Similarity Solutions for Relativistic Accelerating Fluid Plates of Embedding Class One Using Symbolic Computation

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Abstract

A non linear partial differential equation in general relativity is solved by the Lie continuous point group similarity transform method using symbolic computation. Four new similarity solutions for relativistic plane symmetric accelerating perfect fluid distribution of embedding class one are derived through symbolic algebra packages. The solutions so obtained are also analyzed physically. The energy density ($\rho$) and pressure ($p$) satisfies the physical conditions ($\rho \geq p > 0$ for $t \geq 0$).

Keywords: symbolic computation; relativistic accelerating fluid; similarity solutions

1 Introduction

Since the inception of integral and differential calculus, the differential equations have played remarkable role in dealing with almost every physical situation. The physical problems pertaining to science and engineering, can
be expressed by ordinary and partial differential equations most of the time. The differential equations describing the realistic situations normally are of non-linear type and corresponding solution are not obtainable easily even by computer oriented methods. Many times, exact solutions of such equations are preferred over a numerical one. Quite commonly, exact analytical solution of system of partial differential equations in applied fields like fluid dynamics and general relativity are extremely valuable as they provide more insight into extreme cases which is not possible through numerical treatment. In this regard there have been some theories based upon coordinates – transformation which enables researchers to get exact solution or reduce the degree of difficulty by reducing the order of the equation. This approach essentially requires the solution of a system of determining equations which are, in general, coupled homogeneous partial differential equations. Finding the determining equations is quite a complex and tedious task. The number of these determining equations may sometime be very large. In such cases, finding the system of equations manually becomes quite daunting and some times untreatable.

Occurrence of large mathematical expressions in such problems is also quite frequent. There is a possible chance of human error while manipulating large analytical expressions manually. On the other hand, Computer Algebra Systems (CAS) are becoming a necessity in scientific computations wherever large algebraic expressions are to be manipulated. This is due to the fact that if the computations are carried out through CAS, the solution is not only achieved faster but also the chances of human errors are tremendously reduced.

CAS are computer programs which carry out symbolic as well as numerical computations of common mathematical expressions and are proven to be extremely useful in solution of many physical problems [Corless et al.4, Lange et al.8, Matsson and Nordstrom9, Scott et al.13]. Computer algebra can save both time and effort in solving a wide range of problems. In general, the solution obtained through CAS is much more accurate than the manual computations. A lot of problems, which were stopped in the past due to their huge size, can be solved using these systems [Corless et al.5, Pratibha10]. There are several such packages available, but most commonly used are Mathematica, MAPLE, Derive and Reduce. The utility of the MAPLE package in solving ordinary differential equations with regular singular points is demonstrated in Pratibha11. Here we demonstrate the use of symbolic algebra systems in obtaining new exact solution of highly non-linear partial differential equation, describing non conformal accelerating perfect fluid distribution of embedding class one, through the method of similarity transformation. Through the use of interactive user control of evaluation of expressions in MAPLE/Derive, we have minimized the problem of intermediate expression swell which makes the problems manageable.

Lately, some articles on the idea that space-time can be considered as a
four-dimensional space embedded in a higher dimensional flat space [Anchor-
doqui and Bergliaffa[1], Randal and Sundaram[12]] have renewed researchers’
interest in the topic where such spaces are related to Brane theory. It is well
known that the perfect fluid distributions of class one can be divided in two
categories. In the first category, which are conformally flat with vanishing
conformal curvature tensor (Petrov type – O), while the second category pos-
sesses non vanishing Weyl conformal curvature tensor or non-conformally flat
(Petrov type – D). All the Zeldowich fluids of second category were found in
Gupta[6]. A class of non-static analogues of Kohler-Chao solution of second
category is also obtained in Gupta[7].

We derive some new type – D relativistic fluid solutions of embedding class
one, defined in Barnes[2] to describe non-conformally flat plane symmetric
accelerating perfect fluid distributions using similarity transformation method
(STM) given by Bluman[3]. The solutions with non-zero acceleration are very
rare. In this regard we have obtained new solutions which possess acceleration.

2 The field equations for plane symmetry

A metric of embedding class one can be expressed as

\[ ds^2 = -(dz^1)^2 - (dz^2)^2 - (dz^3)^2 + (dz^4)^2 - (dz^5)^2 \]  

(1)

In order to introduce plane symmetry, let us set

\[ z^1 = t \theta \cos \phi, z^2 = t \theta \sin \phi, z^3 = \frac{\theta^2}{2} t + v, \]

\[ z^4 = (\frac{\theta^2}{2} + 1) t + v, z^5 = r, \]

\[ v = v(r, t) \]  

(2)

which transforms (1) to

\[ ds^2 = -dr^2 - t^2 (d\theta^2 + \theta^2 d\phi^2) + (1 + 2\dot{v}) dt^2 + 2v' dr dt \]  

(3)

where

\[ v' = \frac{\partial}{\partial r}, \quad \dot{v} = \frac{\partial}{\partial t} \]

Einstein’s field equations for perfect fluid distributions can be expressed as

\[ 8\pi T^i_j = -R^i_j + \frac{1}{2} \delta^i_j R = 8\pi [(p + \rho)v^i v_j - \delta^i_j p], \]  

(4)

where \( v^i v_i = 1 \) and \( \rho, p \) and \( v_i \) are energy density, pressure and flow vector
respectively. The Einstein’s field equations (4) for the metric (3) read as
\[ 8\pi T^1_1 = -\frac{2(\ddot{v} + v\dot{v}')}{tP^2} + \frac{1}{t^2 P} = 8\pi [(p + \rho)v^1v_1 - p] \quad (5) \]

\[ 8\pi T^2_2 = 8\pi T^3_3 = \frac{1}{P^2} \left[ \frac{1}{t} \left\{ (1 + 2\dot{v})v'' - \ddot{v} - 2v\dot{v}' \right\} + \dot{v}'^2 - \ddot{v}v'' \right] = -8\pi p \quad (6) \]

\[ 8\pi T^4_4 = \frac{2}{tP^2} \left[ (1 + 2\dot{v})v'' - v\dot{v}' \right] + \frac{1}{t^2 P} = 8\pi [(p + \rho)v^4v_4 - p] \quad (7) \]

\[ 8\pi T^1_4 = \frac{2}{tP^2} \left[ v'\ddot{v} - (1 + 2\dot{v})\dot{v}' \right] = 8\pi [(p + \rho)v^1v_4] \quad (8) \]

\[ 8\pi T^4_1 = \frac{2}{tP^2} \left[ v'\dot{v}'' + \dot{v}' \right] = 8\pi [(p + \rho)v^4v_1] \quad (9) \]

\[ v^1v_1 + v^4v_4 = 1, v^2 = v^3 = v^3 = 0 \quad (10) \]

where

\[ P = 1 + 2\dot{v} + v'^2 \quad (11) \]

However \( p, \rho \) are expressible in terms of \( T^i_j \)'s as

\[ p = -T^2_2, \quad \rho = T^1_1 + T^4_4 - T^2_2 \quad (12) \]

Eliminating \( v_i \) from equation (5) to (10) one gets

\[ (3w - 8\pi p)(w - 8\pi p) = 0 \quad (13) \]

where

\[ w = \frac{1}{Pt^2} \left[ \dot{v}'^2 - \ddot{v}v'' + \frac{1}{t} \left\{ (1 + 2\dot{v})v'' - \ddot{v} - 2v\dot{v}' \right\} + \frac{1}{t^2} (1 + 2\dot{v} + v'^2) \right] \]

**Case I:** \( 3w - 8\pi \rho = 0; \) this implies the vanishing of conformal curvature tensor and the corresponding fluid distributions are termed as conformally flat fluid distributions. All the solutions of this category have been exhausted by Stephani[14].

**Case II:** \( w - 8\pi p = 0; \) this implies the non-conformally flat fluid distributions, many of such solutions have been obtained as mentioned in the introduction but many are yet to be derived. For the later case we come across a partial differential equation to be satisfied by \( v(r, t) \) as

\[ t^2 \dot{v}'^2 - t^2 \ddot{v}v'' + t \left\{ (1 + 2\dot{v})v'' - \ddot{v} - 2v\dot{v}' \right\} + (1 + 2\dot{v} + v'^2) = 0, \quad (14) \]
and the expressions for pressure and density, using (12) and (14), can be given as

\[ 8\pi p = \frac{1}{t^2 P} \]  
\[ 8\pi \rho = \frac{1}{t^2 P} + \frac{2(\dot{v}v'' - \dot{v}^2)}{P^2} \]  

### 3 Solutions of the field equations

Similarity method of solving partial differential equations is extremely powerful to get the solutions even for a highly non-linear equation. In the process of getting the groups of transformation under which the given equation is to be invariant, we come across the Lie symmetries of equation (14) as follows:

Suppose that

\[ H = t^2 \dot{v}^2 - t^2 \ddot{v}v'' + t \{(1 + 2\dot{v})v'' - \dot{v} - 2v\dot{v}'\} + (1 + 2\dot{v} + v'^2) = 0 \]  

The similarity method requires the invariance of the above equation under the Lie group of transformations and hence needs to satisfy the following equation given in Bluman[3]

\[ \eta \left( \frac{\partial H}{\partial r} \right) + \xi \left( \frac{\partial H}{\partial t} \right) + \tau \left( \frac{\partial H}{\partial v} \right) = 0, \]  

where \( \eta, \xi, \tau, [\eta_r], [\eta_t], [\eta_{rr}], [\eta_{rt}], [\eta_{tt}] \) are infinitesimals with the associated derivatives of \( H \) and are given by the following expressions

\[ [\eta_r] = \frac{\partial \eta}{\partial r} + \left( \frac{\partial \eta}{\partial u} - \frac{\partial \xi}{\partial r} \right) \theta_r - \frac{\partial \tau}{\partial r} \theta_t - \frac{\partial \xi}{\partial u} \theta^2_r - \frac{\partial \tau}{\partial u} \theta_t \theta_r \]  

\[ [\eta_t] = \frac{\partial \eta}{\partial t} + \left( \frac{\partial \eta}{\partial u} - \frac{\partial \tau}{\partial t} \right) \theta_t - \frac{\partial \xi}{\partial t} \theta_r - \frac{\partial \tau}{\partial u} \theta^2_t - \frac{\partial \xi}{\partial u} \theta_t \theta_r \]  

\[ [\eta_{rr}] = \frac{\partial^2 \eta}{\partial r^2} + \left( 2 \frac{\partial^2 \eta}{\partial r \partial u} - \frac{\partial^2 \xi}{\partial r^2} \right) \theta_r - \frac{\partial^2 \tau}{\partial r^2} \theta_t + \left[ \frac{\partial^2 \eta}{\partial u^2} - 2 \frac{\partial \xi}{\partial r \partial u} \right] \theta^2_r \]  

\[-2 \frac{\partial^2 \tau}{\partial r \partial u} \theta_r \theta_t - \frac{\partial^2 \xi}{\partial u^2} \theta^3_r \theta_t \theta_r + \left( \frac{\partial \eta}{\partial u} - 2 \frac{\partial \xi}{\partial r} \right) \theta_{rt} \]  

\[-2 \frac{\partial \tau}{\partial r} \theta_{rt} - 3 \frac{\partial \xi}{\partial u} \theta_{r \theta \theta} - \frac{\partial \tau}{\partial u} \theta_{r \theta} \theta_t - 2 \frac{\partial \tau}{\partial u} \theta_{r \theta} \theta_r \]
\[ [\eta] = \frac{\partial^3 \eta}{\partial t^2} + \left( 2 \frac{\partial \eta}{\partial t} - \frac{\partial^2 \tau}{\partial t^2} \right) \theta_t - \frac{\partial^2 \xi}{\partial u^2} \theta_r + \left[ \frac{\partial^2 \eta}{\partial u^2} - 2 \frac{\partial^2 \tau}{\partial t \partial u} \right] \theta_t^2 - 2 \frac{\partial^2 \xi}{\partial t \partial u} \theta_t \theta_r - \frac{\partial^2 \xi}{\partial u^2} \theta_r^2 + \left( \frac{\partial \eta}{\partial u} - 2 \frac{\partial \tau}{\partial t} \right) \theta_{tt} \]
\[ -2 \frac{\partial \xi}{\partial t} \theta_{rt} - 3 \frac{\partial \tau}{\partial t} \theta_{tt} - \frac{\partial \xi}{\partial u} \theta_{tt} \theta_r - 2 \frac{\partial \xi}{\partial u} \theta_{rt} \theta_t \]

\[ [\eta_r] = \frac{\partial^2 \eta}{\partial t \partial r} + \left( \frac{\partial^2 \eta}{\partial r \partial u} - \frac{\partial^2 \tau}{\partial t \partial r} \right) \theta_t + \left[ \frac{\partial^2 \eta}{\partial u^2} - \frac{\partial^2 \xi}{\partial r \partial u} - \frac{\partial^2 \tau}{\partial t \partial u} \right] \theta_t \theta_r - \frac{\partial^2 \xi}{\partial u^2} \theta_r^2 \theta_t + \left( \frac{\partial \eta}{\partial r} - \frac{\partial \xi}{\partial r} - \frac{\partial \tau}{\partial t} \right) \theta_{tt} - \frac{\partial \xi}{\partial t} \theta_{rr} \]
\[ - \frac{\partial \tau}{\partial u} \theta_{rt} \theta_t - 2 \frac{\partial \xi}{\partial u} \theta_{rt} \theta_r - \frac{\partial \tau}{\partial u} \theta_{tt} \theta_r - \frac{\partial \xi}{\partial u} \theta_{rr} \theta_r \]

On making use of equation (17) and the above expressions (19) into (18), we obtain determining equations of the group by equating the coefficients of \( \theta \) and its derivatives to zero (Appendix A). Obtaining these equations is quite a tedious task, if carried out manually. Moreover, chances of human mistakes are always there. We obtain the determining equations using symbolic computation packages. On solving these equations, we get the expressions for infinitesimals \( (\xi, \eta, \tau) \) as

\[ \tau = \alpha t \]
\[ \xi = \alpha r + \beta t + \psi \]
\[ \gamma = \alpha v + \alpha t^3 + \beta r + \delta, \]

where \( \alpha, \beta, \psi, \delta \) and \( c \) are arbitrary constants.

Solution of the equation (14) can be obtained by solving the following equations:

\[ \frac{dr}{\alpha r + \beta t + \psi} = \frac{dt}{\alpha t} = \frac{dv}{\alpha v + \alpha t^3 + \beta r + \delta} \]

(21)

The following sub cases occur: (i) \( \alpha = 0 \), (ii) \( \alpha \neq 0 \)

Case 1: \( \alpha = 0 \),

Equation (21) yields
\[ v = f(t) + \frac{ct^3 + \delta}{\beta t + \psi} r + \frac{\beta r^2}{2(\beta t + \psi)} \]  \quad (22)

Inserting the equation (22) into (17), we get second order differential equation in \( f(t) \). The expression for \( f(t) \) are obtained as

\[
f(t) = \frac{c\delta(4\beta^3t^3 - 7\psi^3 - 7\beta t\psi^2)}{4\beta^4(\beta t + \psi)} + \frac{\delta^2}{2\beta(\beta t + \psi)} \frac{1}{(8\beta^3 t^3 + \psi^3)} + \frac{24\beta^3}{64\beta^7} + \frac{(240c^2\beta^4t^4\psi^2 - 640\beta^6t^2 - 320\beta^6\psi^2 - 600c^2\beta^3t^3\psi^3 + 64c^2\beta^6t^6 + 1227c^2\psi^6 + 144c^2\beta t^5\psi + 1347c^2\beta t\psi^5 - 960\beta^7t\psi)}{1280\beta^7(\beta t + \psi)}
\]

where \( C_1 \) and \( C_2 \) are constants of integration. Using MAPLE/Derive, we obtained expressions for pressure and density as

\[ 8\pi p = \frac{1}{t^2} [A(r, t)]^{-1}, \]  \quad (24)

and

\[ 8\pi \rho = \frac{1}{t^2} [A(r, t)]^{-1} \]

\[
8\delta^5 \left[ 8\beta^5(\beta t + \psi)(2\beta t + \psi)C_1 + 24\beta^4cr(2\beta t + \psi) - 9c^2\psi^2(2\beta t + \psi) \ln(2\beta t + \psi) - 24ct^4(\beta t^3 - 2\delta) \right] + \frac{6c\psi^2(5ct^3 - 4\delta) + 12c^2 t^2 \psi^2 \beta^2 - 18c^2 t^2 \psi^3 \beta - 12c^2 \psi^4}{t^3(2\beta t + \psi)} \]

\[
(9c^2 t^2 \psi \beta + 9c^2 \psi^3) \ln(2\beta t + \psi) + 6c \beta^3(3t^3 + 4\delta) + 6c t^2 \psi^2 \beta - 12c^2 \psi^3 \right] \]

where

\[ A(r, t) = \frac{6c^2 t^2 r}{(\beta t + \psi)} + \frac{6c t^2 \delta}{\beta(\beta t + \psi)} - \frac{9c^2 t^2 \psi^2 \ln(\psi + 2\beta t)}{4\beta^4} + \frac{3c^2 t^2 (\beta^3 t^3 + \psi^2 t \beta - 2\psi^3)}{2\beta^3(\beta t + \psi)} + 2t^2 C_1 \]

**Reduction to Normal Form:**

We developed a procedure in MAPLE/Derive to normalize the metric (3) to the following normal form

\[ ds^2 = -(\beta t + \psi)^2 dR^2 - t^2 d\sigma^2 - \chi(r, t) dt^2 \]  \quad (26)
through the transformation

\[ r = (\beta t + \psi)R + \frac{3c\psi^2(\beta t + \psi)\ln(\beta t + \psi)}{\beta^4} \]

\[ -\frac{\delta}{\beta} - \frac{c(t^2\beta^3 - 3t^2\beta^2\psi - 4t\beta\psi^2 + 2\psi^3)}{2\beta^4} \] (27)

The corresponding expressions for pressure and density are obtained and simplified as

\[ 8\pi p = \frac{1}{t^2} [\chi(r, t)]^{-1}, \] (28)

and

\[ 8\pi \rho = \frac{1}{t^2} [\chi(r, t)]^{-1} + [\chi(r, t)]^{-2} \left[ \frac{36tc^2\psi^2\ln(\beta t + \psi)}{\beta^3(\beta t + \psi)} - \frac{9tc^2\psi^2\ln(2\beta t + \psi)}{2\beta^3(\beta t + \psi)} + \frac{12Rtc\beta}{(\beta t + \psi)C_1} \right] \]

\[ + \frac{4t\beta}{(\beta t + \psi)C_1} - \frac{3c^2t\psi(5\beta t\psi + 15\beta^2t^2 - 2\psi^2)}{\beta^3(2\beta t + \psi)(\beta t + \psi)} \] (29)

where

\[ \chi(r, t) = \frac{18c^2\psi^2t^2\ln(\beta t + \psi)}{\beta^4} - \frac{9c^2\psi^2t^2\ln(2\beta t + \psi)}{4\beta^4} + 2t^2C_1 \]

\[ + 6Rt^2c + \frac{3c^2t^2(2\psi^2 + 3\beta^2t^2 - 9\beta t\psi)}{2\beta^4} \]

Deduction of \( \chi(r, t) \) is quite a tedious and time consuming task if carried out manually. The use of MAPLE/Derive procedure makes it convenient to find the above expressions. The velocity vector and acceleration are obtained as

\[ v^1 = \frac{\chi^{1/2}}{3ct^2} \left[ \frac{1}{t} - \frac{1}{2} \chi^{-1} \right] \left[ B(r, t) + \frac{1}{2} \dot{\chi}t \right]^{1/2} \] (30)

\[ v^2 = v^3 = 0 \] (31)

\[ v^4 = \chi^{-1/2} \left[ \frac{B(r, t) + \frac{1}{2} \dot{\chi}t}{B(r, t) + \chi} \right]^{1/2} \] (32)

where
\[
B(r, t) = \frac{\beta t \chi}{\beta t + \psi} + \frac{9c^2 t^6}{(\beta t + \psi)^2} \left[ 1 + \frac{t}{2\beta t + \psi} \right]
\]  
(33)

\[
\dot{v}_1 = \frac{t^2}{(2 + B(r, t))} \left[ \frac{B(r, t) + \frac{1}{2} \dot{x} t}{B(r, t) + \chi} \right] \left[ 3c + \frac{\chi}{3c t^6} (\beta t + \psi)^2 - \frac{1}{4} \dot{x}^2 \right]
\]  
(34)

\[
\dot{v}_2 = \dot{v}_3 = 0
\]  
(35)

\[
\dot{v}_4 = \frac{\chi(2 - t \dot{x} \chi^{-1})}{6c t(2 + B(r, t))} \left[ \frac{B(r, t) + \frac{1}{2} \dot{x} t}{B(r, t) + \chi} \right] \left[ 3c + \frac{\chi}{3c t^6} (\beta t + \psi)^2 - \frac{1}{4} \dot{x}^2 \chi^{-1} \right]
\]  
(36)

**Case (ii):** when \( \alpha \neq 0 \),

The equation (21) yields

\[
v = \frac{c}{2\alpha} t^3 - \frac{\beta^2}{2\alpha^2} t \log^2 |t| + \left( \frac{\beta \psi}{\alpha^2} - \frac{\delta}{\alpha} \right) + \frac{\beta r}{\alpha} \log |t| + \frac{\beta \varphi}{\alpha^2} \log |t|
\]  
\[+ t F \left[ \frac{r}{t} - \frac{\beta}{\alpha} \log |t| + \frac{\psi}{\alpha t} \right]
\]  
(37)

We insert the equation (37) into (14) and get

\[
\frac{2\beta^2}{\alpha^2} + 1 + \frac{3\beta}{\alpha} u + 2F - \frac{3\beta}{\alpha} F - 2uF + F^2 - 2F + \frac{\beta}{\alpha} F F + F \left[ 1 - \frac{2\beta^2}{\alpha^2} - \frac{\beta}{\alpha} u - u^2 \right] = 0
\]  
(38)

where

\[
u = \frac{r}{t} - \frac{\beta}{\alpha} \log |t| + \frac{\psi}{\alpha t}
\]  
(39)

Equation (38) can be reduced to the first order differential equation by use of similarity transformation method and we get the following Lie symmetries

\[
\eta = k u
\]

\[
\xi = k
\]

where \( k \) is a constant, which yields two independent solutions

\[
\frac{u^2}{2} - F = k_1, \text{ and } u - F = k_2,
\]
where \( k_1 \) and \( k_2 \) being constant. Considering

\[
U = \frac{u^2}{2} - F, \quad V = u - \bar{F}
\]

Now by making use of (41) into (38), the latter can be reduced to the following form

\[
\frac{dV}{dU} = \frac{2(1 - 2U) + V(V + \frac{2\beta}{\alpha})}{V \left( (1 - 2U) - \frac{\beta}{\alpha}(V + \frac{2\beta}{\alpha}) \right)}.
\]

Equation (42) is a first order differential equation, which on integration provides a relation between \( V \) and \( U \) and hence among \( u, F \) and \( \bar{F} \). In fact the equation (42) is substantially difficult to solve. So we will consider a particular case \( \beta = 0 \), and consequently equation (38) and (42) now read as

\[
1 + 2F - 2uF + \bar{F}^2 + \bar{F}[2F + 1 - u^2] = 0,
\]

and

\[
V(1 - 2U)dV - \left\{ 2(1 - 2U) + V^2 \right\} dU = 0
\]

respectively.

The equation (44) is an exact equation and admits the integral

\[
V^2(1 - 2U) + (1 - 2U)^2 = k\]

On inserting (41) into (44), we get

\[
(u - \bar{F})^2(2F + 1 - u^2) + (2F + 1 - u^2)^2 = k
\]

It can easily be verified that (45) is a first integral of (44). The solution plan for the equation (46), can be divided into two parts (i) \( k=0 \) , (ii) \( k \neq 0 \)

**Case k = 0:** The equation (46) reads as

\[
(2F + 1 - u^2) \left[ (u - \bar{F})^2 + (2F + 1 - u^2) \right] = 0
\]

Vanishing of the first factor yields

\[
F = \frac{u^2}{2} - \frac{1}{2}
\]

The simplified form of expressions for pressure and density are obtained as

\[
8\pi\rho = 3 \left( 8\pi p \right) = \frac{\alpha}{t^4 c}
\]
This represents conformally flat disordered radiation of Tolman. However, vanishing of the second factor gives rise the differential equation

\[(u - F)^2 + (2F + 1 - u^2) = 0\]  \hspace{1cm} (50)

Equation (50) can be re-arranged as

\[(u - F)(u^2 - 2F - 1)^{-1/2} = \pm 1,\]  \hspace{1cm} (51)

which immediately admits the integral

\[F = \frac{1}{2} \left[c^2 + 2cu + 1\right]\]  \hspace{1cm} (52)

The simplified expressions for pressure and density, as obtained in MAPLE/Derive, are given by

\[8\pi \rho = 8\pi p = \frac{1}{3} \frac{\alpha}{t^4 c},\]

which is a Zeldowich fluid.

**Case** \(k \neq 0\): A substitution,

\[X = \frac{1}{2} \left[u^2 - 2F - 1\right]\]  \hspace{1cm} (53)

sends the equation (47) to the form

\[\sqrt{\frac{2X}{4X^2 - k}} dX = \pm du,\]  \hspace{1cm} (54)

which can yield two different solutions corresponding to positive and negative values of the constant \(k\) as follows:

1. For \(k = K^2\), equation (54) can be transformed to

\[\sqrt{K} \int \frac{\cosh \theta}{\cosh \theta} d\theta = \pm du\]  \hspace{1cm} (55)

Through the transformation \(X = \frac{K}{2 \cosh \theta}\), (55) on integration gives

\[\frac{1}{\sqrt{2}} \left[F(\alpha, \frac{1}{\sqrt{2}}) - 2E(\alpha, \frac{1}{\sqrt{2}})\right] + \frac{\sinh \theta}{\sqrt{\cosh \theta}} = \pm \frac{(u + c)}{\sqrt{K}},\]

where \(F\) and \(E\) are elliptic function of first and second kind respectively, with

\[\alpha = \sin^{-1} \sqrt{\frac{\cosh \theta - 1}{\cosh \theta}}.\]  \hspace{1cm} (56)
2. For \( k = -K^2 \), the equation (55) can be sent to the form

\[
\sqrt{|K|} \int \sqrt{\sinh \theta} \, d\theta = \pm du.
\]  

(57)

Through the relation \( X = \frac{K}{2} \sinh \theta \), and on integration, we get

\[
\frac{1}{\sqrt{2}} \left[ F'(\alpha, \frac{1}{\sqrt{2}}) - 2E(\alpha, \frac{1}{\sqrt{2}}) \right] + \frac{\sqrt{\sinh \theta} \left(1 + \cosh \theta \right)}{1 + \sinh \theta} = \pm \frac{(u + c)}{\sqrt{K}},
\]

where \( F \) and \( E \) are elliptic function of first and second kind respectively, with

\[
\alpha = \cos^{-1} \left( \frac{1 - \sinh \theta}{1 + \sinh \theta} \right)
\]  

(58)

Reduction to Normal Form

The equation (3) with (39) can be transformed to the normal form by setting

\[
ds^2 = -\eta^{-2} dR^2 - t^2 d\sigma^2 + [1 + 2\dot{\nu} - \left(\frac{\beta}{\alpha} \log |t| + F\right)^2] dt^2,
\]

(59)

where \( \eta = e^{-\frac{1}{2} \int \frac{du}{F - u - (\beta/\alpha)}} \),

through the transformation

\[
dR = \eta \left[ dr - \left\{ \frac{\beta}{\alpha} \log |t| + F \right\} dt \right],
\]

(60)

\( \eta \) being the integrating factor to make the right hand side an exact expression. The corresponding expression for pressure and density can be furnished as

\[
8\pi p = \frac{1}{t^2} [G(r, t)]
\]

(61)

\[
8\pi \rho = \frac{1}{t^2} [G(r, t)] \{1 + G(r, t).C(r, t)\}
\]

(62)

where

\[
G(r, t) = \left[ 1 + \frac{3\alpha t^2}{\alpha} + \frac{2\beta}{\alpha} u - 2F u - \frac{2\beta}{\alpha} F + 2F + F^2 \right]^{-1},
\]

(63)

\[
C(r, t) = \left[ \frac{\beta}{\alpha^2}(F - 1) + \frac{\beta}{\alpha}(F u - F) + \frac{3\alpha t^2}{\alpha} F \right],
\]

(64)

while the velocity vectors and acceleration are
\[ v^1 = -\frac{(\dot{G}t + 2G)}{tG'} \left[ \frac{G^2C - \dot{G}t}{G^2C - 2G} \right]^{1/2} \] (65)

\[ v^4 = \left[ \frac{G^2C - \dot{G}t}{G^2C - 2G} \right]^{1/2} \] (66)

and

\[ \dot{v}_1 = \frac{G^2C - \dot{G}t}{t(G^2C - 2G)^2} \left[ tG'(1 + 2\dot{v} - \frac{\beta}{\alpha} \log |t| - F) - \frac{(\dot{G}t)^2 - 4G^2}{tn^2G''} \right], \] (67)

\[ \dot{v}_4 = -\frac{(\dot{G}t + 2G)}{tG'} \dot{v}_1, \] (68)

and

\[ \dot{v}_2 = \dot{v}_3 = 0. \] (69)

It is clear that the acceleration is non-zero in general.

4 Conclusions

A highly non-linear partial differential equation in general relativity is solved by the method of similarity transformation using computer algebra system. Most of the solutions are accelerating. In particular, the solution corresponding to \( \alpha = 0 \) is physically valid and satisfies the physical conditions (\( \rho \geq p > 0 \) for \( t \geq 0 \)). The solutions corresponding to \( k \neq 0 \) are also physically compliant, however they could only be expressed implicitly.

Throughout the algebraic manipulations, we have made extensive use of symbolic computation package, which not only saves time but also minimize the chances of errors if compared with manual computations. Through the use of interactive user control of evaluation of expressions in CAS, we have minimized the problem of intermediate expression swell and simplified the expressions in a way as we do manually.

A Appendix: Determining Equations

The determining equations used in section 3 were obtained by equating the coefficients of \( \theta \) and its various derivatives to zero. The program output for the determining equations is reproduced here:

\[ -6 \left( \frac{\partial^2}{\partial v \partial r} \tau(r, t, v) \right) t^3 + \tau(r, t, v) = 0 \] (70)
\[-2 \left( \frac{\partial^2}{\partial r \partial v} \xi(r, t, v) \right) t^2 + 2 \left( \frac{\partial^2}{\partial v^2} \eta(r, t, v) \right) t^2 - \left( \frac{\partial^2}{\partial v^2} \eta(r, t, v) \right) t^2 + \tau(r, t, v) - \left( \frac{\partial^2}{\partial r^2} \eta(r, t, v) \right) t^2 + 2 \left( \frac{\partial}{\partial r} \eta(r, t, v) \right) t \]
\[+2 \left( \frac{\partial^2}{\partial v \partial r} \xi(r, t, v) \right) t^2 - \left( \frac{\partial}{\partial r} \eta(r, t, v) \right) t = 0 \tag{71} \]

\[\frac{\partial^2}{\partial v^2} \tau(r, t, v) = 0 \tag{72} \]

\[\frac{\partial^2}{\partial v^2} \xi(r, t, v) = 0 \tag{73} \]

\[\frac{\partial^2}{\partial v \partial r} \tau(r, t, v) = 0 \tag{74} \]

\[\left( \frac{\partial^2}{\partial r} \xi(r, t, v) \right) t - 2 \left( \frac{\partial^2}{\partial v \partial r} \xi(r, t, v) \right) t + \left( \frac{\partial^2}{\partial v^2} \xi(r, t, v) \right) t \]
\[-\left( \frac{\partial}{\partial v} \xi(r, t, v) \right) t = 0 \tag{75} \]

\[- \left( \frac{\partial^2}{\partial r^2} \eta(r, t, v) \right) t^2 + \left( \frac{\partial}{\partial v} \eta(r, t, v) \right) t - 2 \left( \frac{\partial}{\partial r} \xi(r, t, v) \right) t = 0 \tag{76} \]

\[- \left( \frac{\partial}{\partial v} \eta(r, t, v) \right) t - \tau(r, t, v) + \left( \frac{\partial^2}{\partial v^2} \eta(r, t, v) \right) t^2 \]
\[+2 \left( \frac{\partial}{\partial v} \xi(r, t, v) \right) t = 0 \tag{77} \]

\[- \left( \frac{\partial}{\partial v} \eta(r, t, v) \right) t - \left( \frac{\partial^2}{\partial v^2} \eta(r, t, v) \right) t + \left( \frac{\partial^2}{\partial v^2} \eta(r, t, v) \right) t \]
\[+ \left( \frac{\partial}{\partial v} \tau(r, t, v) \right) t + \left( \frac{\partial}{\partial v} \xi(r, t, v) \right) t = 0 \tag{78} \]

\[- \left( \frac{\partial^2}{\partial r^2} \tau(r, t, v) \right) t + \left( \frac{\partial}{\partial v} \tau(r, t, v) \right) t = 0 \tag{79} \]

\[-2 \left( \frac{\partial}{\partial r} \tau(r, t, v) \right) + 2 \left( \frac{\partial^2}{\partial v \partial r} \eta(r, t, v) \right) t + 3 \left( \frac{\partial}{\partial v} \xi(r, t, v) \right) \]
\[- \left( \frac{\partial^2}{\partial v^2} \xi(r, t, v) \right) t = 0 \tag{80} \]

\[4 \left( \frac{\partial^2}{\partial v \partial r} \xi(r, t, v) \right) - 2 \left( \frac{\partial^2}{\partial v v} \tau(r, t, v) \right) + 2 \left( \frac{\partial^2}{\partial v^2} \tau(r, t, v) \right) \]
\[- \left( \frac{\partial^2}{\partial v^2} \eta(r, t, v) \right) = 0 \tag{81} \]

\[\left( \frac{\partial^2}{\partial t^2} \xi(r, t, v) \right) - \left( \frac{\partial^2}{\partial v \partial t} \xi(r, t, v) \right) = 0 \tag{82} \]

\[2 \left( \frac{\partial^2}{\partial v \partial t} \tau(r, t, v) \right) t + 4 \left( \frac{\partial}{\partial v} \tau(r, t, v) \right) - \left( \frac{\partial^2}{\partial v^2} \eta(r, t, v) \right) t = 0 \tag{83} \]

\[\left( \frac{\partial}{\partial v} \xi(r, t, v) \right) + \left( \frac{\partial^2}{\partial t^2} \xi(r, t, v) \right) t = 0 \tag{84} \]
\[ t^2 (-\frac{\partial^2}{\partial r^2} \tau(r, t, v)) t^2 + (\frac{\partial}{\partial r} \tau(r, t, v)) t + 2 (\frac{\partial^2}{\partial v \partial t} \eta(r, t, v)) t^2 - 2 \tau(r, t, v) + 4 (\frac{\partial}{\partial r} \xi(r, t, v)) t - 2 (\frac{\partial}{\partial v} \eta(r, t, v)) t = 0 \] (85)

\[ -2 (\frac{\partial}{\partial v} \eta(r, t, v)) t^2 + (\frac{\partial}{\partial r} \tau(r, t, v)) t^2 + (\frac{\partial}{\partial t} \eta(r, t, v)) t + (\frac{\partial^2}{\partial v \partial r} \tau(r, t, v)) t^2 - 2 (\frac{\partial^2}{\partial v \partial t} \tau(r, t, v)) t^2 + 6 (\frac{\partial^2}{\partial v \partial \eta} \eta(r, t, v)) t^2 + 2 (\frac{\partial^2}{\partial v \partial \tau} \eta(r, t, v)) t^2 - 4 (\frac{\partial}{\partial r} \xi(r, t, v)) t^2 = 0 \] (86)

\[ (\frac{\partial}{\partial v} \xi(r, t, v)) + (\frac{\partial^2}{\partial v^2} \xi(r, t, v)) t - (\frac{\partial^2}{\partial v^2} \xi(r, t, v)) t = 0 \] (87)

\[ -\frac{\partial}{\partial v} \tau(r, t, v) + \frac{\partial^2}{\partial v^2} \tau(r, t, v) t = 0 \] (88)

\[ 4 (\frac{\partial}{\partial r} \xi(r, t, v)) + 2 (\frac{\partial^2}{\partial v \partial r} \eta(r, t, v)) t - 3 (\frac{\partial^2}{\partial v \partial r} \tau(r, t, v)) t - 2 (\frac{\partial^2}{\partial v \partial \eta} \eta(r, t, v)) t = 0 \] (89)

\[ -\frac{\partial}{\partial r} \tau(r, t, v) - 2 (\frac{\partial}{\partial r} \xi(r, t, v)) + (\frac{\partial^2}{\partial v \partial r} \tau(r, t, v)) t - 2 (\frac{\partial^2}{\partial v \partial \eta} \eta(r, t, v)) t = 0 \] (90)

\[ 5 (\frac{\partial}{\partial r} \xi(r, t, v)) - (\frac{\partial^2}{\partial r^2} \tau(r, t, v)) t + 2 (\frac{\partial}{\partial r} \eta(r, t, v)) + 3 (\frac{\partial^2}{\partial v \partial r} \eta(r, t, v)) t + 3 (\frac{\partial^2}{\partial v \partial \eta} \eta(r, t, v)) t = 0 \] (91)

\[ 2 (\frac{\partial}{\partial r} \tau(r, t, v)) t + 2 (\frac{\partial}{\partial v} \tau(r, t, v)) t + 2 (\frac{\partial^2}{\partial v \partial \tau} \tau(r, t, v)) t^2 + (\frac{\partial^2}{\partial v^2} \tau(r, t, v)) t^2 = 0 \] (92)

\[ -\frac{\partial^2}{\partial v \partial \tau} \xi(r, t, v) t^2 + \tau(r, t, v) - (\frac{\partial^2}{\partial v^2} \eta(r, t, v)) t^2 + (\frac{\partial}{\partial v} \tau(r, t, v)) t + (\frac{\partial^2}{\partial v \partial \eta} \eta(r, t, v)) t^2 + 2 (\frac{\partial^2}{\partial v \partial \tau} \xi(r, t, v)) t^2 = 0 \] (93)

\[ -3 (\frac{\partial}{\partial r} \tau(r, t, v)) + (\frac{\partial^2}{\partial r \partial v} \eta(r, t, v)) t + 2 (\frac{\partial}{\partial r} \xi(r, t, v)) t + (\frac{\partial^2}{\partial r \partial \xi} \xi(r, t, v)) t - 2 (\frac{\partial}{\partial r} \eta(r, t, v)) t - 2 (\frac{\partial}{\partial r} \tau(r, t, v)) t = 0 \] (94)

\[ -\frac{\partial^2}{\partial v \partial \tau} \tau(r, t, v) + (\frac{\partial^2}{\partial v \partial \xi} \eta(r, t, v)) = 0 \] (95)

\[ 2 (\frac{\partial^2}{\partial v \partial \tau} \tau(r, t, v)) t + (\frac{\partial^2}{\partial v \partial \xi} \tau(r, t, v)) t + 2 (\frac{\partial}{\partial v} \tau(r, t, v)) t - 2 (\frac{\partial}{\partial v} \xi(r, t, v)) t - 2 (\frac{\partial}{\partial v} \eta(r, t, v)) t = 0 \] (96)
\[
\begin{align*}
-\left(\frac{\partial^2}{\partial v^2} \eta(r, t, v)\right) t + \left(\frac{\partial^2}{\partial v \partial r} \xi(r, t, v)\right) t + \left(\frac{\partial^2}{\partial v \partial t} \tau(r, t, v)\right) t \\
+3 \left(\frac{\partial}{\partial v} \tau(r, t, v)\right) = 0 \\
(\frac{\partial}{\partial v} \xi(r, t, v)) + \left(\frac{\partial^2}{\partial v \partial t} \xi(r, t, v)\right) t = 0
\end{align*}
\]
References


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