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Solutions of Duffing - van der Pol Equation Using Decomposition Method

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Abstract

In this paper, Duffing-van der Pol (DVP) differential equation is solved as a second order nonlinear ordinary differential equation by Adomian's decomposition method (ADM), then the equation is converted to a system of first order differential equations and solved using the same method. The Lindsted's method (LM) is used to compare the solutions of ADM and show that converting the differential equation to a system of equations in Adomian's method, gives more accurate answers in a shorter time of computations.

Keywords: Adomian's decomposition method, Duffing - van-der Pol differential equation, system of differential equations.

1 Introduction

Since the beginning of the 1980s, Adomian [1,2] has presented and developed a so-called decomposition method for solving linear and nonlinear problems such as ordinary differential equations and integral equations. Adomian decomposition method consists of splitting the given equation into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, identifying the initial and/or boundary conditions and the terms involving the independent variable alone as

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initial approximation, decomposing the nonlinear function in terms of special polynomials called Adomian's polynomials, and finding the successive terms of the series solution by recurrent relation using Adomian's polynomials.

Several authors have proposed a variety of modifications to ADM. Wazwaz proposed a powerful modification of ADM that accelerates the rapid convergence of the series solution [3, 4]. E. Babolian et al. introduced the restart method to solve the equation $f(x)=0$ [5], and the integral equations [6]. H. Jafari et. al used a correction of decomposition method for ordinary and nonlinear systems of equations and show that the correction accelerates the convergence [7,8].

The classical DVP oscillator appears in many physical problems and is governed by the nonlinear differential equation

$$\ddot{x}-\mu(1-x^2)\dot{x}+x+\alpha x^3=0; \quad x(0)=x_0,\, \dot{x}(0)=\dot{x}_0, \tag{1}$$

where the overdot represents the derivative with respect to time, μ and α are two positive coefficients. It describes electrical circuits and has many applications in science, engineering and also displays a rich variety of nonlinear dynamical behaviors.

In this paper, equation (1) is solved directly, using ADM. Then we convert (1) to a system of first order differential equations and solved the system using ADM. As a criterion, to compare the solutions, we used the solution obtained by LM.

In order to compare two applied Adomian’s scheme, we calculated the running time of the programs need to obtain equal order polynomial solutions in the approaches.

2 General description of ADM

Consider the functional equation

$$Fu=g \tag{2}$$

We need to find u such that fulfill the equation (2). Suppose $F=L+R+N$ that L and R are invertible and noninvertible linear parts of F and N is it’s nonlinear part.

So, equation (2) takes the form

$$(L+R+N)u=g \tag{3}$$

Defining L^{-1} as the inverse of the operator L and applying it on both sides of (3), we obtain

$$u=L^{-1}g-L^{-1}Ru-L^{-1}Nu \tag{4}$$

The basis of ADM is considering u as the series

$$u = \sum_{n=0}^{\infty} u_n \tag{5}$$

And defining the nonlinear term as the sum of Adomian's polynomials

$$Nu = \sum_{n=0}^{\infty} A_n (u_o, \ldots, u_n) \tag{6}$$

Suppose $\sum_{n=0}^{\infty} u_n$ is a convergent series and $Nu = f(u)$ that f is an analytic function, then we have

$$f\,(\sum_{n=0}^{\infty} u_n \lambda^n) = \sum_{n=0}^{\infty} A_n \lambda^n ,$$

derivating n times with respect to λ, yields

$$\frac{d^n}{d\lambda^n} f(\sum_{n=0}^{\infty} u_n \lambda^n)\,|_{\lambda=0} = n!\,A_n ,$$

from which we obtain the Adomian's polynomial A_n as follow

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} f(\sum_{n=0}^{\infty} u_n \lambda^n)\,|_{\lambda=0} \tag{7}$$

Using (4) and (5) in (3), one obtain

$$\sum_{n=0}^{\infty} u_n = L^{-1}g - L^{-1}R(\sum_{n=0}^{\infty} u_n) - L^{-1}N(\sum_{n=0}^{\infty} A_n) \tag{8}$$

Choosing $u_0 = L^{-1}g$, from (7) we obtain

$$u_1 = -L^{-1}Ru_0 - L^{-1}A_0$$

$$u_2 = -L^{-1}Ru_1 - L^{-1}A_1$$

$$\vdots$$

$$u_n = -L^{-1}Ru_{n-1} - L^{-1}A_{n-1}$$

As $u = \sum_{n=0}^{\infty} u_n$ is a convergent series, we have an approximate series solution of equation (2) as φ_n

$$\varphi_n = \sum_{i=0}^{n-1} u_i \quad , \quad \lim_{n\to\infty} \varphi_n = u \tag{9}$$

3 ADM to a system of ODEs

Most applied problems are described by second-order or higher-order differential equations. A differential equation of order n, can be written as

$$u^{(n)} = f(t, u, u', \ldots, u^{(n-1)}), \quad u^{(n)}(0) = u_{0,n}, \quad n \geq 2 \tag{10}$$

Using $u^{(i)} = y_{i+1}$ this equation converts to a system of ordinary first-order differential equations as follow

$$\begin{aligned} y'_1 &= f_1(t, y_1, \ldots, y_n) \\ y'_2 &= f_2(t, y_1, \ldots, y_n) \\ &\vdots \\ y'_n &= f_n(t, y_1, \ldots, y_n) \end{aligned} \tag{11}$$

where each equation represents the first derivative of a function as a map depending on the independent variable x, and n unknown functions $f_1, \ldots, f_n$. Defining the operator L as the first order derivative with respect to t, then $i-th$ equation of the system (11) can be represented as the common form

$$Ly_i = f_i(t, y_1, \ldots, y_n)$$

Applying the inverse of L, $L^{-1}(.) = \int_0^t (.)dt$, the equation (10) can be written as

$$y_i = y_i(0) + \int_0^t (L_i(t, y_1, \ldots, y_n) + N_i(t, y_1, \ldots, y_n))dt \tag{12}$$

that called canonical form in Adomain schema. In order to apply ADM, we let

$$y_i = \sum_{j=0}^{\infty} y_{ij} \tag{13}$$

$$L_i(t, y_1, \ldots, y_n) = \sum_{k=1}^{n} \sum_{j=0}^{\infty} a_k y_{kj} \tag{14}$$

$$N_i(t, y_1, \ldots, y_n) = \sum_{j=0}^{\infty} A_{ij}(f_{i0}, \ldots, f_{ij}) \tag{15}$$

where $a_k, k = 0,1,\ldots,n$ are scalars.
Substituting (13), (14) and (15) into (12), we have

$$\sum_{j=1}^{\infty} y_{ij} = y_1(0) + \int_0^t \sum_{k=1}^{n} \sum_{j=0}^{\infty} a_k y_{kj} dt + \int_0^t \sum_{j=0}^{\infty} A_{ij}(f_{i0}, \ldots, f_{ij}) dt$$

from which, we define

$$y_{i0} = y_i(0) \tag{16}$$

$$y_{i,j+1} = \int_0^t \sum_{k=1}^{n} a_k y_{kj} dt + \int_0^t A_{ij}(f_{i0}, \ldots, f_{ij}) dt \quad , \quad j = 0,1,\ldots \ . \tag{17}$$

In practice, all terms of series (13), cannot be determined. So we consider approximate solution, calculating following truncated series

$$\varphi_{ik} \approx y_i(t) = \sum_{m=0}^{k-1} y_{ij}(t) , \qquad \lim_{k \to \infty} \varphi_{ik} = y_i(t) . \tag{18}$$

Our procedure leads to a system of second kind Volterra integral equations, so referring to [9] convergence of the method is proved.

4 The problem and solutions

DVP equation has the common form given in (1). We consider a special version of this problem as follow

$$\ddot{x} - 0.1(1 - x^2)\dot{x} + x + 0.01x^3 = 0 \quad ; \quad x(0) = 2 \quad , \quad \dot{x}(0) = 0, \tag{19}$$

Defining $y_1 = x$ and $y_2 = \dot{x}$, the problem (19) converts to the problem system of differential equations

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = 0.1(1 - y_1^2) y_2 - y_1 - 0.01 y_1^3 \end{cases} \quad ; \quad y_1(0) = 2 \quad , \quad y_2(0) = 0 \tag{20}$$

In this section equation (19) and system of equations (20) are solved using ADM as described in sections 2 and 3. As there isn't any exact solution to compare these approaches of ADM, we use Lindsted's perturbation method, as an approximate analytical method to obtain an acceptable solution as a criterion of comparison.

In LM, a solution by the form $x_0(\tau)+\mu x_1(\tau)+\mu^2 x_2(\tau)+\cdots$ uses to convert the problem to a set of solvable differential equations. Problem (19) is solved using this method to obtain the solution as follow

$$x(t) = A\cos\omega t + \frac{\alpha}{4}\cos 3\omega t + \mu(\frac{3}{4}\sin\omega t - \frac{1}{4}\sin 3\omega t) + O(\mu^2) \tag{21}$$

with $A = 2-\frac{1}{2}\alpha,\ \omega = 1+\frac{3}{2}\alpha-\frac{27}{16}\alpha^2-\frac{1}{16}\mu^2+O(\mu^2).$

A detailed description of LM is presented in [10].

4.1 ADM to (19)

Following the procedure explained in section 2 and considering 5 terms of (9) we obtain the polynomial

$$x_{ADM}(t) \approx \varphi_4(t) = 2-1.04t^2+0.104t^3+0.089t^4+\cdots+0.00053t^8 \tag{22}$$

as approximate solution of (1). Note that this polynomial is of order 8. running time of the program that used to obtain (22) on a laptop with 1GB of ram with a 2.00GHz CPU, is 0.313 seconds. The plot of $x(t)$ and $x_{ADM}(t)$ in the figure 2 shows that the solution of ADM for DVP problem diverges for $t>1$.

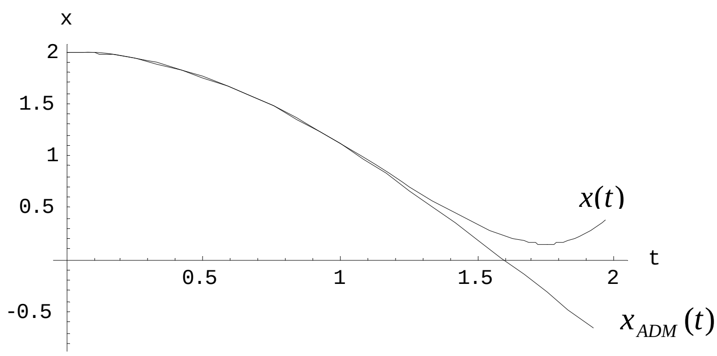

Figure 2-Plot of $x(t)$ and $x_{ADM}(t)$

Numerical results of $x(t)$ and $x_{ADM}(t)$ for $0 \le t \le 2$ and absolute errors of ADM are listed in table 1.

Table 1. Numerical results and errors of ADM.

t	LM	ADM	Abs. Error
0.0	2.00000	1.99750	0.00250
0.1	1.98971	1.98724	0.00247
0.2	1.95936	1.95697	0.00239
0.3	1.90980	1.90758	0.00222
0.4	1.84202	1.84008	0.00193
0.5	1.75705	1.75552	0.00153
0.6	1.65598	1.65493	0.00105
0.7	1.53999	1.53937	0.00062
0.8	1.41039	1.40982	0.00056
0.9	1.26872	1.26726	0.00146
1.0	1.11696	1.11267	0.00429
1.1	0.95770	0.94704	0.01065
1.2	0.79445	0.77147	0.02298
1.3	0.63206	0.58715	0.04491
1.4	0.47712	0.39545	0.08166
1.5	0.33856	0.19795	0.14061

4.2 ADM to (20)

Following the procedure of section 2, and considering 9 terms of (18) we obtain the following 8-th order polynomial as approximate solution of (20).

$$x_{SADM}(t) \approx \varphi_4(t) = 2 - 1.04t^2 + 0.104t^3 + 0.089t^4 + \cdots - 0.0048t^8 \tag{23}$$

By this approach the running time was 0.109 seconds.

The plot of $x(t)$ is presented in the figure 2 and shows that the solution of ADM for VDPD equation diverges for $t > 1$.

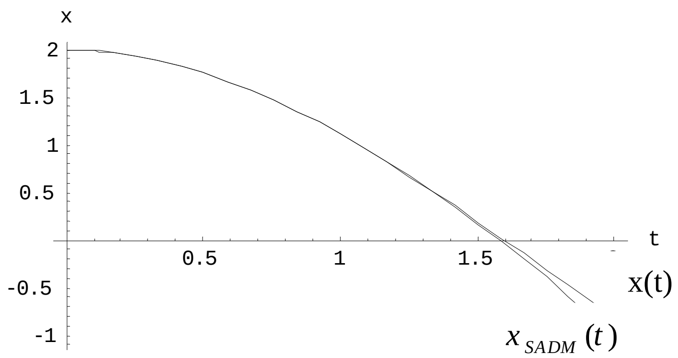

Figure - Plot of $x(t)$ and $x_{SADM}(t)$

Numerical results of $x(t)$ and $x_{SADM}(t)$ for $0 \le t \le 2$ and absolute errors of ADM are listed in table 2.

Table 2. Numerical results and errors of ADM.

t	LM	ADM	Abs. Error
0.0	2.00000	1.99750	0.00250
0.1	1.98971	1.98724	0.00247
0.2	1.95936	1.95697	0.00239
0.3	1.90980	1.90758	0.00222
0.4	1.84202	1.84008	0.00193
0.5	1.75702	1.75552	0.00150
0.6	1.65586	1.65493	0.00092
0.7	1.53958	1.53937	0.00021
0.8	1.40922	1.40982	0.00059
0.9	1.26581	1.26726	0.00145
1.0	1.11033	1.11267	0.00233
1.1	0.94373	0.94704	0.00330
1.2	0.76686	0.77147	0.00460
1.3	0.58037	0.58715	0.00677
1.4	0.38462	0.39545	0.01083
1.5	0.17946	0.19795	0.01848

Comparing the errors of two Adomian approaches in tables 1 and 2, we see that ADM for a nonlinear differential equation and it's related system of equations, gives the results with equal errors in the beginning of the solution interval, while in the end of the interval, converting the equation to a system of differential equations, tends to more accurate solutions.

5 Conclutions

The study show that ADM can be used to solve DVP problem and obtain the solutions by acceptable errors. Converting the equation to a system of first order differential equations and solving the system by ADM, gives more accurate results in a lower time of computations in comparison with direct application of ADM. This can be relates to lowering the calculations by reducing the order of differential equations.

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