N=4 Supersymmetric Non Linear Sigma Models and Generalized Monodromy Matrix

T. Lhallabi
Lab/UFR-Physique des Hautes Energies
Faculté des Sciences de Rabat, Morocco

A. Moujib
Groupement National de Physique des Hautes Energies, GNPHE
Siege focal, Lab/UFR-HEP, Rabat, Morocco
asmoujib25@gmail.com

J. Zerouaoui
VACBT, Virtual African Centre for Basic Science and Technology
Focal point Lab/UFR-PHE, Fac Sciences, Rabat, Morocco

Abstract

Two-dimensional $N = 4$ Supersymmetric non linear sigma-models with non compact Lie groups $O(d,d)/O(d) \times O(d)$ models are constructed. These large symmetries lead to basis solutions with well defined transformations properties of $T$-duality. Under this latter, we construct the generalized Monodromy Matrix $\hat{M}(\omega)$ of two-dimensional string effective action by introducing the general integrability conditions.

Keywords: Two-dimensional non linear sigma models, $T$-duality transformations, Integrability properties, Monodromy Matrix

1 Introduction

String theory should provide answers to the questions related to the evolution of the universe. Indeed, in the cosmological scenario one considers Einstein and matter field equations obtained from the string effective action when the metric and background fields depend only on time coordinate usually identified as the cosmic time. The scalar dilaton appears in the massless spectrum of the theory.
and it is tempting to identify this field as the one responsible for inflation in early epochs [1]. For many years the possibility of describing inflation in terms of string theory seemed very hard to follow. Because the only string vacua that were understood were highly supersymmetric ones with many massless scalar fields called moduli possessing potential energy functions that vanish identically to all orders of perturbation theory [2].

On the other hand, the tree level string effective action compactified on a $d$-dimensional torus $T^d$ is known to be invariant under the noncompact global symmetry group $O(d, d)$ [2, 3]. In fact, two dimensional field theories play an important role in describing a variety of physical systems [4]-[6] and the reduction to two space-time dimensions leads to encounter rich symmetry structure [4]. Some of these models possess the interesting property of integrability which corresponds to exactly solvable models [6]. The effective two-dimensional action has been studied to bring out it’s integrability properties [7]-[9] and appears when some aspects of black hole physics, colliding plane waves as well as special types of cosmological models are considered [7, 10]. In this context, one may recall that the construction of the Monodromy matrix under $T$-duality and by introducing the integrability conditions which depend on spectral parameter turns out to be one of the principal objectives in the study of integrable systems [4, 11]. In general, equivalent string backgrounds are related by duality transformations that corresponds to a discrete subgroup of a non-compact global symmetry of the effective supergravity actions.

In string theory, there were many attempts to develop a consistent theory of string cosmology where inflation plays an important role [12]. The solutions of the non-linear sigma model equations in a Friedman-Robertson-Walker background for the graviton, dilaton and antisymmetric tensor have been found [12, 13].

The purpose of this article is to

- discuss the $O(d, d)$ symmetry of the two-dimensional $N = 4$ supersymmetric sigma-model where the maximal compact subgroup is $O(d) \times O(d)$. Moreover, we develop the $N = 4$ superfield equations of the $O(d, d)$ $N = 4$ supersymmetric model which can be solved for some special superpotentials. In fact, by considering the dependence only on time of all cosmological superfields.

- Construct the generalized Monodromy matrix $\hat{M}(\omega)$ of a two-dimensional string effective action obtained from a $D$-dimensional effective action, which is compactified on $T^d$, by considering the general integrability conditions. In this process the special properties of the generalized Monodromy matrix are given in terms of general functions depending on spectral parameter.
2 \ N=4\ Supersymmetric\ non-linear\ \sigma\text{-model}

with

\( O(d, d)/O(d) \times O(d) \)

The group \( O(d, d) \) is the non-compact pseudo-orthogonal group in \( 2d \) dimensions where it’s representation is given in [14].

Let us define \( M \in O(d, d) \) the symmetric matrix written as follows

\[
M = \begin{pmatrix}
G^{-1} & -G^{-1}B \\
BG^{-1} & G - BG^{-1}B
\end{pmatrix}
\]  

(2.1)

where \( G \) and \( B \) are the background fields of \( NS - NS \) sector in string theory.

The \( N = 4 \) supersymmetric non-linear sigma-model action with such matrix notation is given by

\[
I = \int d\xi \sqrt{g} e^{-\Phi} \left\{ R + \left( \nabla^+ + \tilde{\Phi} \right) \left( \nabla^- - \tilde{\Phi} \right) + \frac{1}{8} Tr \left[ \nabla^+ + M^{-1} \nabla^- M \right] \right\}
\]  

(2.2)

where

\[
\tilde{\Phi} = \Phi - \frac{1}{2} \ln \det G_{ij}
\]  

(2.3)

is the shifted dilaton superfield. This action is invariant under the global \( O(d, d) \) transformation [15] namely

\[
M \rightarrow UMU^T \\
g_{MN} \rightarrow g_{MN} \\
\Phi \rightarrow \tilde{\Phi}
\]  

(2.4)

In general, there will be additional terms in the \( N = 4 \) supersymmetric action (2.2) corresponding to \( d \) abelian gauge superfields coming from the original metric and another set of \( d \) gauge superfields coming from the antisymmetric tensor as a result of dimensional reduction [10]. These gauge superfields are dropped since we are in two-dimensional superspace [16].

In fact, if we consider a spatially closed flat homogeneous superspace, the \( N = 4 \) supersymmetric manifestly \( O(d, d) \) invariant action (2.2) with the introduction of a general superpotential \( V \) and the use of the conformal gauge.

\[
g_{MN} = \eta_{MN}
\]  

(2.5)

takes the following form

\[
I = \int d\xi^0 e^{-\Phi} \left\{ V + \left( \nabla^+ + \tilde{\Phi} \right) \left( \nabla^- - \tilde{\Phi} \right) + \frac{1}{8} Tr \left[ \eta \nabla^+ M \eta \nabla^- M \right] \right\}
\]  

(2.6)
with \( d\xi^0 = dt^2 \theta^+ d\theta^- dU \) since we have considering all superfields depending only on time. In this way the harmonic covariant derivatives are reduced as
\[
\nabla^\pm \mp = \partial^\pm \mp + \theta^\pm \theta^\mp \partial_t
\] (2.7)

Therefore, the first equation of motion follows from reintroducing \( G_{00} \) in the action (2.6) and from setting to zero the corresponding variation which leads to the zero energy condition namely
\[
(\nabla^+ \Phi)(\nabla^- \Phi) + \frac{1}{8} Tr [\eta \nabla^+ M \eta \nabla^- M] - V = 0
\] (2.8)

The variation of the action (2.6) with respect to the superfield \( \bar{\Phi} \) leads to
\[
(\nabla^+ \nabla^- \Phi) + (\nabla^+ \Phi)(\nabla^- \Phi) + \frac{1}{8} Tr [\eta \nabla^+ M \eta \nabla^- M] - \frac{\partial V}{\partial \bar{\Phi}} + V = 0
\] (2.9)

If we expand these equations in terms of \( \theta^\pm \) components by using the expressions (2.7), we obtain the important equation of motion of the dilaton field \( \varphi \) in the ordinary case [14] completed by the terms corresponding to the supersymmetric partners of the dilaton and auxiliary fields and the matrix \( M \) [17]. This is given by
\[
\left( \dot{\varphi} \right)^2 + \frac{1}{8} Tr [\eta M \eta M] + \frac{\partial V}{\partial \varphi} \dot{\varphi} + \ldots = 0
\] (2.10)

On the other hand, in order to obtain the variation of the action (2.6) with respect to \( M \), we use the following infinitesimal variation of \( M \)
\[
\delta M = M \varepsilon + \varepsilon^T M
\] (2.11)

Therefore, at first order in the superparameter \( \varepsilon \) the variation of the action (2.6) leads to
\[
\nabla^+ J^- - + \nabla^- J^+ = 0
\] (2.12)

with
\[
J^+ = e^{-\Phi} Tr [\eta M \eta \nabla^+ M]
\]
\[
J^- = e^{-\Phi} Tr [\eta M \eta \nabla^- M]
\] (2.13)

where we have used the constraints of [18] the constraint (2.12) becomes
\[
\nabla^+ \nabla^- J = 0
\] (2.14)
and the supercurrent is given by

\[ J = \frac{1}{2} e^{-\Phi} Tr[\eta M \eta M] \]  (2.15)

Note that for cosmological considerations we assume that all superfields are time dependent only. Therefore, the superfield equation for the matrix \( M \) namely (2.12) admits the first integral

\[ e^{-\Phi}(M\eta \dot{M}) \equiv \text{cste} = A \]  (2.16)

The substitution of (2.16) into the equation (2.8) leads to the first order equation for the dilaton superfield which is given in [14, 18]. Consequently, the general solution of (2.16) is obtained in terms of \( \tau \) [14, 18] as

\[ M(t) = e^{-A\eta \tau} M(t_0) \]  (2.17)

However, in order to include time dependent cosmological solutions of two dimensional super black-holes, these equations have to be solved for some special superpotential [18].

Then, the solution of (2.17) for \( M \) reads [18]

\[ M(t) = \exp(CA\eta \ln \frac{T-t}{T-t_0})M(t_0) \]  (2.18)

We note that different super black hole solutions can be identified for particular choices of the super constants \( T \) and \( \Lambda \) [18, 19] The above \( M \) matrix with it’s matricial background fields and \( O(d, d) \) transformations which leave the dimensionally reduced action invariant globally under \( O(d, d) \) group and locally under \( O(d) \times O(d) \) play an important role in the study of integrable systems.

The integrability of dimensionally reduced gravity and Supergravity to two dimensions has been studied extensively by introducing the spectral parameter and constructing a set of currents which are invariant under a local \( O(d) \times O(d) \) transformation and satisfy the zero curvature condition [20]. In this context, the generalized Monodromy Matrix is constructed under \( T \)-duality symmetry by considering the general integrability conditions in terms of general functions \( f \) and \( g \).

3 Two-Dimensional String Effective Action

We are interested by the two-dimensional \( \sigma \)-model coupled to gravity [15, 21] namely

\[ S = \int dx^2 \sqrt{-g} e^{-\Phi} \left[ R + (\partial \Phi)^2 + \frac{1}{8} Tr(\partial_{\alpha} M^{-1} \partial^\alpha M) \right] \]  (3.1)
In this expression, $g$ is the determinant of the metric $g_{\alpha \beta}$ where $\alpha, \beta = 0, 1$ are the two- dimensional space time indices and $R$ is the associated Ricci scalar curvature and $d = D - 2$. The action (3.1) is invariant under global $O(d,d)$ transformations (2.4) and the variation with respect to $M$ leads to the conservation law

$$\partial_\alpha \left[ e^{-\Phi} \sqrt{-g} g^{\alpha \beta} M^{-1} \partial_\beta M \right] = 0$$

(3.2)

In order to make apparent a local $O(d) \times O(d)$ as well as a global $O(d,d)$ transformations, it is convenient to introduce a triangular matrix $V$ contained in $O(d,d)/O(d) \times O(d)$ of the following form

$$V = \begin{pmatrix} E^{-1} & 0 \\ BE^{-1} & E^T \end{pmatrix}$$

(3.3)

such that $M = VV^T$ with $(E^T E)_{ij} = G_{ij}$ where $E$ is the vielbein in the internal space. Consequently, the matrix $V$ parametrizing the coset $O(d,d)/O(d) \times O(d)$ transforms non trivially under global $O(d,d)$ and local $O(d) \times O(d)$ namely

$$V \longrightarrow \Omega^T V h(x)$$

(3.4)

where $\Omega \in O(d,d)$ and $h(x) \in O(d) \times O(d)$. However, from the matrix $V$, we can construct the following current [10, 22]

$$V^{-1} \partial_\alpha V = P_\alpha + Q_\alpha$$

(3.5)

which belongs to the Lie algebra of $O(d,d)$. In such decomposition $Q_\alpha$ belongs to the Lie algebra of the maximally compact subgroup $O(d) \times O(d)$ and $P_\alpha$ to the complement [23]. Therefore

$$P_\alpha = \frac{1}{2} \left[ V^{-1} \partial_\alpha V + (V^{-1} \partial_\alpha V)^T \right]$$

(3.6)

$$Q_\alpha = \frac{1}{2} \left[ V^{-1} \partial_\alpha V - (V^{-1} \partial_\alpha V)^T \right]$$

Now we can show that

$$Tr \left[ \partial_\alpha M^{-1} \partial_\beta M \right] = -4 Tr \left[ P_\alpha P_\beta \right]$$

(3.7)

4 Classical Integrability and General Monodromy Matrix

In this section we consider the case of two dimensional sigma model in flat space time defined on the coset $O(d,d)/O(d) \times O(d)$. The integrability conditions
following from the currents (3.5) corresponds to the zero curvature condition namely
\[
\partial_\alpha [V^{-1} \partial_\beta V] - \partial_\beta [V^{-1} \partial_\alpha V] + [(V^{-1} \partial_\alpha V), (V^{-1} \partial_\beta V)] = 0 \quad (4.1)
\]
Now, let us consider the generalized current decomposition with arbitrary functions of the spectral parameter \( t \) as follows [24]
\[
\hat{V}^{-1} \partial_\alpha \hat{V} = Q_\alpha + f(t) P_\alpha + g(t) \varepsilon_{\alpha\beta} P_\beta \quad (4.2)
\]
where \( f(t) \) and \( g(t) \) are general functions satisfying the following conditions
\[
\begin{align*}
  f(t = 0) &= 1, & \lim_{t \to +\infty} f(t) &= -1 \\
  g(t = 0) &= 0, & \lim_{t \to +\infty} g(t) &= 0
\end{align*} \quad (4.3, 4.4)
\]
in flat space-time. Let us note that these functions have to possess singularities for \( t = \pm 1 \) in order to recover the case of ref [10].
The zero curvature in (4.1) leads to the integrability condition of the spectral parameter namely
\[
(\partial_\beta t) P_\alpha - (\partial_\alpha t) P_\beta = (f' + g')^{-1} \left\{ (f + g)(\partial_\beta P_\alpha - \partial_\alpha P_\beta + 2 [P_\alpha, P_\beta]) + (4fg - 1) [P_\alpha, P_\beta] + [Q_\alpha, Q_\beta] \right\} \quad (4.5)
\]
which can be simplified by using the light cone indices to the following equations
\[
\partial_\pm t = P_0^{-1} (P_\pm \pm H(t)) \quad (4.6)
\]
with
\[
H(t) = (f' + g')^{-1} \left\{ (f + g)(\partial_1 P_0 - \partial_0 P_1 + 2 [P_0, P_1]) + (4fg - 1) [P_0, P_1] + [Q_0, Q_1] \right\} \quad (4.7)
\]
These currents that are subject of integrability condition can be incorporated in the matrix \( \hat{V} \) as
\[
\hat{V}^{-1} \partial_\pm \hat{V} = (f(t) + g(t)) \begin{pmatrix} -E^{-1} & 0 \\ 0 & E^{-1} \end{pmatrix} \partial_\pm E \quad (4.8)
\]
Let us note that in this parametrization, the antisymmetric tensor field is chosen to be zero and the matrices \( E \) and \( G \) are assumed to be diagonal [10, 18]. The dynamics is defined via the Lie algebra decomposition (3.5) and in addition to the matter fields, the model contains a dilaton field \( \rho \) which is given in the conformal gauge [23] by
\[
\rho(x) = \rho_+(x^+) + \rho_-(x^-) = e^{-\bar{\Phi}} \quad (4.9)
\]
The consistency of matter fields equation [23] requires that the spectral parameter $t$ itself be subject to a very similar system of linear differential equations as follows

$$t^{-1}\partial_{\pm}t = (f(t) + g(t)) \rho^{-1}\partial_{\pm}\rho$$

(4.10)

that relates $\rho$ and $t$ as a Bäcklund duality [25]. Indeed

$$\partial_{\pm}t = t (f(t) + g(t)) e^\Phi \partial_{\pm} e^{-\Phi}$$

(4.11)

Thus, from the current form, we assume that

$$f(t) + g(t) = \sqrt{\omega - \rho - \omega + \rho + \frac{e^{-\Phi}}{1 - N^2(t)}}$$

(4.12)

where $\omega$ is a function of spectral parameter and represent the constant of integration of $t$ ($\omega$ is namely as a hidden spectral parameter) given by

$$\omega = -\rho_+ + \frac{e^{-\Phi}}{1 - N^2(t)}$$

(4.13)

where $N(t)$ must verify that $N(t) \neq \pm 1$ with $N(t) = f(t) + g(t)$. Therefore, the generalized monodromy matrix has to be of the form

$$\hat{\mathcal{M}} = \hat{V}(x,t)\hat{V}(x,\frac{1}{t}) = \begin{pmatrix} \mathcal{M}(\omega) & 0 \\ 0 & \mathcal{M}^{-1}(\omega) \end{pmatrix}$$

(4.14)

where $\mathcal{M}(\omega)$ is diagonal with

$$\mathcal{M}_i(\omega) = \frac{\omega_i - \omega}{\omega_i + \omega}$$

(4.15)

and

$$\omega_i = -\rho_+ + \frac{e^{-\Phi}}{1 - N^2(t_i)}$$

(4.16)

Finally this generalized monodromy matrix can be used to examine general currents and general integrability conditions in some cosmological models.
5 Conclusion

In this paper, we have studied the case of the \( O(d,d) \) group where the moduli superfields parametrize the \( O(d,d)/O(d) \times O(d) \) coset. The manifestly \( O(d,d) \) invariant \( N = 4 \) supersymmetric equation of motion are derived by introducing a general superpotential \( V \). By considering the dependence only on time of all superfields, we have considered cosmological and super black-hole solutions. Finally, with these developments, the \( N = 4 \) supergravitational anomalies in the two-dimensional super-black hole background can be studied.

Furthermore, we have constructed the generalized monodromy matrix of two-dimensional string effective action by using the general integrability conditions which are expressed in terms of general functions \( f \) and \( g \) depending on the spectral parameter and satisfying conditions (4.3). In fact, we have shown that the generalized monodromy matrix transforms non-trivially under the non-compact \( T \)-duality group. Furthermore, it is necessary to note that we have realized a connection between \( T \)-duality symmetry and general integrability properties by introducing a spectral parameter \( t \) which is space-time dependent. Indeed, we have envisaged a vielbein \( E(x,t) \) in order to define the action under consideration and examine its invariance globally under \( G \) and locally under \( H \). By including \( E \), we have introduced an infinite potentials family of matrices parametrized by general functions depending on a continuous spectral parameter necessary to obtain the generalized monodromy matrix in ordinary case.

Finally, our general results have likely exploitations and more applications in string theory and cosmological models [26]-[29] such the Becchi-Rouet-Stora-Tyutin quantization of bosonic string on \( AdS_5 \times S^5 \) and analysis of black-holes with solutions. Furthermore the integrable models like calogero model for two particles and \( N \) particles and others cosmological models with non-compact Lie groups can be studied. With these results, a very large and rich class of models can be treated from this process.

Acknowledgement 1 We are thankful to the anonymous referee for several constructive suggestions. We would like to thank the program Protars III D12/25, CNRST for it’s contribution in this research work.

References


N=4 Supersymmetric non linear sigma models


[18] T. Lhallabi, A. Moujib, N=4 Supersymmetric non linear sigma models with non Compact Lie groups,


Received: April, 2010