Rotational Invariance as an Additional Constraint on Entanglement

Koji Nagata

Future University Hakodate
kojimna@yahoo.co.jp

Tadao Nakamura

Keio University Science and Technology

Abstract

Recently, separability inequalities are derived with multiqubit states [S. M. Roy, Phys. Rev. Lett. 94, 010402 (2005)]. We introduce new kind of entanglement, which is rotationally invariant. We derive quadratic separability inequalities with multipartite states. The quadratic separability inequality can be used as a witness of rotationally invariant two-partite entanglement. We discuss that the quadratic separability inequality implies standard Bell inequalities. It is also proved that when the two measured observables are assumed to precisely anticommute, a stronger quadratic inequality can be used as a witness of rotationally variant two-partite entanglement. We discuss that the stronger quadratic inequality implies an inequality stronger than standard Bell inequalities. Our argumentations imply that standard Bell inequalities are derived under the assumption that every quantum state is rotationally invariant. We discuss the implication of anticommute along with rotational invariance.

PACS: 03.67.Mn, 03.65.Ca

1 Introduction

The results contained in this paper are minor modifications of another paper [1] where the presenter of this paper is one of the co-authors.
We explain what are the main progress in comparison to the paper [1] is
done as follows.

The paper [1] mainly aimed to present argumentation to detect full n-
partite entanglement. This paper adds how to present argumentation to detect
two-partite entanglement.

We discuss the relation between the Roy work [2] and the paper [1]. We
discuss that we can derive the Roy inequality from the argumentation presented
in [1].

We discuss importance of the Roy work by using rotational invariance.
So we distinguish entanglement. One is rotationally invariant entanglement.
Another is rotationally variant entanglement.

We discuss the implication of anticommute. It means that the quantum
theory is described by space with the existence of the orientation of reference
frames. We agree. Therefore we can consider rotationally variant entangle-
ment. If the quantum theory would be described by space without the existence
of the orientation of reference frames, we doubt the validity of anticommute
proposition. In this case, only rotationally invariant entanglement exists.

Quantum information theory [3, 4] relies on utilizing entangled state. Also
there is much research of nature of entangled states related to local realistic
theories [5, 6, 7]. Separable state and entangled state were defined in 1989
[8]. And it was discussed very much which state is separable or entangled
[9, 10, 11, 12, 13, 14, 15, 16, 17].

The enormous research of quantum information theory [3, 4] relies on utilizing rotationally variant entangled pure state (See (1)). The method of de-
tection of this kind of entanglement is discussed very much (cf. [18]).

Assume $\rho$ is usual rotationally variant $n$-partite entangled state along with
the quantum theory. Then,

$$U^1 \otimes U^2 \otimes \cdots \otimes U^n \rho U^{1\dagger} \otimes U^{2\dagger} \otimes \cdots \otimes U^{n\dagger} \neq \rho,$$

where $U^j$ is some local unitary operator with respect to party $j$. Therefore, $\rho$
does depend on the orientation of reference frames of each of local observers.
Here, we introduce new kind of entanglement. Assume $\rho$ is rotationally invari-
ant $n$-partite entangled state. Then,

$$U^1 \otimes U^2 \otimes \cdots \otimes U^n \rho U^{1\dagger} \otimes U^{2\dagger} \otimes \cdots \otimes U^{n\dagger} = \rho,$$

where $U^j$ is any local unitary operator with respect to party $j$. Therefore,
$\rho$ does not depend on the orientation of reference frames of each of local ob-servers.
Much experimental data reports detection of both kind of entanglement successfully [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

Recently, it is shown [2] that separability of $n$-qubit quantum states implies new inequalities on Bell correlations which are stronger than standard Bell inequalities [31, 33, 34, 35, 36, 32] by a factor of $2^{(n-1)/2}$. The assumption to derive such inequalities is that the system is described by rotationally invariant separable state. Thus, a violation of such inequalities implies rotationally invariant entanglement even though it is not full multiqubit rotationally invariant entanglement. We see that these separability inequalities are useful to detect rotationally invariant two-partite entanglement.

In most of the real experiments, we have to deal with higher dimensional systems rather than qubit systems. For example, when polarizations of photons from a nonideal source are measured by imperfect detectors, it is difficult to claim strictly that the observed correlations are obtained by measuring subsystems with only two-dimensional Hilbert spaces, due to the ambiguity in the number of photons. The arguments about higher dimensional systems will thus be necessary in order to establish tests applicable to real experiments without making auxiliary assumptions as to the dimension of the measured space or as to measured observables [1].

In this paper, we introduce new kind of entanglement, which is rotationally invariant. We derive quadratic separability inequalities with multipartite states in an arbitrary dimensional space. Our methodology enables us to derive a quadratic separability inequality as tests for rotationally invariant two-partite entanglement in various Bell-type correlation experiments on the systems that may not be identified as a collection of qubits, e.g., those involving photons measured by incomplete detectors. We discuss that the quadratic separability inequality implies standard Bell inequalities. It is also proved that when the two measured observables are assumed to precisely anticommute, a stronger quadratic inequality can be used as a witness of rotationally variant two-partite entanglement. We discuss that the stronger quadratic inequality implies an inequality stronger than standard Bell inequalities. Our arguments imply that standard Bell inequalities are derived under the assumption that every quantum state is rotationally invariant. Further, we see the implication of anticommute. It means that the quantum theory is described by space with the existence of the orientation of reference frames. If the quantum theory would be described by space without the existence of the orientation of reference frames, we doubt the validity of anticommute proposition.
2 Detection of multipartite rotationally invariant entanglement

In what follows, we derive tight quadratic inequalities for hybrid separable-inseparable states with respect to partition $\alpha_1, \ldots, \alpha_k$ of an arbitrary dimensional space. We assume that for each particle $j$, either of two observables $A_j$ or $A'_j$ is chosen and is measured, where $-1 \leq A_j, A'_j \leq 1, \forall j$.

The Bell-Mermin operators take a simple form when we view on a complex plane using a function $f(x, y) = \frac{1}{\sqrt{2}}e^{-i\pi/4}(x + iy), x, y \in \mathbb{R}$. Note that this function is invertible, as $x = \Re f - \Im f, y = \Re f + \Im f$. The Bell-Mermin operators $B_{N_n}$ and $B'_{N_n}$ are defined by [34, 35, 36, 31, 32]

$$f(B_{N_n}, B'_{N_n}) = \otimes_{j=1}^n f(A_j, A'_j), \quad (3)$$

where $N_n = \{1, 2, \ldots, n\}$. We also define $B_{\alpha}$ for any subset $\alpha \subset N_n$ by

$$f(B_{\alpha}, B'_{\alpha}) = \otimes_{j \in \alpha} f(A_j, A'_j). \quad (4)$$

It is easy to see, when $\alpha, \beta(\subset N_n)$ are disjoint, that

$$f(B_{\alpha \cup \beta}, B'_{\alpha \cup \beta}) = f(B_{\alpha}, B'_{\alpha}) \otimes f(B_{\beta}, B'_{\beta}), \quad (5)$$

which leads to

$$B_{\alpha \cup \beta} = 1/2(B_{\alpha}B'_{\beta} + B'_{\alpha}B_{\beta}) + 1/2(B_{\alpha}B_{\beta} - B'_{\alpha}B'_{\beta}),$$

$$B'_{\alpha \cup \beta} = 1/2(B_{\alpha}B'_{\beta} + B'_{\alpha}B_{\beta}) - 1/2(B_{\alpha}B_{\beta} - B'_{\alpha}B'_{\beta}). \quad (6)$$

We calculate an upper bound of $\langle B_{N_n} \rangle^2 + \langle B'_{N_n} \rangle^2$ for states of the form $\otimes_{i=1}^k \rho^{\alpha_i}$. We then conclude [1] that, for any state $\rho$ that is $k$-separable with respect to $\alpha_1, \ldots, \alpha_k$,

$$\langle B_{N_n} \rangle^2 + \langle B'_{N_n} \rangle^2 \leq 2^{n+m-2k+1}. \quad (7)$$

Here $m$ is the number of particles which are not entangled with any particles. The maximum depends only on two parameters $k$ and $m$ but not on the detailed configuration of the partition. Clearly, the bound (7) is optimal.

The inequality for testing full n-partite entanglement for $n \geq 3$ is obtained by maximizing the right-hand side of (7) under the condition $k \geq 2$. Noting that $m \leq k - 1$ when $k < n$, we obtain

$$\langle B_{N_n} \rangle^2 + \langle B'_{N_n} \rangle^2 \leq 2^{n-2}. \quad (8)$$

Violations of the inequality (8) imply rotationally invariant full $n$-partite entanglement. The same inequality (8) was derived in Ref. [37].
3 Detection of rotationally invariant entanglement

The inequality for testing rotationally invariant two-partite entanglement is obtained by specific case of the right-hand side of (7) under the condition $k = n$ and $m = k - 1$. We have

$$\langle B_{N_n} \rangle^2 + \langle B'_{N_n} \rangle^2 \leq 1 \quad (9)$$

Violations of the inequality (9) imply rotationally invariant two-partite entanglement.

The inequality (9) also implies

$$|\text{tr}[\rho B_{N_n}]| \leq 1. \quad (10)$$

It is known that $|\text{tr}[\rho B_{N_n}]| \leq 1$ when the system is fully separable [34, 35, 36]. This inequality on Bell correlations is same as standard Bell inequalities [31, 33, 34, 35, 36, 32]. Hence, a violation of a Bell inequality implies rotationally invariant two-partite entanglement.

4 Detection of rotationally variant entanglement

In previous section, we derived the threshold value (i.e., 1) of $\langle B_{N_n} \rangle^2 + \langle B'_{N_n} \rangle^2$ for use as rotationally invariant two-partite entanglement witness over all observables satisfying $-1 \leq A_j, A'_j \leq 1$. Now, let us consider an additional assumption that the two measured observables anticommute, i.e., $\{A_j, A'_j\} = 0 \forall j$. This assumption introduces the orientation of reference frames in measurement setting apparatus. This assumption is approximately fulfilled within the accuracy of the measurement apparatus in the common experimental situations, e.g., when we measure Pauli operators $\sigma_x$ and $\sigma_y$ for each particle. With this assumption, the threshold value of $\langle B_{N_n} \rangle^2 + \langle B'_{N_n} \rangle^2$ becomes as small as $2^{1-n}$ as is shown below. This implies that we can use a stronger quadratic inequality as tests for rotationally variant two-partite entanglement in this case.

Suppose that $\{A_j, A'_j\} = 0$ and $-1 \leq A_j, A'_j \leq 1, \forall j$. We obtain

$$\langle B_{\alpha} \rangle^2 + \langle B'_{\alpha} \rangle^2 \leq 2^{|\alpha|-1}, (|\alpha| \geq 1). \quad (11)$$
Similar to the argument that derives (7), applying the relation (11), we conclude [1]

\[(\text{tr}[\rho B_{N_n}])^2 + (\text{tr}[\rho B'_{N_n}])^2 \leq 2^{n-2k+1}.\] (12)

The inequality for testing rotationally variant two-partite entanglement is obtained by the specific case of the right-hand side of (12) under the condition \(k = n\). We obtain

\[\langle B_{N_n} \rangle^2 + \langle B'_{N_n} \rangle^2 \leq 2^{1-n}.\] (13)

We give an example that the relation (13) is stronger than (9) as a witness of rotationally variant two-partite entanglement for multiqubit systems. We assume that \(A_j = \vec{a}_j \cdot \vec{\sigma}, A'_j = \vec{a}'_j \cdot \vec{\sigma}\), where \(\vec{a}_j\) and \(\vec{a}'_j\) are normalized vectors in \(\mathbb{R}^3\) and \(\vec{\sigma}\) is the vector of Pauli matrices, i.e., \(\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)\). The condition \(\{A_j, A'_j\} = 0\) leads to \(\vec{a}_j \cdot \vec{a}'_j = 0\). Let us consider the following multiqubit states [10]:

\[\rho = x|\Phi_n\rangle\langle\Phi_n| + \frac{1-x}{2^n}I,\] (14)

where \(I\) is the identity operator for the \(2^n\)-dimensional space and \(|\Phi_n\rangle\) is an \(n\)-qubit GHZ state [38], i.e.,

\[|\Phi_n\rangle = \frac{1}{\sqrt{2}}(|+1, +2, \ldots, +n\rangle + |−1, −2, \ldots, −n\rangle).\] (15)

It is easy to see that the maximum of \(\langle B_{N_n} \rangle^2 + \langle B'_{N_n} \rangle^2\) is \(2^{n-1}x^2\) with \(\vec{a}_j \cdot \vec{a}'_j = 0 \forall j\) (See [34, 35, 36]). Hence, assuming that \(x\) is in the range of

\[2^{(1-n)} < x \leq \sqrt{2^{1-n}},\] (16)

we can confirm rotationally variant two-partite entanglement from (13), which cannot be confirmed by (9). Hence if the measurement setups are precisely chosen as \(\{A_j, A'_j\} = 0 \forall j\), then one can use a stronger inequality as tests for rotationally variant two-partite entanglement in comparison with the relation (9). The strength grows exponentially with the number of particles \(n\).

The inequality (13) also implies

\[|\text{tr}[\rho B_{N_n}]| \leq \sqrt{2^{1-n}}.\] (17)

This inequality on Bell correlations is stronger than standard Bell inequalities [31, 33, 34, 35, 36, 32] by a factor of \(2^{(n-1)/2}\).
5 Conclusions

In conclusion, we have introduced new kind of entanglement, which is rotationally invariant. We have derived quadratic separability inequalities with multipartite states in an arbitrary dimensional space. Our methodology has enabled us to derive a quadratic separability inequality as tests for rotationally invariant two-partite entanglement in various Bell-type correlation experiments on the systems that may not be identified as a collection of qubits, e.g., those involving photons measured by incomplete detectors. We have discussed that the quadratic separability inequality implies standard Bell inequalities. It has been also proved that when the two measured observables are assumed to precisely anticommute, a stronger quadratic inequality can be used as a witness of rotationally variant two-partite entanglement. We have discussed that the stronger quadratic inequality implies an inequality stronger than standard Bell inequalities. Our argumentations have implied that standard Bell inequalities are derived under the assumption that every quantum state is rotationally invariant. Further, we have seen the implication of anticommute. It has meant that the quantum theory is described by space with the existence of the orientation of reference frames. If the quantum theory would be described by space without the existence of the orientation of reference frames, we doubt the validity of anticommute proposition.

In real experimental situations, we cannot claim that \( \{A_j, A'_j\} = 0 \) with arbitrary precision. The relevance of the bounds claiming two-partite entanglement assuming that \( \langle\{A_j, A'_j\}\rangle \leq \epsilon \), where \( \epsilon \) means experimental errors, would be worth further investigations.

References


Rotational invariance as an additional constraint on entanglement


Received: September, 2010