The t-J Model for the Antiferromagnetic Triangular Lattice Configuration

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Abstract

From our family of first-order Lagrangian and applying the super-symmetric version of the Faddeev-Jackiw symplectic formalism for a t-J model, the constrained structure for triangular lattice antiferromagnetic is found. In this context, the Hubbard operators are used as field variables. In this approach the Hubbard X-operators are used as field variables, and they satisfy the commutation rules of the graded algebra spl(2,1). This model is also analyzed by using the path-integral formalism, and so the correlation generating functional and the effective Lagrangian are constructed. On the other hand, we must introduce appropriate ghost field in order to obtain a renormalizable model. It is shown how propagators and vertices can be renormalized to each order. In particular, the renormalized antiferromagnetic magnon propagator coming from our formalism is studied. As an example the thermal softening of the magnon frequency is computed.

1 Introduction

It is natural to treat the electronic correlation effects [1], through the Hubbard operator representation, in which these operators are directly the field variables with which we write the Lagrangian of the model [2, 3]. In this approach
the Hubbard $\hat{X}$-operators, treated as indivisible objects, representing the real physical excitations and any decoupling scheme is used.

From our particular family of first-order constrained Lagrangians, obtained using the Faddeev-Jackiw (FJ) symplectic method [4] in its supersymmetric version [5, 6], we can study the t-J model in the framework of the slave-particle representations [7]. With this study we have found two kind of solutions of physical interest. In the context of the path-integral formalism, the different family of constrained Lagrangians can be mapped in the two slave-particle representations; the slave-boson and the slave-fermion representations (see Ref.[3]). The slave-boson representation privileges the fermion dynamics, and therefore seems to be more adequate to describe a Fermi liquid state [8, 9, 10]. Instead, the slave-fermion representation seems to give a good response when the system is closed to the antiferromagnetic order [11, 12].

It is important to understand the physics of the high-$T_c$ superconductors, and to solve how to move from one representation to the other. This is because in high-$T_c$ superconductors, both the Fermi liquid and the magnetic order states seem to be present.

Other problem appearing in these kind of models is to define the dynamics of fermions in the constrained Hilbert space, when the double occupancy of lattice sites is excluded. In this case also a convenient representation is given in terms of slave-particles [7].

Also, the slave-particle models exhibit a local gauge invariance which is destroyed in the mean field approximation. This local gauge invariance has associated a first-class constraint which is difficult to handle in the path-integral formalism.

The t-J model has three possible states on a lattice site: $|\alpha> = |0>$, $|+>$, $|->$. These states correspond respectively to an empty site, an occupied site with an electron of spin-up, or an occupied site with an electron of spin-down. Double occupancy is forbidden in the t-J model. In terms of these states the Hubbard $\hat{X}$-operators are defined as:

$$\hat{X}^{\alpha\beta}_i = i<\alpha | i\beta > .$$ (1)

when one of the index is zero and the other different from zero, the corresponding $\hat{X}$-operator is fermion-like, otherwise boson-like.

In this model the spin and charge degrees of freedom are present, for this reason the Hubbard $\hat{X}$-operators represent the real physical excitations, and verifies the graded algebra spl(2,1) given by:

$$[\hat{X}^{\alpha\beta}_i , \hat{X}^{\gamma\delta}_j]_\pm = \delta_{ij}(\delta^{\beta\gamma} \hat{X}^{\alpha\delta}_i \pm \delta^{\alpha\delta} \hat{X}^{\gamma\beta}_i) .$$ (2)
where the indices $\alpha, \beta, \gamma, \delta$ run in the values $+, -, 0$. In equation (2), the $+$ sign must be used when both operators are fermion-like, otherwise it corresponds the $-$ sign, and $i, j$ denote the site indices.

In order to describe the dynamics of the t-J model, the purpose is to find the family of first-order Lagrangians, written in terms of field variables which verify the graded commutation rules (2). In this way the family of Lagrangian can be mapped in the slave-fermion representation, written in terms of fermion-like and boson-like Hubbard $\hat{X}$-operators. The family of Lagrangians and the constraint structure of the model will be determined by using the Faddeev-Jackiw (FJ) symplectic method \[4\]. The two representations corresponds to different initial conditions imposed on the differential equation system produced when the FJ symplectic method is implemented. Moreover, the set of constraints is also provided by the symplectic formalism and it is second-class one \[2, 3\].

Later on, using the technique of path-integral, the correlation generating functional is written in terms of the above mentioned effective Lagrangian, which results non-polynomial. Moreover, the ghost fields needed to render the model renormalizable are introduced. We are able to build the standard Feynman diagrammatics of the model through appropriate propagators and vertices.

Finally, the boson self-energy and the renormalized expression for magnon propagator in the antiferromagnetic configuration is deduced, with the intention of implementing it in a triangular lattice, caracteristic of the cobaltates \[13\]. In particular, the softening energy effect is computed and analyzed.

It is assumed that we are close to an undoped regime where the system is an antiferromagnetic insulator, in this condition there is a small number of holes and it can be assumed that the hole density is constant. it must be determined later by consistency for a given value of the chemical potencial $\mu$.

\section{Definitions and previous results}

We sum up the main results obtained in Ref. \[3\], which are the starting point for our perturbative formalism.

As it is well known, the Faddeev-Jackiw (FJ) symplectic quantization method \[4\] is formulated on actions only containing first order time derivatives. So, we consider the following first order Lagrangian written in terms of the Hubbard $X$-variables:

\[ L = \sum_i a_{i\alpha\beta}(X)\dot{X}_i^{\alpha\beta} - V^{(0)}(X) \]
\[
\sum_{i} a_{\alpha\beta}(X) \dot{X}_{i}^{\alpha\beta} - (H(X) + \lambda^{a} \Omega_{a}). \tag{3}
\]

where the coefficients \(a_{\alpha\beta}\) are unknown and they are determined in such a way that a graded algebra (2) for the Hubbard \(X\)-operators must be verified, \(V^{(0)}\) is the symplectic potential and \(\lambda^{a}\) are appropriate Lagrange multipliers for the constraints \(\Omega_{a}\). It is important to remark that at this level the \(X\)-variables must be treated as classical fields.

From the equation (3) the constraints \(\Omega_{a}\) verify:

\[
\Omega_{a} = \frac{\partial V^{(0)}}{\partial \lambda^{a}} \tag{4}
\]

In the equation (3) \(H(X)\) is the usual t-J Hamiltonian:

\[
H(X) = \sum_{i,j,\sigma} t_{ij} X_{i}^{\sigma 0} X_{j}^{0\sigma} + \frac{1}{2} \sum_{ij} J_{ij} (X_{i}^{-} X_{j}^{+} - X_{i}^{+} X_{j}^{-}) - \mu \sum_{i,\sigma} X_{i}^{\sigma \sigma} \tag{5}
\]

where a term depending on the chemical potential \(\mu\) was added.

In equation (5) \(t_{ij}\) and \(J_{ij}\) are respectively the hopping and the effective exchange parameters between sites \(i\) and \(j\). The indices \(\alpha, \beta\) take the values \(0\) (empty state) or spin index \(\sigma = \pm\), up and down states respectively. So, the five Hubbard \(X\)-variables, \((X^{++}, X^{+}, X^{++}, X^{--})\) and \(X^{00}\), are boson-like and the four Hubbard \(X\)-variables, \((X^{0+}, X^{0-}, X^{+0}, X^{-0})\), are fermion-like.

If we apply the Faddeev-Jackiw symplectic algorithm on the first-order Lagrangian (3) \([4]\), by using the matrix \(M_{AB}\) one can obtain a particular solution of the differential equations giving the values for the coefficients \(a_{\alpha\beta}\) and the constraints \(\Omega_{a}\).

Therefore, the symplectic supermatrix associated to the Lagrangian (3) can be formally written as:

\[
M_{AB} = \left( \begin{array}{ccc}
\frac{\partial a_{\alpha\delta}}{\partial X^{\gamma\sigma}} & \frac{\partial a_{\alpha\beta}}{\partial X^{\gamma\sigma}} & \frac{\partial \lambda_{b}}{\partial X^{\gamma\sigma}} \\
\frac{\partial \Omega_{a}}{\partial X^{\gamma\sigma}} & 0 & \frac{\partial \lambda_{b}}{\partial X^{\gamma\sigma}} \\
\end{array} \right), \tag{6}
\]

where the compound indices \(A = \{(\alpha\beta), a\}\) and \(B = \{({\gamma\delta}) , b\}\) run in the different ranges of the complete set of variables defining the extended configuration space. In order to obtain an invertible symplectic supermatrix, the problem is to determine both, the Lagrangian coefficients \(a_{\alpha\beta}(X)\) and how many constraints \(\Omega_{a}\) are provided by the symplectic (FJ) algorithm.

It is easy to show that the invertible symplectic supermatrix \(M_{AB}(X)\) given in eq. (6) is a square \(6 \times 6\) dimensional one, and it can be written in the form:
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\[ M_{AB} = \begin{pmatrix} A_{bb} & B_{bf} \\ C_{fb} & D_{ff} \end{pmatrix}, \quad (7) \]

whose Bose-Bose parts \( A_{bb} \) and Fermi-Fermi parts \( D_{ff} \) are even elements of the Grassmann algebra and whose Bose-Fermi parts \( B_{bf} \) and Fermi-Bose parts \( C_{fb} \) are odd elements. As it is well known [15] the inverse \((M_{AB})^{-1}\) exists if and only if \( A_{bb} \) and \( D_{ff} \) have an inverse. In the present case the Bose-Bose part \( A_{bb} \) is an ordinary non-singular 4 \( \times \) 4 dimensional matrix and the Fermi-Fermi part \( D_{ff} \) is an ordinary non-singular 2 \( \times \) 2 dimensional matrix.

By using the equation \( M_{AB}(M_{BC})^{-1} = \delta_{A}^{C} \), and taking into account the equation:

\[ (M_{BC})^{-1} = -i(-1)^{|\varepsilon_{B}|} [\hat{B}, \hat{C}]_{\pm}, \quad (8) \]

where \(|\varepsilon_{B}|\) is the Fermi grading of the variable \( B \), differential equations on the Lagrangian coefficients \( a_{\alpha\beta}(X) \) and on the constraints \( \Omega_{a} \) are obtained.

Since there are only four bosonic fields, only two bosonic constraints are possible (Ref.[14]), and they may be written as:

\[ \Omega_{1} = X^{++} + X^{--} + (X^{0+}X^{+0} + X^{0-}X^{-0}) - 1 = 0, \quad (9) \]

\[ \Omega_{2} = X^{+-}X^{--} + \frac{1}{4}(X^{++} - X^{--})^{2} - (1 - \frac{1}{2}(X^{++} + X^{--}))^{2} + (X^{0+}X^{+0} + X^{0-}X^{-0}) = 0, \quad (10) \]

On the other hand, the fermionic constraints result:

\[ \Xi_{1} = X^{0+}X^{--} - X^{0-}X^{+-} = 0, \quad (11) \]

\[ \Xi_{2} = X^{+0}X^{--} - X^{0}X^{+} = 0, \quad (12) \]

\[ \Xi_{3} = X^{0+}X^{+-} - X^{0}X^{++} = 0, \quad (13) \]

\[ \Xi_{4} = X^{+0}X^{--} - X^{0}X^{++} = 0. \quad (14) \]

Only two of the fermionic constraints (11-14) must be considered as independent. In fact, from eqs. (13) and (14) and using the eq.(10), it is easy to
show that the eqs. (11) and (12) can be recovered. From this last fact, in our t-J model there two bosonic constraints given by eqs. (9) and (10), and two fermionic constraints given by eqs. (13) and (14). The completeness condition is obtained as one of the bosonic constraints (eq. (9)). Therefore, in eq. (9) \( \rho \) must be identified with the hole density, namely, proportional to the number of holes \( X^{00} \).

We remember that such a condition has an important physical meaning, and it must be imposed to avoid at quantum level the configuration with double occupancy at each site.

Therefore, we must emphasize that by means of our approach the completeness condition appears as necessary by consistency. Moreover, the coefficients \( a_{\alpha \beta} \) result:

\[
a_{+ -} = i \, F(u, v, \rho) \, X^{++}, \\
a_{- +} = a^*_{+ -} = -i \, F(u, v, \rho) \, X^{-+}, \\
a_{- 0} = \frac{i}{2} X^{0-}, \quad a_{0 -} = \frac{i}{2} X^{-0} \\
a_{+ 0} = \frac{i}{2} X^{0+}, \quad a_{0 +} = \frac{i}{2} X^{+0}
\]  

(15)  

(16)  

(17)  

(18)

where \( F(u, v, \rho) \) is a real function: \( F(u, v, \rho) = \frac{(1 + \rho) u + \alpha}{(2 - v)^2 - 4 \rho - u^2} \), being \( \alpha \) an arbitrary and non trivial integration constant.

Without loosing generality, we choose \( \alpha = -1 \). So, we can write the Lagrangian (3) as:

\[
L(X, \dot{X}) = i \sum_i \frac{(1 + \rho_i) u_i - 1}{(2 - v_i)^2 - 4 \rho_i - u_i^2} \left( X^{++}_i X^{--}_i - X^{+-}_i X^{-+}_i \right) \\
+ \frac{i}{2} \sum_{i, \sigma} \left( X^{0\sigma}_i \dot{X}^{0\sigma}_i + X^{0\sigma}_i \dot{X}^{0\sigma}_i \right) - H_{t-J}(X),
\]

(19)

Now, it is useful to write the boson-like Hubbard \( X \)-operators in terms of the real components \( S_\alpha \) (\( \alpha = 1, 2, 3 \)) of a vector field \( S \) and the fermion-like Hubbard \( X \)-operators in terms of suitable component spinors (Grassmann variables):

\[
X^{++}_i = \frac{1}{2s} \left( 1 - \rho_i \right) \left( s + S_{i3} \right),
\]

(20)
\begin{align*}
X_i^- &= \frac{1}{2s} (1 - \rho_i) (s - S_{i3}) , \\
X_i^+ &= \frac{1}{2s} (1 - \rho_i) (S_{i1} + i S_{i2}) , \\
X_i^- &= \frac{1}{2s} (1 - \rho_i) (S_{i1} - i S_{i2}) , \\
X_i^0 &= \Psi_{i+} , \quad X_i^0 &= \Psi^*_{i+} , \\
X_i^+ &= \Psi_{i-} , \quad X_i^- &= \Psi^*_{i-} ,
\end{align*}

where \( s \) is a constant and the hole density in the new variables writes
\( \rho_i = \Psi^*_{i+} \Psi_{i+} + \Psi^*_{i-} \Psi_{i-} \). Accounting the fermionic constraints \((11-14)\) it results
\( (1 - \rho_i) (1 + \rho_i) = 1 \).

The real vector field \( S \) can be identified with the spin only when \( \rho = 0 \), i.e., in the pure bosonic case.

The remaining bosonic constraint \((10)\) as function of the real vector field variable \( S \) writes:
\[ \Omega_2 = S_1^2 + S_2^2 + S_3^2 - s^2 = 0 . \]

Analogously, the two fermionic constraints \((13)\) and \((14)\) can be written:
\[ \Xi_3 = \Psi^*_- (S_1 + i S_2) - \Psi^*_+ (s + S_3) = 0 , \]  \( \Xi_4 = \Psi_- (S_1 - i S_2) - \Psi_+ (s + S_3) = 0 . \]

As already mentioned, the Lagrangian can be mapped into the slave-fermion representation is useful when the system is closed to a ferromagnetic or an antiferromagnetic configuration. Starting from a non-polynomial Lagrangian a perturbative formalism can be developed in the framework of the path-integral.

### 3 Path Integral formalism

In this section, our starting point in the partition function \( Z \) written in the new set of variables:
\[ Z = \int \mathcal{D}S_{i1} \mathcal{D}S_{i2} \mathcal{D}S_{i3} \mathcal{D}\Psi_{i\alpha} \mathcal{D}\Psi^*_{i\bar{\alpha}} \delta(\Omega_2)\delta(\Xi_3)\delta(\Xi_4) \]
\[ \cdot (s\text{det}\mathcal{M}_{AB})^\frac{1}{2} \left( \frac{\partial X}{\partial S} \right)_i \exp \left( i \int dt \ L^E(S,\Psi) \right) , \]  
\hspace{1cm} (29)

Given the particular solution for the coefficients \( a_{\alpha\beta} \) of the Lagrangian (3) leads to the following Euclidean Lagrangian:

\[ L^E = -\frac{i}{2s} \sum_i \frac{S_{i1} \dot{S}_{i2} - S_{i2} \dot{S}_{i1}}{s + S_{i3}} + \sum_{i,\sigma} \Psi_{i\alpha} \dot{\Psi}^*_{i\bar{\alpha}} + H_{t-J} , \]
\hspace{1cm} (30)

and the set of second-class constraints:

\[ \Omega = S_1^2 + S_2^2 + S_3^2 - s^2 = 0 , \]  
\hspace{1cm} (31)

\[ \Xi_3 \equiv \Xi^* = \Psi^*_+ (S_1 + iS_2) - \Psi^*_+ (s + S_3) = 0 , \]  
\hspace{1cm} (32)

\[ \Xi_4 \equiv \Xi = \Psi^- (S_1 - iS_2) - \Psi^+ (s + S_3) = 0 . \]  
\hspace{1cm} (33)

We note that the functional \( \mathcal{M}_{AB} \) appearing in eq. (29), is the superdeterminant of the Faddeev-Jackiw symplectic supermatrix. Such a superdeterminant is computed and after some algebra we get:

\[ \mathcal{M}_{AB} = \frac{4s^2(1 + \rho)^2}{(1 - \rho)^2(s + S_3)^2} . \]  
\hspace{1cm} (34)

where \( \rho_i = \Psi^*_{i+} \Psi_{i+} + \Psi^*_{i-} \Psi_{i-} \) is the hole density.

Analogously, the super Jacobian of the transformation \( \frac{\partial X}{\partial S} \) is given by:

\[ \frac{\partial X}{\partial S} = -i(1 - \rho)^3 \frac{2s^3}{s} . \]  
\hspace{1cm} (35)

Form de eqs. (34) and (35) it can be see that both functionals are field variable dependent.

We must remark once that only when the hole density \( \rho \) vanishes (pure bosonic case), the real vector \( \mathbf{S} \) can be related with the spin.

Equation (29) can be written in an alternative way by using the integral representation for the delta functions on the constraints \( \phi \), which is written:

\[ \delta(\phi) = \int \mathcal{D}\chi \exp \left( i \int dt \chi \phi \right) , \]  
\hspace{1cm} (36)

where the quantities \( \chi \) are suitable bosonic or fermionic Lagrange multipliers.
The correlation generating functional is obtained integrating out the fermionic constraints (32) and (33) and by using the integral representation for the delta function on the non-linear bosonic constraints (31). Therefore, the partition function (29) can be written:

\[
Z = \int DS_{i1} \ D S_{i2} \ D S_{i3} \ D \Psi_{i-} \ D \Psi_{i-}^{*} \ D \lambda_{i} \ (s \text{det} \mathcal{M}_{AB})_{i}^{1/2} \ \exp \left( - \int dt \ L_{eff}^{E}(S, \Psi) \right),
\]

(37)

where \( L_{eff}^{E}(S, \Psi) \) is defined by:

\[
L_{eff}^{E}(S, \Psi) = -\frac{i}{2s} \frac{1}{1 - \rho} \sum_{i} \frac{S_{i2} \dot{S}_{i1} - S_{i1} \dot{S}_{i2}}{s + S_{i3}} - \sum_{i} \lambda_{i}(S_{i1}^{2} + S_{i2}^{2} + S_{i3}^{2} - s^{2})
\]

\[- s \sum_{i} \frac{1}{s + S_{i3}}(\dot{\Psi}_{i-}^{*} \Psi_{i-} - \dot{\Psi}_{i-} \Psi_{i-}^{*}) - 2s \mu \sum_{i, \sigma} \frac{1}{s + S_{i3}} \Psi_{i-}^{*} \Psi_{i-}
\]

\[+ \sum_{i,j} t_{ij} \Psi_{i-}^{*} \Psi_{j-}
\]

\[
- \frac{1}{8s^{2}} \sum_{i,j} J_{ij}(1 - \rho_{i})(1 - \rho_{j})[S_{i1} S_{j1} + S_{i2} S_{j2} + S_{i3} S_{j3} - s^{2}].
\]

(38)

being \( J_{ij} < 0 \) for an antiferromagnetic state (\( J_{ij} > 0 \) for ferromagnetic state).

The first term in Eq. (38) shows the non-polynomial structure of the kinetic part of the Lagrangian.

The simplectic supermatrix \( \mathcal{M}_{AB} \) writes:

\[
\mathcal{M}_{AB} = \begin{pmatrix}
0 & \frac{1 - \rho}{s(s + S_{1})} & \frac{(1 - \rho)S_{2}}{2s(s + S_{3})^{2}} & -2S_{1} & 0 & 0 \\
\frac{1 - \rho}{s(s + S_{1})} & 0 & \frac{1 - \rho}{s(s + S_{3})^{2}} & -2S_{2} & 0 & 0 \\
\frac{(1 - \rho)S_{2}}{2s(s + S_{3})^{2}} & \frac{(1 - \rho)S_{1}}{2s(s + S_{3})^{2}} & 2S_{3} & i \frac{s}{s + S_{3}} & \Psi^{*}_{-} & i \frac{s}{s + S_{3}} \Psi_{-} \\
2S_{1} & 2S_{2} & 2S_{3} & 0 & 0 & 0 \\
0 & 0 & -i \frac{s}{s + S_{3}} & \Psi^{*}_{-} & 0 & 0 \\
0 & 0 & -i \frac{s}{s + S_{3}} & \Psi_{-} & 0 & -i \frac{2s}{s + S_{3}}
\end{pmatrix}.
\]

(39)

From now on the system fluctuating around a antiferromagnetic state (\( J_{ij} < 0 \)) can be assumed. In such conditions, the components of the real vector field \( \mathbf{S} \) are close to be the spin variables, and so in both cases the vector \( \mathbf{S} \) is written:

\[
\mathbf{S} = (0, 0, s') + (\tilde{S}_{1}, \tilde{S}_{2}, \tilde{S}_{3})
\]

(40)

where \( \tilde{S}_{1}, \tilde{S}_{2}, \tilde{S}_{3} \) are the fluctuations. To simplify notation hereafter the tilde over the fluctuations is omitted.
The next step is to get the Feynman rules, which are necessary to rewrite the superdeterminant of the symplectic supermatrix $M_{AB}$ appearing in the partition function equation (37) as a path-integral over Faddeev-Popov ghost superfields $(\theta_\alpha, Z_i \text{ con } \alpha = 1, 2, 3, 4, \ i = 1, 2)$.

By simple calculation is:

$$(s\text{det} M_{AB}) = \frac{\det A}{\det^{-1} D}.$$  \hfill (41)

where $A$ is the Bose-Bose parts and $D$ is the Fermi-Fermi parts of $M_{AB}$. From Eq. (39) is seems that $A$ is a real antisymmetric $4 \times 4$ dimensional matrix, defining $I_4(A) = 4(\det A)^{1/2}$ it can be written as:

$$I_4 = \int D\theta_\alpha \exp\left( -\int_0^\beta d\tau \theta^T A \theta \right),$$  \hfill (42)

where $\theta_\alpha$ are four real Grassmann numbers or ghost fields.

Similarly, the $(\det D)^{-1/2}$ of the $2 \times 2$ dimensional matrix $D$ is written:

$$(\det D)^{-1/2} = \int DZ^* DZ \exp\left( -\int_0^\beta d\tau z^* C z \right),$$  \hfill (43)

where $Z = Z_1 + iZ_2, Z^* = Z_1 - iZ_2$ are complex scalar fields, and $C = \frac{2s}{s+s'}$ ($iC = -D_{12} = -D_{21}$).

Because of this the Lagrangian $L_{\text{ghost}}$ for the bosonic ghost fields $\theta_\alpha$ and fermionic ghost fields $Z$ is given by:

$$L_{\text{ghost}} = \theta^T A \theta + Z^* C Z,$$  \hfill (44)

and the total Lagrangian writes:

$$L = L_{\text{eff}}^E + L_{\text{ghost}}.$$

(45)

Once the effective Lagrangian (38) and the matrix elements of the two matrices, $A$ and $D$ are written in terms of the fluctuations (40) the total Lagrangian (45) is ready to construct the diagrammatics in a perturbative way.

The matrix elements of the two matrices ($A$ and $D$) are:

$$A_{12} = -A_{21} = \frac{(1-\rho)}{s(s+s')} \sum_{n=0}(-1)^n \left( \frac{S_3}{s+s'} \right)^n,$$

$$A_{13} = -A_{31} = \frac{(1-\rho)S_2}{2s(s+s')^2} \sum_{n=0}(-1)^n (n+1) \left( \frac{S_3}{s+s'} \right)^n,$$

$$A_{14} = -A_{41} = -2S_1,$$  \hfill (46)\hfill (47)\hfill (48)
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\[ A_{23} = -A_{32} = -\frac{(1 - \rho)S_1}{2s(s + s')^2} \sum_{n=0}^\infty (-1)^n(n + 1) \left( \frac{S_{i3}}{s + s'} \right)^n, \]  
(49)

\[ A_{24} = -A_{42} = -2S_2, \]  
(50)

\[ A_{34} = -A_{43} = -2(s' + S_3), \]  
(51)

and

\[ C = \frac{2s}{(s + s')} \sum_{n=0}^\infty (-1)^n \left( \frac{S_3}{s + s'} \right)^n, \]  
(52)

4 Diagrammatics to one loop and Feynman rules for the antiferromagnetic configuration

In this section, we begin by analysing the antiferromagnetic case \((J_{ij} < 0)\).

As it is usual in the antiferromagnetics configuration a rotation of spins on the second sublattice by 180° about the \(S_1\) axis is performed (Mattis, 1981):

\[ S_{j1} \to S_{j1}, S_{j2} \to -S_{j2}, S_{j3} \to -S_{j3} \text{ and } \Psi_{j\sigma} \to \Psi_{j\bar{\sigma}}, \]  
(53)

where \(\sigma \to \bar{\sigma}\) implies \(\pm \to \mp\).

This canonical transformation changes the antiferromagnetic configuration into a ferromagnetic one with all spins up, and so it is not necessary to distinguish between sublattices. It can be seen that the effective Lagrangian (38) is not invariant under such transformation, because the non invariance of the \(t - J\) Hamiltonian.

Therefore in this case the effective Lagrangian (38) in terms of the fluctuations (40) takes the form:

\[
L_{eff}^E = \frac{i}{2s}(1 - \rho) \sum_i \left( \frac{S_{i1} \dot{S}_{i2} - S_{i2} \dot{S}_{i1}}{s + s'} \left[ 1 + \sum_{n=1}^\infty (-1)^n \left( \frac{S_{i3}}{s + s'} \right)^n \right] \right) \\
- \frac{s}{s + s'} \sum_i \left( \dot{\Psi}_{i-} \Psi_{i-} + \dot{\Psi}_{i-} \Psi_{i-}^* \right) \left[ 1 + \sum_{n=1}^\infty (-1)^n \left( \frac{S_{i3}}{s + s'} \right)^n \right] \\
- \frac{2s \mu}{s + s'} \sum_i \Psi_{i-}^* \dot{\Psi}_{i-} \left[ 1 + \sum_{n=1}^\infty (-1)^n \left( \frac{S_{i3}}{s + s'} \right)^n \right] \\
+ \frac{1}{2(s + s')} \sum_{i,j} t_{ij} \dot{\Psi}_{i-} \Psi_{j-} \left[ S_{i1} - iS_{i2} + S_{j1} + iS_{j2} + H.c \right]
\]
\[ + \frac{1}{2(s + s')} \sum_{i,j} t_{ij} \Psi_i \Psi_j^* \left[ (S_{i1} - iS_{i2}) \left( \sum_{n=1}^{\infty} (-1)^n \left( \frac{S_{i3}}{s + s'} \right)^n \right) \right. \\
+ \left( S_{j1} + iS_{j2} \right) \left( \sum_{n=1}^{\infty} (-1)^n \left( \frac{S_{j3}}{s + s'} \right)^n \right) + H.c \right] \\
- \frac{1}{8s^2} J' \sum_{i,I} \left[ S_{i1}S_{(i+I)1} - S_{i2}S_{(i+I)2} - S_{i3}S_{(i+I)3} + S_{i1}^2 + S_{i2}^2 + S_{i3}^2 \right] \\
- 2s' \sum_i \lambda_i S_{i3} - \sum_i \lambda_i [S_{i1}^2 + S_{i2}^2 + S_{i3}^2] \right) , \quad (54) \]

where \( J' < 0 \) was defined \( J' = J(1 - \rho)^2 \), and \( \sum_I \) indicates sum over nearest-neighbor sites.

It is easy to show that the symplectic supermatrix \( \mathcal{M}_{AB} \) does not change under the canonical rotation (53).
Antiferromagnetic triangular lattice configuration

Analogously, the Lagrangian for the ghost fields takes the form:

\[ L_{\text{ghost}}(\theta_\alpha, Z) = \theta^T_a (G^{\alpha\beta})^{-1} \theta_\beta + \theta^T_a \Gamma^{\alpha\beta} V^a \theta_\beta + \frac{1}{n!} \sum_{n=2}^\infty \theta^T_a \Gamma^{\alpha\beta}_{a_1...a_n} V^{a_1}...V^{a_n} \theta_\beta + Z^*(G)^{-1} Z + \frac{1}{n!} \sum_{n=1}^\infty Z^* \Delta_{a_1...a_n} V^{a_1}...V^{a_n} Z. \]  

(55)

From the eqs. (54) and (55) propagators and vertices can be defined straightforward:

i) Propagators:

By defining a new extended four component bosonic vector field \( V^a \) as: \( V^a = (S_1, S_2, S_3, \lambda) \), the 4 \( \times \) 4, bilinear bosonic part \( L^B(V) \) of the Lagrangian given in (54) can be written:

\[ L^B(V) = \sum_{i,j} V^a_i D_{(0)ab}^{-1} V^b_j, \]  

(56)

where the 4 \( \times \) 4 matricial propagator is:

\[ D_{ab}^{(0)}(q, \omega_n) = \begin{pmatrix} -J'z \alpha^2 \frac{(1-\gamma_q)}{\omega_n^2+\omega_q^2} \beta^2 & s \cdot \alpha \frac{\omega_n}{\omega_n^2+\omega_q^2} \beta & 0 & 0 \\ -s \cdot \alpha \frac{\omega_n}{\omega_n^2+\omega_q^2} \beta & -J'z \alpha^2 \frac{(1-\gamma_q)}{\omega_n^2+\omega_q^2} \beta^2 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2s} \\ 0 & 0 & -\frac{1}{2s} \frac{J'z(1-\gamma_q)}{32s^2s'^2} \\ \end{pmatrix} \]  

(57)

where \( \alpha = (s + s') \) and \( \beta = (1 + \rho) \).

The antiferromagnetic free boson propagator is represented by a continuous wavy line connecting two generic points \( a \) and \( b \):

\[ a \xrightarrow{\text{wavy line}} b \]

Figure 1: Propagator \( D_{0}^{ab} \) of the boson field \( V^a \)

The quantities \( q \) and \( \omega_n \) are respectively the momentum and the Matsubara frequency of the bosonic field.

In equation (57) was defined:

\[ \omega_q = \frac{zJ'}{8s}(s + s')(1 + \rho) \sqrt{1 - \gamma_q^2}. \]  

(58)
where \( z \) is the number of first nearest-neighbor sites; \( z \gamma_q = \sum_I \exp(iq.I) \); and \( I \) is the lattice vector.

Consequently, the antiferromagnetic free magnon propagator (57) remains defined by:

\[
D_{(0)}^{-+} = \left< T(S^-(\tau) S^+(0) \right> = \frac{1}{2} \left( D_{(0)}^{11} + D_{(0)}^{22} + i(D_{(0)}^{12} - D_{(0)}^{21}) \right) = -s(s + s')(1 + \rho) \left( \frac{J' z(s + s')}{8s}(1 + \rho) + i\omega_n \right) \frac{1}{\omega_q^2 + \omega_n^2} .
\]

(59)

On the other hand, the fermionic sector in the antiferromagnetic configuration really differs from the ferromagnetic case and it must be carefully analyzed. As we will see in this case the main problem is to give the mechanism for the fermion propagation.

From (54) the bilinear fermionic part \( L^F(\Psi) \) of the Lagrangian reads:

\[
L^F(\Psi) = \sum_{k,\nu_n} \Psi^*_-(k,\nu_n) G^{-1}_0 \Psi_-(k,\nu_n) ,
\]

(60)

where we have named:

\[
G^{-1}_0 = \frac{2s}{s + s'}(i\nu_n - \mu).
\]

(61)

The inverse of this scalar function given by:

\[
G_0 = \frac{s + s'}{2s} \frac{1}{i\nu_n - \mu} ,
\]

(62)

is a (non-propagating) functional which only depends on the Matsubara frequency \( \nu_n \). So, the situation for the fermionic sector is very different to the case of the ferromagnetic free fermion propagator.

The antiferromagnetic free fermion propagator is represented by a line connecting two generic points:

Figure 2: Propagator \( G_0 \) of the fermion field \( \Psi_- \)

where \( k \) and \( \nu_n \) are respectively the momentum and the Matsubara frequency of the fermion field.

We associate with the ghost field \( \theta_\alpha \), the propagator:

\[
G_{\alpha\beta} = (1 + \rho)s(s + s')(\delta^1_{\alpha} \delta^2_{\beta} - \delta^2_{\alpha} \delta^1_{\beta}) + \frac{1}{2s'}(\delta^3_{\alpha} \delta^4_{\beta} - \delta^4_{\alpha} \delta^3_{\beta}) ,
\]

(63)
that is represented by a dashed line connecting two generic points, as it is shown in figure 3.

\[
\begin{array}{ccc}
\alpha & & \theta \\
\hline
& \text{---} & \\
\beta
\end{array}
\]

Figure 3: Propagator \( G_{\alpha\beta} \) associated with the ghost field \( \theta_\alpha \)

We associate with the ghost complex scalar field \( Z \), the propagator:

\[
G = \frac{s + s'}{2s},
\]

that is represented by a dotted line:

\[
\ldots Z \ldots
\]

Figure 4: Propagator \( G \) associated with the ghost complex scalar field \( Z \)

ii) Vertices:

The expressions of the three-leg and four-leg different vertices containing physical fields are respectively:

The three-leg boson vertex \( F_{abc} \) defined by:

\[
F_{abc}(\omega_1, \omega_2, \omega_3) = - \left[ \frac{(1 - \rho)}{2s(s + s')} \right] \left[ (\omega_2 - \omega_1) (\delta_a^1 \delta_b^2 - \delta_a^2 \delta_b^1) \delta_c^3 \\
+ (\omega_3 - \omega_1) (\delta_a^1 \delta_c^2 - \delta_a^2 \delta_c^1) \delta_b^3 + (\omega_3 - \omega_2) (\delta_b^1 \delta_c^2 - \delta_b^2 \delta_c^1) \delta_a^3 \\
+ 2 \left[ \delta_a^4 (\delta_b^1 \delta_c^1 + \delta_b^2 \delta_c^2 + \delta_b^3 \delta_c^3) + \delta_b^4 (\delta_a^1 \delta_c^1 + \delta_a^2 \delta_c^2 + \delta_a^3 \delta_c^3) \\
+ \delta_c^4 (\delta_a^1 \delta_b^1 + \delta_a^2 \delta_b^2 + \delta_a^3 \delta_b^3) \right] \right].
\]

The four-leg boson vertex \( F_{abcd} \) defined by:

\[
F_{abcd}(\omega_1, \omega_2, \omega_3, \omega_4) = \left[ \frac{(1 - \rho)}{s(s + s')^2} \right] \left[ (\omega_2 - \omega_1) (\delta_a^1 \delta_b^2 - \delta_a^2 \delta_b^1) \delta_c^3 \delta_d^3 \\
+ (\omega_3 - \omega_1) (\delta_a^1 \delta_c^2 - \delta_a^2 \delta_c^1) \delta_b^3 \delta_d^3 + (\omega_4 - \omega_1) (\delta_b^1 \delta_d^2 - \delta_b^2 \delta_d^1) \delta_a^3 \delta_c^3 \\
+ (\omega_3 - \omega_2) (\delta_b^1 \delta_c^2 - \delta_b^2 \delta_c^1) \delta_a^3 \delta_d^3 + (\omega_4 - \omega_2) (\delta_c^1 \delta_d^2 - \delta_c^2 \delta_d^1) \delta_a^3 \delta_b^3 \\
+ (\omega_4 - \omega_3) (\delta_c^1 \delta_b^2 - \delta_c^2 \delta_b^1) \delta_a^3 \delta_d^3 \right].
\]

Looking at the Lagrangian (54) it can be seen that in the antiferromagnetic configuration the boson-fermion interaction vertex can be written:
\[ L_{int}^{B-F} = \frac{1}{n!} \Psi_\nu^* (\nu', k') U_{a_1 \ldots a_n} V^{a_1 \ldots} V^{a_n} \Psi_\nu (\nu, k), \quad (67) \]

where \( U_{a_1 \ldots a_n} \) is the vertex corresponding to one and two bosonic legs with two fermionic legs. In particular, the three-leg vertex \( U_a \) (one boson-two fermions) explicitly reads:

\[ U_a = \frac{1}{s + s'} \left[ (\varepsilon(k) + \varepsilon(k'))\delta^1_a + i(\varepsilon(k') - \varepsilon(k))\delta^2_a - \frac{s}{s + s'} \left[ i(\nu_n + \nu'_n) - 2\mu \right] \delta^3_a \right], \quad (68) \]

The four-leg vertex \( U_{ab} \) (two bosons-two fermions) defined by:

\[ U_{ab} = \left[ \frac{-1}{(s + s')^2} (\varepsilon_{k'} + \varepsilon_k) \left[ \delta^1_a \delta^3_b + \delta^3_a \delta^1_b \right] + \frac{i}{(s + s')^2} (\varepsilon_{k'} - \varepsilon_k) \left[ \delta^2_a \delta^3_b + \delta^3_a \delta^2_b \right] + \frac{2s}{(s + s')^3} \left[ i(\nu + \nu') - 2\mu \right] \delta^3_a \delta^3_b \right]. \quad (69) \]

The eqs. (68) and (69) show that the boson-fermion interaction for the antiferromagnetic configuration has also a different structure with respect to the ferromagnetic one.

Since the remaining vertices containing more than four bosons, its construction can be done systematically.

Finally the vertices containing ghost fields are:

The three-leg vertex \( \Gamma^{\alpha\beta}_{a} \) (one-boson, two-ghost \( \theta \)) defined by:

\[ \Gamma^{\alpha\beta}_{a} = -2(\delta^\alpha_a \delta^\beta_1 - \delta^\beta_1 \delta^\alpha_a) - \frac{1 - \rho}{2 s (s + s')^2} (\delta^\alpha_2 \delta^\beta_3 - \delta^\beta_3 \delta^\alpha_2), \quad (70) \]

\[ \Gamma^{\alpha\beta}_{a} = -2(\delta^\beta_1 \delta^\alpha_3 - \delta^\alpha_3 \delta^\beta_1) + \frac{1 - \rho}{2 s (s + s')^2} (\delta^\alpha_2 \delta^\beta_3 - \delta^\beta_3 \delta^\alpha_2), \quad (71) \]

\[ \Gamma^{\alpha\beta}_{a} = -2(\delta^\beta_1 \delta^\alpha_3 - \delta^\alpha_3 \delta^\beta_1) + \frac{1 - \rho}{s (s + s')^2} (\delta^\alpha_2 \delta^\beta_3 - \delta^\beta_3 \delta^\alpha_2), \quad (72) \]

and \( \Gamma^{\alpha\beta}_{a} = 0 \).

The three-leg vertex \( \Delta_a \) (one-boson, two-ghost \( Z \)) defined by:

\[ \Delta_a = -\frac{2s}{(s + s')^2} \delta^3_a \quad (73) \]
The remaining vertices containing two ghost fields and more than one boson can be systematically constructed.

All vertices are shown in the following figure:

![Figure 5: Vertices of the model](image)

We have all the elements to build the diagrammatic to one loop, which can be used to calculate renormalized quantities such as propagators, vertices and self-energies.

For example, the total fermion (or boson) bself-energy $\Sigma(k, i\nu_n)$ is obtained by the sum of the contributions of six one-loop diagrams constructed with only physical bosons and fermions (see figure 6), and six contributions coming from the ghost loops (see figure 7):

![Figure 6: One loops diagrams constructed with only physical fields](image)

and whose analytical expressions are respectively:
\[ \Pi^{(1)}_{ab} (\omega, q) = \frac{1}{2N_s} \sum_{\omega', q'} F_{abc} (\omega, \omega') D_{(0)}^{de} (\omega', q') F_{ebf} (\omega, \omega') D_{(0)}^{fc} (\omega' - \omega, q' - q), \]  
(74)

\[ \Pi^{(2)}_{ab} (\omega, q) = \frac{1}{2N_s} \sum_{\omega', q'} F_{acb} (\omega) D_{(0)}^{cd} (0) F_{def} (\omega') D_{(0)}^{ef} (\omega', q'), \]  
(75)

\[ \Pi^{(3)}_{ab} (\omega, q) = \frac{1}{2N_s} \sum_{\omega', q'} F_{acdb} (\omega, \omega') D_{(0)}^{cd} (0) U_d G_0 (k, \nu_n), \]  
(76)

\[ \Pi^{(4)}_{ab} (\nu, k) = (-1) \frac{1}{N_s} \sum_{\nu, k} U_a G_0 (k, \nu_n) U_b G_0 (k - q, \nu_n - \omega_n), \]  
(77)

\[ \Pi^{(5)}_{ab} (\nu, k) = (-1) \frac{1}{2N_s} \sum_{\nu, k} F_{acb} (\omega) D_{(0)}^{cd} (\omega') U_d G_0 (k, \nu_n), \]  
(78)

\[ \Pi^{(6)}_{ab} (\nu, k) = (-1) \frac{1}{2N_s} \sum_{\nu, k} U_{ab} G_0 (k, \nu_n), \]  
(79)

Figure 7: One loops diagrams coming from the ghost loops
\[\Pi_{ab}^{(7)}(\omega, q) = (-1) \frac{1}{2N_s} \sum_{\omega', q'} \Gamma_{a}^{\beta\alpha}(\omega, \omega') \mathcal{G}_{\alpha\gamma}(\omega', q') \Gamma_{b}^{\gamma\delta}(\omega, \omega') \mathcal{G}_{\delta\beta}(\omega' - \omega, q' - q) ,\]  
\[\Pi_{ab}^{(8)}(\omega, q) = (-1) \frac{1}{2N_s} \sum_{\omega', q'} F_{acb}(\omega) D_{0(0)}^{cd}(0) \Gamma_{d}^{\beta\alpha}(\omega') \mathcal{G}_{a\beta}(\omega', q') ,\]  
\[\Pi_{ab}^{(9)}(\omega, q) = (-1) \frac{1}{2N_s} \sum_{\omega', q'} \Gamma_{ab}^{\beta\alpha}(\omega, \omega') \mathcal{G}_{\alpha\beta}(\omega', q') ,\]  
\[\Pi_{ab}^{(10)}(\omega, q) = \frac{1}{N_s} \sum_{\omega', q'} \Delta_a \mathcal{G} \Delta_b \mathcal{G} ,\]  
\[\Pi_{ab}^{(11)}(\omega, q) = \frac{1}{2N_s} \sum_{\omega', q'} F_{acb}(\omega) D_{0(0)}^{cd}(\omega') \Delta_d \mathcal{G} ,\]  
\[\Pi_{ab}^{(12)}(\omega, q) = \frac{1}{2N_s} \sum_{\omega', q'} \Delta_{ab} \mathcal{G} ,\]  
where \(N_s\) is the lattice number of sites. Note that in the eqs.(80-82) the minus sign have been taken into account for the fermionic ghost one-loop.

Using the diagrammatics given, and by straightforward computation, finite expression of the different components of the fermion and boson self-energy (74-85) can be found. It can be shown that the infinities (constant divergences) appearing in the eqs.(74-79) originated when the sum over the Matsubara frequency \(\omega'\) is carried out, are mutually canceled with the infinities appearing in the eqs.(80-85). Thus, for example a renormalized expression for the boson self-energy \(\Pi_{ab}^{(R)}\) and the fermion self-energy \(\Sigma_{-1}(k, \nu_n)\) can be obtained. It is important to remark that in the model the divergences only appear in one-loop calculations. It can be seen that in more than one-loop calculations, the diagrams containing ghost fields give finite contributions to the renormalized expressions of the n-point functions.

Finally, by means of the Dyson equation can also define:
\[(\mathcal{D}^{(R)})^{-1}_{ab} = (\mathcal{D}^{(R)}_{0(0)})_{ab}^{-1} - \Pi_{ab}^{(R)} ,\]  
it is possible to find the expression for the renormalized antiferromagnetic magnon propagator.

By definition, the renormalized antiferromagnetic magnon propagator \(D_{(R)}^{-+}\) is given by:
\[D_{(R)}^{-+} = \frac{1}{2} \left( D_{(R)}^{11} + D_{(R)}^{22} + i(D_{(R)}^{12} - D_{(R)}^{21}) \right) .\]
where from the Dyson eq.(86), the renormalized boson propagator components are given by:

\[ D^{11}_{(R)} = D^{22}_{(R)} = 2s^2 \frac{[(1 - \rho)\omega_q - 2s^2\Pi_{11}^{(R)}]}{[(1 - \rho)\omega_q - 2s^2\Pi_{11}^{(R)}]^2 + [(1 - \rho)\omega_n + 2s^2\Pi_{12}^{(R)}]^2}, \]  

(88)

\[ D^{12}_{(R)} = -D^{21}_{(R)} = 2s^2 \frac{[(1 - \rho)\omega_n + 2s^2\Pi_{12}^{(R)}]}{[(1 - \rho)\omega_q - 2s^2\Pi_{11}^{(R)}]^2 + [(1 - \rho)\omega_n + 2s^2\Pi_{12}^{(R)}]^2}. \]  

(89)

Since we are treating the antiferromagnetic case, in the above expressions was taken \( s' = s \).

It can be seen that the contribution to the components \( \Pi_{11}^{(R)} \) and \( \Pi_{12}^{(R)} \) of the fermionic boson self-energy arise from the diagrams corresponding to the eqs.(74,75) and eqs.(78-79).

Moreover, the sum of the contributions of the tadpole graphs give an irrelevant constant depending on the total fermion energy \( \sum_k \varepsilon_k n_F(\varepsilon_k - \mu) \).

With the aim to confront some prediction of our model with others previous well known results obtained for instance from the non linear spin wave model [16], it is useful to compute the correction to the magnon energy \( \omega_q \) or renormalized spin-wave energy.

Therefore, by computing explicitly the matrix elements (88) and (89) using the renormalized expressions of the self-energy, the renormalized antiferromagnetic magnon propagator \( D^{\pm}_{(R)} \) results:

\[ D^{\pm}_{(R)} = \frac{1}{2} \left( D^{11}_{(R)} + D^{22}_{(R)} + i(D^{12}_{(R)} - D^{21}_{(R)}) \right) = 2s^2 (1 + \rho) \frac{1}{\omega_q - i\omega_n - P(q(\omega_n))}, \]  

(90)

where in Eq.(90) was defined:

\[ P_q(\omega_n) = \frac{(1 + \rho)}{N_s} \left( \sum_{q'} n_B(\omega_{q'}) (\omega_{q'} - \omega_{q'-q}) + i\omega_n \sum_{q'} n_B(\omega_{q'}) \right). \]  

(91)

without taken into account the constant total fermion energy.

In eq. (91) the Bose occupation number \( n_B(\omega_q) \) was introduced. Moreover, accounting physical requirements in the calculation of the renormalized boson self-energy \( \Pi_{ab} \) when the sum over Matsubara frequency’s are carried out, only the single pole at \( \omega_q > 0 \) (see equation (59)) must be taken into account.
Now, carrying out the analytic continuation \( i\omega_n = \omega + i\delta \), the thermal correction to the antiferromagnetic magnon energy \( \omega_q \) can be found, and is given by:

\[
\Delta(\omega_q) = \lim_{\delta \to 0} \text{Re} \, P_q(i\omega_n = \omega_q + i\delta) = \frac{(1 + \rho)}{2s^2N_s} \sum_{q'} (\omega_{q'} - \omega_{q'-q} + \omega_q) n_B(\omega_{q'}) .
\]  

(92)

Since:

\[
n_B(\omega_{q'}) = n_B(\omega_{-q'}) \quad \text{and} \quad \sin(qt) = -\sin(-qt)
\]

(93)

and taking into account that:

\[
\sum_{q'} \cos(qt I_1)n_B(\omega_{q'}) = \sum_{q'} \cos(qt I_2)n_B(\omega_{q'}) = \sum_{q'} \cos(qt I_3)n_B(\omega_{q'})
\]

(94)

it can be shown that:

\[
\sum_{q'} \omega_q \omega_{q'} n_B(\omega_q) = \alpha \sum_{q'} (\omega_q + \omega_{q'} - \omega_{q'-q}) n_B(\omega_{q'}) .
\]

(95)

Therefore, the renormalized magnon energy is given by:

\[
\omega_q(T) = \omega_q \left[ 1 - \frac{4}{J' z N_s} \sum_{q'} \omega_{q'} n_B(\omega_{q'}) \right],
\]

(96)

showing clearly the thermal softening of the magnon frequency.

At this stage, it is possible to confront this result of our model with others well known previously obtained from the non linear spin wave model in the pure bosonic case [16]. The finite contributions of the one loop diagrams (74,75) correspond in the scheme of the non linear spin wave model to the direct and exchange contributions to non linear magnon Hamiltonian (see for instance ref. [16]). That is to say, mathematically corresponds to consider that the self-energy expressions (74-79) are analytical functions in \(-\omega_q\).
5 Antiferromagnetic configuration. Self-energy calculations in a triangular lattice

We consider a triangular lattice, characteristic of the cobaltates. As was stated before, \( z \gamma_q = \sum_I \exp(iq.I) \), where \( I \) is the lattice vector, and \( z \) is the number of first nearest-neighbor sites. In this configuration \( z = 6 \), and the vectors lattice \( I_i \) are:

\[
\begin{align*}
I_1 & = (1, 0) \\
I_2 & = (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \\
I_3 & = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})
\end{align*}
\]  

as shown in figure 8.

![Figure 8: The vectors of the triangular lattice](image)

Replacing \( z \gamma_q \) in eq.(58), results:

\[
\omega_q = \alpha - \beta \sum_{i=1}^{3} \cos(q.I_i) \tag{98}
\]

being:

\[
\alpha = \frac{J_z}{4} (1 - \rho) \quad , \quad \beta = \frac{2\alpha}{z} \tag{99}
\]

At this stage, it is important to confront eq. (62) obtained from the bilinear fermionic part, with others previous results given in the literature related with the spin-polaron theories (Martinez and Horsch, 1991; Schmitt-Rink et al., 1988). Like in these theories our starting point was to assume an antiferromagnetic order state. This physical assumption is directly connected
with the fact that the fermionic modes are not propagating. Therefore, the prescriptions for the propagation of the fermionic modes must be given.

The usual way to solve the propagation of fermions is by means of the Dyson equation. As known the Dyson theorem allows to compute the inverse of the corrected fermion propagator in terms of the free fermion propagator and the self-energy. Therefore the propagator:

$$G(k, \nu_n) = [G_0^{-1}(\nu_n) - \Sigma(k, \nu_n)]^{-1},$$

(100)

can be evaluated in a straightforward way within the self-consistent Born approximation scheme.

On the other hand, it is easy to show that in the one-loop computation of the fermion self-energy $\Sigma(k, i\nu_n)$ only one contribution coming from the three-leg vertex $U_a$ is significant. Due to the form of the free boson propagator (57) the part coming from the four-leg vertex $U_{ab}$ vanishes.

Therefore the self-energy $\Sigma(k, i\nu_n)$ is given by:

$$\Sigma(k, i\nu_n) = \frac{1}{N_s} \sum_{\omega, q} U_a D_{(0)}^{ba}(\omega, q) U_b G(\nu + \omega, k + q)$$

$$= \sum_q (f(k, q) + \omega g(k, q)) \sum_\omega \frac{G(\nu + \omega, k + q)}{\omega^2 + \omega_q^2}$$

$$\times G(\nu + \omega, k + q),$$

(101)

where was defined:

$$f(k, q) = \frac{J'_z(1 + \rho)^2}{4N_s} \left( \varepsilon_k^2 + \varepsilon_{k'}^2 - 2\gamma_q \varepsilon_k \varepsilon_{k'} \right),$$

(102)

$$g(k, q) = \frac{-2is(1 + \rho)}{(s + s')N_s} (\varepsilon_{k'}^2 - \varepsilon_k^2).$$

(103)

By using standard techniques the following expression for the fermionic self-energy at zero temperature is found:

$$\Sigma(k, i\nu_n) = \frac{(1 + \rho)}{2N_s} t^2 s^2$$

$$\times \sum_q \frac{\left(\text{sign} \gamma_q \gamma_k \sqrt{1 - \sqrt{(1 - \gamma_q^2)}} - \gamma_{k+q} \sqrt{1 + \sqrt{(1 - \gamma_q^2)}}\right)^2}{\sqrt{(1 - \gamma_q^2)}}$$

$$\times \frac{1}{i\nu_n - \omega_q - \mu - \Sigma(k + q, i\nu_n - \omega_q)},$$

(104)
where was used the relation $\varepsilon_k = -z t \gamma_k$.

Now by defining:

$$u_q = \sqrt{\left(\frac{1 + \sqrt{1 - \gamma_q^2}}{2\sqrt{1 - \gamma_q^2}}\right)} ,$$  

(105)

$$v_q = -(\text{sign} \gamma_q) \sqrt{\left(\frac{1 - \sqrt{1 - \gamma_q^2}}{2\sqrt{1 - \gamma_q^2}}\right)} ,$$  

(106)

the eq. (104) takes the final form:

$$\Sigma(k, i\nu_n) = \frac{(1 + \rho)}{2N_s} t^2 z^2 \sum_q \frac{(u_q \gamma_{k+q} + v_q \gamma_k)^2}{i\nu_n - \omega_q - \mu - \Sigma(k + q, i\nu_n - \omega_q)} .$$  

(107)

The expression (107) is useful in the strong coupling case ($t > J$). Moreover, in order to describe a metallic phase where the holes move coherently on the lattice, it is necessary to solve the self-consistent equation (107), which must be carried out numerically.

Once an appropriate self-energy function $\Sigma(k, i\nu_n)$ is found, the propagator $G(k, \nu)$ remains well defined, and so it is possible to compute numerically the spectral function defined by $A(k, \nu) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} G(k, \nu + i\varepsilon)$.

It can be seen that the eq. (107) is the generalization for finite values of holes to the equivalent equation coming from the spin-polaron theories. In fact this is a strong proof of the correctness of our quantum procedure developed in the t-J model.

On the other hand, due to the interaction of three or more bosons in the antiferromagnetic case remaining invariant with respect to the ferromagnetic one, the expressions for the n-leg boson vertices are unchanged. The same thing happens with the diagrammatics containing ghost fields because the symplectic supermatrix is invariant under canonical rotation, and so the ghost Lagrangian remains unchanged.

About the boson self-energy the situation is rather different to that given for the ferromagnetic configuration. Also in the antiferromagnetic case it is possible to obtain renormalized expressions for the boson self-energy by computing the finite one-loop contributions of the different processes. Of course, all the one-loop divergences of such diagrams are cancelled again by the ghost fields.

However, in the antiferromagnetic configuration the contributions of the three diagrams containing one fermionic loop complicate the boson self-energy
expression. In this case the associated renormalized matrix $\Pi_{ab}^R(q, \omega_n)$ takes the form:

$$\Pi_{ab}^R(q, \omega_n) = \begin{pmatrix}
\Pi_{11}^R & \Pi_{12}^R & \Pi_{13}^R & 0 \\
-\Pi_{12}^R & \Pi_{22}^R & \Pi_{23}^R & 0 \\
\Pi_{13}^R & -\Pi_{23}^R & \Pi_{33}^R & \Pi_{34}^R \\
0 & 0 & \Pi_{34}^R & \Pi_{44}^R
\end{pmatrix}.$$  \hfill (108)

Again, the renormalised antiferromagnetic magnon propagator is obtained in terms of the matrix elements $\Pi_{ab}^R(q, \omega_n)$ by using the Dyson equation.

Finally, all the results obtained in the ferromagnetic configuration for the renormalized n-point functions, can be rewritten in this case by only changing in the equations the free propagator by the free propagator (57).

In summary, from the above results remains clear that the main differences with respect to the ferromagnetic case are essentially originated by two different situations. On one hand by the different forms of the free boson propagators. Contrary to the ferromagnetic case, the antiferromagnetic magnon propagator (59) has two single poles. Moreover, the antiferromagnetic magnon is written in terms of the hole density $\rho$ then, at lowest order it contains only static hole density effects. Consequently, when the hole density $\rho$ is exactly equal to zero such magnon must be understood as the antiferromagnetic magnon at zero doping.

On the other hand, in the antiferromagnetic configuration a priori the fermions are non-propagating particles (see eq. (62)). However the prescription for propagation can be given without ambiguity by means of the Dyson equation, and the fermionic self-energy $\Sigma(k, i\nu_n)$, must be computed numerically by using eq. (107).

Analogously, by considering the fermion self-energy the renormalized expression for the fermion propagator can be constructed. It is easy to show that up to one-loop, the non vanishing contributions to the fermion self-energy are given by the diagrams whose analytical expressions are:

$$\Sigma^{(2)}(k, \nu) = \frac{1}{N_s} \sum_{q,\omega_n} U_{ab} D_{(0)}^{ab}$$  \hfill (109)

$$\Sigma^{(3)}(k, \nu) = \frac{1}{N_s} \sum_{q,\omega_n} U_a D_{(0)}^{ab} F_{bcd} D_{(0)}^{cd}$$  \hfill (110)

By carrying out the sumation on the Matsubara frequency, the finite expression for the fermion self-energy results:
\[ \Sigma(k, \nu) = \sum^{(2)}(k, \nu) + \sum^{(3)}(k, \nu) = \frac{1}{N_s} \sum_q [(1 + \rho)\varepsilon_{k-q} + i\nu_n - \mu] n_B(\omega_q). \] (111)

Once more, through the Dyson equation the renormalized fermion propagator \( G(k, \nu) \) can be computed and it results:

\[ G(k, \nu) = \frac{1}{(i\nu_n - \mu)[1 - \frac{1}{N_s} \sum_q n_B(\omega_q)] + \varepsilon_k[1 - \frac{(1+\rho)}{N_s} \sum_q \gamma_q n_B(\omega_q)]}. \] (112)

Working in a similar way the different vertices can be dressed and the expressions for the renormalized n-point functions are found.

### 6 Conclusions

As was obtained in the previous section, in the present paper, the path-integral formalism coming from a first-order Lagrangian written in terms of the Hubbard operators is studied. The second-class constrained system can be mapped in the well known decoupled slave-particle representation. In this model, the Hubbard \( X \)-operators used as field variables are the generators of the graded algebra \( \text{spl}(2,1) \). This field variables allow to describe without any decoupling assumption, spin and charge fluctuations on the atomic lattice site. In the framework of the path-integral formalism, the correlation generating functional describing the dynamics of the t-J model was analyzed and the standard Feynman diagrammatics was constructed.

Owing that the correlation generating functional corresponds to a second-class constrained system, the superdeterminant of the symplectic supermatrix is field dependent and the exponentiation of such superdeterminant is realized by introducing Faddeev-Popov super-ghost fields in the effective Lagrangian. It can be shown that ghost fields are needed in order to cancel the divergences appearing in the one-loop computation of physical quantities. In this way, the boson and the fermion self-energies and the different vertices of the model can be renormalized.

From our diagrammatics correct expressions for the free boson and fermion propagators are obtained. By computing the finite values of the self-energy it was possible to obtain the renormalized antiferromagnetic magnon propagator. In particular, the softening of the antiferromagnetic magnon energy obtained from our approach is the generalization for different from zero hole density of the expression obtained by means of the non-linear spin wave model [16].
At this point it is important to remark that our model accounts for the softening effect when only one-loop computations is considered. Do not consider any correction to the vertex is an important fact because in the framework of non-linear spin wave model, the softening of the magnon energy is obtained by including vertex corrections. The vertex corrections cancel scattering processes between magnons in such a way that only the direct and the exchange channels must be considered as physical processes. In our perturbative approach the correct physical processes are directly given to each loop order.

It is important to note that in this model the divergences appear only in the one-loop structure, so the quantities are renormalized to any perturbative order. It can be seen that in calculations at more than one-loop the diagrams containing ghosts give finite contributions to the renormalized expressions of the n-point functions.

References


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