Instantaneous Creation of Scalar Particles in Expanding Universe

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Abstract

The scalar field equation non minimally coupled to gravity is formulated in a Robertson Walker (RW) space-time. Standard normal modes are considered by ortho-normalization of solutions induced by a conserved current. A quantization of the scheme is formulated and a second set of normal modes considered that is essentially the time translated of the first one. The corresponding Bogolubov coefficients are calculated. The balance between one-mode particle creation and annihilation per unit time is defined and obtained in two different ways. The results are proportional to the Hubble parameter \( \dot{R}(t)/R(t) \), the radius of the universe, but are independent of any other physical parameter. The result coincides with the corresponding one relative to spin 1 field in flat RW space-time. The creation (annihilation) term is of relevance near the big bang origin of the universe.

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1 Introduction

The existence of the effect of particle production in expanding universe was originally put into evidence by Parker for spin 0, 1/2 fields [10, 11]. Particle creation is a consequence of field quantization in curved space-time and of the universe expansion. The argument leading to the result can now be found widely discussed and illustrated through different meaningful models [3, 7, 12]. The explicit results are based on the calculations of the Bogolubov coefficients that connect two different sets of normal modes (for a recent application see e.g., [9]).
In many examples the two set of normal modes are given by the “in” and “out” states at time \( t = -\infty \) and \( t = +\infty \) respectively at what times the underlying space-time is generally assumed to be Minkoskian.

Recently, the balance per unit time of created and annihilated spin 1 particles in a flat Robertson-Walker (RW) space-time has been calculated for every time \([15, 16]\). This avoids consideration of time \( t = -\infty \), that is a situation not compatible with the generally admitted big bang origin of the universe.

An important consequence of particle creation is its influence on the expansion law of the universe. A simple model that takes into account such back reaction for spin 1 particle production was proposed and discussed in \([16]\). The scheme leading to that result however holds only for the flat RW space-time. The curved RW space-time cases have not been treated due to the lack, as far as the author knows, of explicit normal modes of the spin 1 field equation. It seems therefore useful to propose a model of instantaneous particle creation that possibly holds for all curvature RW space-time cases.

In the present paper, we study a scalar field non minimally coupled to gravity in a RW space-time. Standard normal modes of the field equation are then considered. The covariant character of the ortho-normalization procedure is ensured by the use of a covariant current. The Bogolubov coefficients are calculated in relation to a second set of normal modes that is in fact obtained by time translation of the first one. The balance of one-mode created and annihilated particles per unit time is explicitly calculated at any time. On the one hand, it seems of some interest the fact that the result is proportional to \( \dot{R}(t)/R(t) \), as for the spin 1 case \([15, 16]\), and that it does not depend on any other physical parameter. On the other hand this seems implausible because it implies the creation (annihilation) of an infinite number of particles of arbitrary mode. If however the result is taken for grant, a large number of particles is expected to be produced at the initial times of a big-bang like cosmological model. The fact that the result is proportional to that relative to the spin 1 case \([15]\), leaves open the question whether the property is a general feature of Bosons field in RW space-time.

2 Assumptions and preliminary results.

In the RW space-time of metric \([13]\):

\[
ds^2 = dt^2 - R^2(t) \left[ \frac{dr^2}{1 - ar^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (a = 0, \pm 1) \quad (1)
\]

the non minimally coupled scalar field equation reads \([3]\)

\[
\nabla_\alpha \nabla^\alpha \phi(x) + [\mathcal{m}_0^2 + \xi \dot{R}(x)]\phi(x) = 0 \quad (2)
\]
where \( \xi \) is a numerical factor, \( m_0 \) the mass of the particles of the field, \( \bar{R}(x) \) the Ricci scalar: 
\[
\bar{R}(x) = 6\bar{R}(t)\bar{R}(t) + \bar{R}(t)^2 + a]/R^2(t).
\]
The term \( \xi \bar{R}\dot{\phi} \) represents the coupling between the scalar and the gravitational field. The eq. (2) can be solved by variable separation by setting
\[
\phi(r, \theta, \varphi, t) = T(t)\chi(\theta, \varphi)S(r) \tag{3}
\]
The separated angular solutions are given by \( \chi = Y_{lm}(\theta, \varphi), \ l = 0, 1, 2, ..., \ m = -l, -l + 1, ..., 0, 1, ..., l, \ Y_{lm} \) the usual spherical harmonics. The separated radial solutions are of the form \( S = S_{kl}(r), \ k \) the separation constant of the radial and time dependence. For \( a = 0 \) the \( S_{kl}'s \) are the Bessel functions of the first type \([1]\); for \( a = 1 \) the \( S_{kl}'s \) result to be the ultra spherical (Gegenbauer) polynomials \([5]\); the specific expression for \( a = 1 \) can be found in \([2, 4]\). (These solutions have been collected in a uniform manner in \([14]\)). The \( Y_{lm}'s \) and the \( S_{kl}'s \) are properly orthogonal functions in their respective functional spaces (See the following). As to the separated time dependence one is left with
\[
\ddot{T} R^2 + 3\bar{R}\dot{T} + [k^2 + m_0^2 R^2 + 6\xi(\bar{R} \dot{R} + \dot{\bar{R}}^2) + a]T = 0 \ (a = 0, \pm 1) \tag{4}
\]
that can be solved once the time evolution of the cosmological background is given. In general, for the solution \( T_k(t) \) of (4), it holds:
\[
R^3(t)[T_k(t)\dot{T}_k^*(t) - \dot{T}_k(t)T_k^*(t)] = \text{const.} = i \tag{5}
\]
where the value of the constant has been chosen for later convenience. A covariant scalar product between solutions \( \phi_1, \phi_2 \) of eq. (2) can be defined by considering the four vector (the four current if \( \phi_1 = \phi_2 \))
\[
J_\mu(\phi_1, \phi_2) = -i(\phi_1 \nabla_\mu \phi_2 - \phi_2 \nabla_\mu \phi_1) \tag{6}
\]
that, as a consequence of eq. (2) with \( \xi \) real, is divergence free: \( \nabla_\mu J^\mu(\phi_1, \phi_2) = 0 \). By Gauss theorem \([8]\) the expression
\[
(\phi_1, \phi_2) = \int_\Sigma J_\mu(\phi_1, \phi_2)\left| g \right|^{1/2} n^\mu d\Sigma \tag{7}
\]
is independent of \( t \) (\( \Sigma \) a Cauchy surface, \( n^\mu \) a future directed unit vector orthogonal to \( \Sigma \)). The expression (8), that follows by choosing \( t = \text{const.} \) and \( n^\mu \equiv (1, 0, 0, 0) \) can then be used to define the scalar product between solutions of the field equation. By using the orthogonality relations of the angular and radial solution and equation (5) one has for \( \phi = \phi_{klm}(t), \ \phi' = \phi_{k'l'm'}(t) \):
\[
(\phi, \phi') = -iR^3 \int d\Omega Y_{lm}Y_{l'm'}^* \int \frac{dr^2}{\sqrt{1 - ar^2}} S_{klm}S_{k'l'm'}^*(T_k \dot{T}_k^* - \dot{T}_k T_k^*) \tag{9}
\]
\[
= -iR^3(T_k \dot{T}_k^* - \dot{T}_k T_k^*)\delta_{ll'}\delta_{mm'}\delta(k - k') \tag{10}
\]
\[
= \delta_{ll'}\delta_{mm'}\delta(k - k') \equiv \delta_{\alpha, \alpha'} \tag{11}
\]
where $\delta(k - k')$ stands for the Dirac delta in case $a = 0, -1$ and for the Kronecker delta $\delta_{nn'}$ for $a = 1$. One can easily check that also

$$
(\phi^*_\alpha, \phi^*_{\alpha'}) = -\delta_{\alpha\alpha'}; \quad (\phi_\alpha, \phi^*_{\alpha'}) = 0, \quad (\alpha \equiv (klm)) \quad (12)
$$

Therefore $\{\phi_\alpha\}$ is a set of normal modes for the field equation (2) that can be used for the quantization of the theory. Notice that by setting $\phi_\alpha(t) = \phi_\alpha(t, r, \theta, \phi)$ and $\phi_{\alpha'}(t') = \phi_{\alpha'}(t', r, \theta, \phi)$ one obtains

$$
(\phi_\alpha(t), \phi^*_{\alpha'}(t')) = -iR^3(t)\left[T_k(t)\dot{T}^*_k(t') - \dot{T}_k(t)T^*_k(t')\right]\delta_{\alpha\alpha'} \quad (13)
$$

that for $t = t'$ and by (5) collapses to (11).

### 3 The quantization scheme.

The quantization of spin 0 field can be implemented in curved space-time by expanding the wave field operator on the creation and annihilation operators $[3, 5, 6, 7]

$$
\phi(x) = \sum_j [a_j \phi_j(x) + a^+_j \phi^*_j(x)] \quad (14)
$$

(in the present scheme $\sum_j = \sum_{lm} \int dk$ for $a = 0, -1$ and $\sum_j = \sum_{nlm}$ for $a = 1$) that satisfy the usual canonical commutation relations: $[a_i, a_k] = [a^+_i, a^+_k] = 0$, $[a_i, a^+_k] = \delta_{ik}$ (here $\delta_{ik} \equiv \delta(k - k')\delta_{ll'}\delta_{mm'}$) where $\{\phi_j(x)\}$ is a set of normal modes, that is a set of solutions of the scalar field equation satisfying (11)-(12).

Suppose $\hat{\phi}_h(x)$ is a second set of normal modes satisfying

$$
(\hat{\phi}^*_\alpha, \hat{\phi}^*_{\alpha'}) = \pm \delta_{\alpha\alpha'} \quad (\hat{\phi}^*_\alpha, \hat{\phi}^*_{\alpha'}) = \mp \delta_{\alpha\alpha'} \quad (\hat{\phi}_\alpha, \hat{\phi}^*_\alpha) = 0 \quad (15)
$$

(It is understood that in (15) it holds only the upper or the lower sign). The wave function operator can be as well developed in the form

$$
\phi(x) = \sum_j [\hat{a}_j \hat{\phi}_j(x) + \hat{a}^+_j \hat{\phi}^*_j(x)] \quad (16)
$$

The two decompositions of $\phi$ define then two vacuum states $|0\rangle$ and $|\hat{0}\rangle$ for which $a_j|0\rangle = 0$, $\hat{a}_j|\hat{0}\rangle = 0$ for every $j$. Comparing (14) and (16) one obtains the transformations between the two sets of normal modes

$$
\hat{\phi}_j = \sum_i [\alpha_{ji} \phi_i + \beta_{ji} \phi^*_i], \quad \alpha_{ji} = (\hat{\phi}_j, \phi_i), \quad \beta_{ji} = -(\hat{\phi}_j, \phi^*_i) \quad (17)
$$

$$
\phi_h = \sum_l [\mp \alpha^*_{hl} \hat{\phi}_l \mp \beta_{hl} \hat{\phi}^*_l] \quad (18)
$$
and between the corresponding creation annihilation operators

\[ a_j = \sum_l [\pm \alpha_{lj} \hat{a}_l \mp \beta^*_{lj} \hat{a}_l^+] \] (19)

\[ \hat{a}_j = \sum_i [\alpha_{ji}^* a_i + \beta_{ji}^* a_i^+] \] (20)

Moreover the Bogolubov coefficients are found to satisfy

\[ \sum_i [\pm \alpha_{ji} \alpha_{li}^* \mp \beta_{ji} \beta_{li}^*] = \delta_{jl} \] (21)

\[ \sum_i [\alpha_{li} \beta_{ji}^* - \beta_{li} \alpha_{ji}^*] = 0 \] (22)

According to the above we have indeed two possible schemes of quantization: the one that correspond to the choice of the upper signs and the other that corresponds to the choice of the lower signs in all equations (14)-(22). A priori the schemes are both possible.

The Bogolubov coefficients \( \alpha_{ji} = (\hat{\phi}_j, \phi_i), \ \beta_{ji} = -(\hat{\phi}_j, \phi_i^*) \) are of relevance because if one considers the expectation of the particle number operators \( N_h = a_h^+ a_h, \ \hat{N}_h = \hat{a}_h^+ \hat{a}_h \) on the empty states, one obtains, by using (19), (20),

\[ \langle \hat{0} | N_h | \hat{0} \rangle = | | a_h | \hat{0} \rangle | |^2 + | | \sum_i \beta_{ih}^* \hat{a}_i^+ | \hat{0} \rangle | |^2 \] (23)

\[ \langle \hat{0} | \hat{N}_h | \hat{0} \rangle = | | \sum_j \beta_{jh}^* a_j^+ | \hat{0} \rangle | |^2 \] (24)

where \( |.| \rangle \langle .| \) is the norm in the Fock space of \( | \hat{0} \rangle \) and \( |.| \rangle \langle .| \_F \) is the norm in the Fock space of \( | \hat{0} \rangle \). The expression (23) represents the fact that the vacuum of \( \hat{\phi}_i \)-modes contains particle in the \( \hat{\phi}_i \)-modes and similarly (24) says that the \( \phi \) vacuum contains particles of the \( \hat{\phi}_h \)-modes.

4 Particle creation.

We now introduce, together with the normal modes \( \{ \phi_i \} \) of Sect. 2 a second set of normal modes. To that end consider the functions

\[ \hat{\phi}_i(t) \equiv \hat{\phi}_i(t, r, \theta, \varphi) \equiv \left[ \frac{R(t + \tau)}{R(t)} \right]^{\frac{3}{2}} \phi_i^*(t + \tau, r, \theta, \varphi) \] (25)

the \( \{ \phi_i \} \) being the normal modes of Sect. 3. By applying the definition (8) with \( g = g(t) \) one obtains

\[ (\hat{\phi}_i, \hat{\phi}_{i'}^*) = -i R^3(t + \tau) [T_k^*(t + \tau) T_k(t + \tau) - \hat{T}_k^*(t + \tau) T_k(t + \tau)] \delta_{ii'} \] (26)

\[ = (i)^2 \delta_{ii'} = -\delta_{ii'} \] (27)
of normal modes. Indeed one has for the Bogolubov coefficients defined in (17)

\[ (\hat{\phi}_h, \hat{\phi}^*_h') = \delta_{hh'}, \quad (\hat{\phi}_h, \hat{\phi}^*_h') = 0 \quad \forall \ h, h' \]  

(28)

\((\delta_{hh'} = \delta(k - k')\delta_{\mu \nu} \delta_{\nu ' \mu '})\). Therefore \(\{\hat{\phi}_i\}\) is a set of normal modes. Notice that we are in the case of the lower signs considered in Sect. 3. One can further check this fact by calculating the Bogolubov coefficients relative to the two sets of normal modes. Indeed one has for the Bogolubov coefficients defined in (17)

\[ - \alpha_{ij} = i \left[ R(t + \tau)R(t) \right]^{3/2} [T_k^*(t + \tau)\dot{T}_k(t) - \dot{T}_k^*(t + \tau)T_k(t)] \delta_{ij} = A_i \delta_{ij} \]  

(29)

\[ \beta_{ji} = i \left[ R(t + \tau)R(t) \right]^{3/2} [T_k^*(t + \tau)\dot{T}_k(t) - \dot{T}_k^*(t + \tau)T_k(t)] \delta_{ij} = B_j \delta_{ij} \]  

(30)

(i \equiv klm) and one can check that \(A_j, B_j\) defined in (29), (30) satisfy

\[ |A_j|^2 - |B_j|^2 = i R^3(t + \tau) [\dot{T}_k(t + \tau)T_k^*(t + \tau) - T_k^*(t + \tau)\dot{T}_k(t + \tau)] \]

(31)

by repeatedly applying eq. (5). Therefore the equation (21) is automatically verified, in the lower sign case, by the Bogolubov coefficients (29), (30).

For what concerns the expectation the number operators the of (created) particles present in \(|\hat{0} >\) but not in \(|0 >\) one has, by (23), (30):

\[ N^+_h = \sum_j |\beta_{jh}|^2 = [R(t + \tau)R(t)]^3 |T_k^*(t + \tau)\dot{T}_k(t) - \dot{T}_k^*(t + \tau)T_k(t)|^2 \]  

(32)

Conversely, to obtain the (annihilated) particles present in \(|0 >\) but not in \(|\hat{0} >\), one can proceed in at least two ways obtaining the expressions

\[ N^-_{1h} = \left[ \frac{R(t + \tau)}{R(t)} \right]^3 R^6(t + \tau) |T_k^*(t + \tau)\dot{T}_k(t) - \dot{T}_k^*(t)T_k(t)|^2 \]  

(33)

\[ N^-_{2h} = \left[ \frac{R(t + 2\tau)}{R(t + \tau)} \right]^3 R^6(t) |T_k^*(t + 2\tau)\dot{T}_k(t + \tau) - \dot{T}_k^*(t + \tau)T_k(t + \tau)|^2 \]  

(34)

\(N^-_{1h}\) is defined by calculating \(\beta_{ji} = - (\hat{\phi}_j, \hat{\phi}^*_i)\) by (8) with \(g\) at time \(t + \tau\); \(N^-_{2h}\) is defined by considering in (8) \(\hat{\phi}_i\) and \(\hat{\phi}_j\) at time \(t + \tau\), but \(g\) at time \(t\). The surviving particles, per unit of time at time \(t\), is according to the two definitions \((h \equiv klm):\)

\[ n_{1h}(t) = \lim_{\tau \to 0} \frac{N^+_h(t) - N^-_{1h}(t)}{\tau} = -6 \frac{\dot{R}(t)}{R(t)} \]  

(35)

\[ n_{1h}(t) = \lim_{\tau \to 0} \frac{N^+_h(t) - N^-_{2h}(t)}{\tau} = +6 \frac{\dot{R}(t)}{R(t)} \]  

(36)

The result (35) is quite immediate while (36) is obtained with some calculations that makes repeated use of the relation (5). If \(\dot{R}(t)\) is positive, then the particle annihilation is dominant in case of \(n_{1i}\) and particle creation is dominant in case of \(n_{2i}\). Conversely if \(\dot{R}\) is negative.
5 Remarks and comments.

In the previous Section the balance per unit time of one-mode created and annihilated scalar particles in a RW space-time model has been calculated in two different ways. In both cases the result does not depend on any physical parameter of the theory except on the radius of the universe. It is also independent of the curvature parameter of the space-time. This has the unpleasant consequence that the total number of created (annihilated) particles, per unit time, of arbitrary mode is infinite. However it seems not negligible the fact that, modulo the algebraic sign, the results (35), (36) are the same of the analogous result for spin 1 field in flat RW space-time [15, 16]. If the previous treatment is taken for grant the number of the created (annihilated) one-mode particles should be very small at a generic time. It seems questionable the possibility of distinguishing, e. g., at the present time in which $\dot{R}(t) > 0$, whether the creation or the annihilation effect is dominant.

Creation of particle produces obviously a back reaction on the gravitational dynamics of a cosmological model. This suggests to modify the Standard Cosmological Model by considering a modification of pressure and energy density of the universe, by a term depending on the creation rate $\dot{R}/R$. A model in this direction has been proposed and studied in [16].

One can note also that in any cosmological model that starts with a big-bang, $R(0) = 0$, as it is generally admitted for our universe, the particle creation (annihilation) term is of relevance. Indeed if, e.g., $R(t) \sim t^\epsilon$, ($\epsilon > 0$), for $t \sim 0$ then the term $\dot{R}/R \sim \epsilon/t$ is responsible for the production of an enormous number of particle near the big-bang.

Coincidence of the result (35) with that of spin 1 field case suggests to possibly extend the study to boson fields of arbitrary spin in RW space-time. Solutions for boson field equation of arbitrary spin have been recently obtained in the RW model [17], but, unfortunately, explicit normal mode solutions have not yet been determined. Finally it would also of interest to know whether a result similar to that of eq. (35) does hold in the fermionic case, and in particular for the Dirac field.

References


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