)}{dz} \\ i\dfrac{du^*(z,t)}{dz} & i\,|u(z,t)|^2 \end{pmatrix} \quad , \tag{40}$$

where $u(z,t)$ is a function dependent on time t and on space z.

Substituting Eqs. (39-40) in the *compatibility equation* (38) we get after straightforward calculations

$$\begin{pmatrix} 0 & \dfrac{\partial u}{\partial t} + i\dfrac{\partial^2 u}{\partial z^2} + 2iu\,|^2\,u \\ \dfrac{\partial u^*}{\partial t} - i\dfrac{\partial^2 u}{\partial z^2} - 2i\,|u|^2\,u^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad . \tag{41}$$

While the diagonal elements in the *compatibility equation* vanish in a trivial way the vanishing of the nondiagonal elements lead to NLSE (up to some normalization constants):

$$\frac{\partial u(z,t)}{\partial t} + i\frac{\partial^2 u(z,t)}{\partial z^2} + 2i\,|u(z,t)|^2\,u(z,t) = 0 \quad , \tag{42}$$

and to the complex conjugate of this equation. We claim, therefore, that the function $u(z,t)$ used in the above analysis represents a solution to NLSE.

Eqs. (33-34) have a simple geometric interpretation as these equations describe *connections* on a two-dimensional *vector bundle* over the (z,t) plane [16]. Eq. (34) describes how to 'parallel translate' the vector $\vec{\psi}(z,t)$ in the z-direction and Eq. (33) describes how to 'parallel translate' $\vec{\psi}(z,t)$ in the t-direction. The matrices $\tilde{\mathrm{M}}$ and $\tilde{\mathrm{H}}$ are the 'connection coefficients'. A connection is defined to have a *zero curvature* if parallel translation between two points is independent of the path connecting the two points. Therefore the *compatibility condition* represents the *zero curvature* condition for the *integrability* of the nonlinear equation. We find that the two dimensional wavefunction $\vec{\psi}(z,t)$ satisfying both Eqs. (33) and (34) with the *compatibility condition* is related to integrable NLSE.

We would like to show now that the above analysis of the 'compatibility condition' is in agreement with the one-soliton solution given in the previous Section. Let us use 'scaled coordinates" by which Eq. (23) can be written as

$$a_c(\tilde{z},\tilde{t}) = A\exp(i\tilde{t})\sec h(\tilde{z}) \quad , \tag{43}$$

where

$$\tilde{z} = \frac{z}{\xi} \quad ; \quad \tilde{t} = \frac{KA^2}{2}t \tag{44}$$

Using Eq. (34) and (39) for time $t=0$ and for a scaled coordinate $\tilde{z}$ we get

$$\frac{\partial}{\partial\tilde{z}}\begin{pmatrix}\psi_1(z)\\ \psi_2(z)\end{pmatrix}=\tilde{M}\begin{pmatrix}\psi_1(z)\\ \psi_2(z)\end{pmatrix}=\begin{pmatrix}0 & -u(\tilde{z})\\ u^*(\tilde{z}) & 0\end{pmatrix}\begin{pmatrix}\psi_1(z)\\ \psi_2(z)\end{pmatrix} \quad . \qquad (45)$$

Eq. (45) is satisfied by substituting

$$\begin{pmatrix}\psi_1(z)\\ \psi_2(z)\end{pmatrix}=\begin{pmatrix}\sec h(\tilde{z})\\ \tanh(\tilde{z}\end{pmatrix} \quad ; \quad u(\tilde{z})=\sec h(\tilde{z}) \qquad (46)$$

obtaining the agreement:

$$\frac{\partial}{\partial\tilde{z}}\begin{pmatrix}\sec h(\tilde{z})\\ \tanh(\tilde{z}\end{pmatrix}=\begin{pmatrix}-\sec h(\tilde{z})\tanh(\tilde{z})\\ \sec h^2(\tilde{z})\end{pmatrix} \quad ;$$

$$\begin{pmatrix}0 & -\sec h(\tilde{z})\\ \sec h(\tilde{z}) & 0\end{pmatrix}\begin{pmatrix}\sec h(\tilde{z})\\ \tanh(\tilde{z}\end{pmatrix}=\begin{pmatrix}-\sec h(\tilde{z})\tanh(\tilde{z})\\ \sec h^2(\tilde{z})\end{pmatrix} \qquad (47)$$

The value of $u(\tilde{z})$ has been given according to Eq. (43) (at time (t=0)) while $\begin{pmatrix}\psi_1(z)\\ \psi_2(z)\end{pmatrix}$ at time $t=0$ have been chosen so that Eq. (34) will be satisfied.

Using Eqs. (33) and Eq. (40) for time $\tilde{t}$ and for a scaled coordinate $\tilde{z}$ we get

$$\frac{\partial}{\partial\tilde{t}}\begin{pmatrix}\psi_1(z,t)\\ \psi_2(z,t)\end{pmatrix}=\tilde{H}\begin{pmatrix}\psi_1(z,t)\\ \psi_2(z,t)\end{pmatrix}=$$

$$\begin{pmatrix}-i\,|\sec h(\tilde{z})|^2 & i\dfrac{d(\sec h(\tilde{z})}{dz}\exp(i\tilde{t})\\ i\dfrac{d(\sec h(\tilde{z})}{dz}\exp(-i\tilde{t}) & i\,|\sec h(z,t)|^2\end{pmatrix}\begin{pmatrix}\psi_1(z,t)\\ \psi_2(z,t)\end{pmatrix} \quad . \qquad (48)$$

The time development of $\begin{pmatrix}\psi_1(z)\\ \psi_2(z)\end{pmatrix}$ according to Eq. (48) is satisfied by substituting

$$\begin{pmatrix}\psi_1(\tilde{z},\tilde{t})\\ \psi_2(\tilde{z},\tilde{t})\end{pmatrix}=\begin{pmatrix}\sec h(\tilde{z})\exp(-i\tilde{t})\\ \tanh(\tilde{z})\end{pmatrix} \quad . \qquad (49)$$

obtaining the agreement

$$\frac{\partial}{\partial\tilde{t}}\begin{pmatrix}\sec h(\tilde{z})\exp(-i\tilde{t})\\ \tanh(\tilde{z})\end{pmatrix}=\begin{pmatrix}-i\sec h(\tilde{z})\exp(-i\tilde{t})\\ 0\end{pmatrix} =$$

$$\begin{pmatrix}-i\sec h^2(\tilde{z}) & i\dfrac{d(\sec h(\tilde{z})}{dz}\exp(-i\tilde{t})\\ i\dfrac{d(\sec h(\tilde{z})}{dz}\exp(i\tilde{t}) & i\sec h^2(z,t)\end{pmatrix}\begin{pmatrix}\sec h(\tilde{z})\exp(-i\tilde{t})\\ \tanh(\tilde{z})\end{pmatrix} \quad . \qquad (50)$$

Here we have used the relations

$$-i\sec h^3(\tilde{z})\exp(-i\tilde{t}) - i\sec h(\tilde{z})\tanh^2(\tilde{z})\exp(-i\tilde{t}) = -i\sec h(\tilde{z})\exp(-i\tilde{t}) \quad ;$$
$$-i\sec h^2(\tilde{z})\tanh(\tilde{z}) + i\sec h^2(\tilde{z})\tanh(\tilde{z}) = 0 \quad . \quad (51)$$

The value of $u(\tilde{z},t)$ has been given according to Eq. (43) while $\begin{pmatrix}\psi_1(z,t)\\ \psi_2(z,t)\end{pmatrix}$ have been chosen so that Eq. (34) is satisfied

6. Summary and conclusions

In the present study we have analyzed QM effects for electromagnetic (EM) waves which satisfy the one-soliton solution of NLSE in a dispersive wave guide using quantum optics methods. We have treated one mode of the EM field which includes a coupling between momentum and frequencies due to dispersion relation. For this purpose a coupled Hamiltonian-Momentum operator with equal-space commutation relations (CR) has been used. The present treatment of the one-soliton state includes QM effects which are developed in analogy with the classical theory. By the Kerr interaction a photon number distribution is obtained with corresponding quantum phases

We have analyzed the relation between the integrability of NLSE and the *compatibility conditio*n by using two equations where one of these equations is related to Hamiltonian and the other to Momentum Hermitian operators. It has been shown that by choosing special Hamiltonian and Momentum matrices the *compatibility-conditio*n leads to NLSE. We have shown the agreement between the integrability condition and the one-soliton solution .

References

[1] H. B. Thacker, Exact integrability in quantum field theory and statistical systems, Rev.Mod.Phys. , 53 (1981), 253-285.

[2] Y. Lai and H . A . Haus, Quantum theory of solitons, Phys.Rev. A, 40 **(**1989**)**, 844-866.

[3] B.Huttner, S. Serulnik and Y.Ben-Aryeh, Quantum analysis of light propagation in a parametric amplifier, Phys.Rev. A, 42 (1990), 5594-5600.

[4] Y. Ben-Aryeh, A. Lux and V. Perinova The concept of equal space commutators in quantum optics, Phys.Lett. A , 165 (1992), 19-27.

[5] Y. Ben-Aryeh, Evanescent waves and super-resolution effects, International Journal of Quantum Information, 3 (1992), 111-115.

[6] H. A. Haus , Electromagnetic Noise and Quantum Optical Measurements, Springer, Berlin (2000).

[7] H. A. Haus and Y.Lai , Quantum theory of soliton squeezing: a linearized approach, J.Opt.Soc.Am. B, 7 **(**1990**),** 386-392.

[8] M. Kitawaga and Y .Yamamoto, Number-phase minimum-uncertainty with reduced number uncertainty in a Kerr nonlinear interferometer, Phys. Rev A, 34 (1986), 3794-3988.

[9] D. K. Mishra, Study of higher order non-classical properties of squeezed Kerr state, Optics Communications, 283 (2010), 3284-3290.

[10] K.Sundar, Amplitude-squeezed quantum states produced by the evolution of a quadrature-squeezed coherent state in a Kerr medium, Phys.Rev A, 53 (1996), 1096-1111.

[11] G .Ariunbold and J.Perina, Quantum statistics of contradirectional Kerr nonlinear couplers, Optics Communications, 176 (2000)**,** 149-154.

[12] F. El-Orany and J. Perina, Higher-order squeezing for the codirectional Kerr nonlinear coupler, Physics Letters A, 333 (2004), + 204-211.

[13] M.Toren and Y. Ben-Aryeh, The problem of propagation in quantum optics, with applications to amplification, coupling of EM modes and distributed feedback lasers, Quantum Optics, 6 (1994), 425-443.

[14] J. Perina and J. Jr. Perina, Quantum statistics and dynamics of nonlinear Couplers, J.Mod. Opt. , 43 (1996), 1951-1971.

[15] M. Remoissenet, Waves called solitons: concepts and experiments, Springer, Berlin (1996)

[16] T. Eguchi, P. B. Gilkey and A. J. Hanson, Gravitation, gauge Theories and differential geometry, Phys.Rep., 66 (1980), 213-393.

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