A Generalized Solvable Quantum Three-Body Calogero System

Paul Bracken

Department of Mathematics
University of Texas
Edinburg, TX 78541-2999, USA
bracken@utpa.edu

Abstract

An exactly solvable non-trivial quantum Calogero three-body problem is presented. It is shown that the model gives a separable equation and can be separated by introducing appropriate coordinate transformations. The procedure yields eigensolutions as well as the corresponding energy spectrum, as well as introducing the idea of fractional statistics.

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1. Introduction.

Fractional statistics has been a subject of great interest recently on account of such applications as the quantum Hall effect among others [1-4]. These statistics appear naturally in two spatial dimensions due to the Abelian nature of the rotation group. The braid group replaces the permutation group in the plane which leads to the possibility of anyonic excitations. Chern-Simons theories in $2 + 1$ dimensions with Abelian gauge field can account for anyonic excitations. In one dimension, statistics and interaction get non-trivially intertwined, and moreover, solvable models do exist which demonstrate the effect statistical interaction explicitly. The Calogero-Sutherland model is in this category [5-8]. With such a diverse array of novel, abstract applications, it should be of interest to undergraduate students to investigate in detail a solvable few body quantum mechanical problem. The problem investigated here was proposed by Meljanac for $N = 3$ particles in $D = 1$ dimensions, and also solved by Quesne [9]. In this instance, the model can be viewed as a generalization of the three-body Calogero model with an additional non-translationally invariant three-body potential. The task will be
to determine the full wavefunction in terms of a radial as well as two angular variables along with their corresponding eigenvalues [10].

The model which is investigated here can be thought of as a generalization of the three-body Calogero problem which contains an additional non-translationally invariant three-body potential [11-13]. The model is one of several which has an underlying conformal $SU(1,1)$ symmetry. It could be thought of as a model consisting of three interacting particles of the same mass $m$ in the harmonic field generated by an infinite mass particle. The wavefunction is to be found in terms of the radial and angular variables, which typically are special functions [14], along with the eigenvalues. Finally, a further generalization of this model to the three-body Calogero-Wolfes problem is given, but the details are just outlined.

2. The Generalized Three-Body Calogero Model.

The generalization of the three-body Calogero model is determined by the Hamiltonian

$$H = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \omega^2(x_1^2 + x_2^2 + x_3^2) + \frac{\mu}{x_1^2 + x_2^2 + x_3^2} + \lambda \left( \frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 - x_3)^2} + \frac{1}{(x_2 - x_3)^2} \right).$$

(1)

The three light particles interact pairwise by means of two-body inverse square potentials joined with a non-translationally invariant three-body potential represented by the second last term in (1). The Hamiltonian will assume an interesting form after it is transformed into $(t, u, v)$ coordinates using the following coordinate transformation,

$$t = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3), \quad u = \frac{1}{\sqrt{2}}(x_1 - x_2), \quad v = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3).$$

(2)

The derivatives in (1) will transform according to

$$\frac{\partial}{\partial x_1} = \frac{1}{\sqrt{3}} \frac{\partial}{\partial t} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial u} + \frac{1}{\sqrt{6}} \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial x_2} = \frac{1}{\sqrt{3}} \frac{\partial}{\partial t} - \frac{1}{\sqrt{2}} \frac{\partial}{\partial u} + \frac{1}{\sqrt{6}} \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial x_3} = \frac{1}{\sqrt{3}} \frac{\partial}{\partial t} - \frac{2}{\sqrt{6}} \frac{\partial}{\partial v}.$$

Remarkably, the form of the Laplacian in (1) remains invariant under (2) and the remaining terms transform easily, for example

$$x_1^2 + x_2^2 + x_3^2 = t^2 + u^2 + v^2, \quad \frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 - x_3)^2} + \frac{1}{(x_2 - x_3)^2} = \frac{9(u^2 + v^2)^2}{2u^2(u^2 - 3v^2)^2}.$$  

(3)

In terms of $(t, u, v)$, the Hamiltonian is

$$H = -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} + \omega^2(t^2 + u^2 + v^2) + \frac{\mu}{t^2 + u^2 + v^2} + \frac{9\lambda(u^2 + v^2)^2}{2(u^2 - 3uv^2)^2}.$$  

(4)

However, this Hamiltonian is not separable in these new coordinates. If now we go to spherical coordinates based on (2), a separable system results

$$t = r \cos \theta, \quad u = r \sin \theta \sin \varphi, \quad v = r \sin \theta \cos \varphi,$$

(5)
The second last term in (4) under (5) becomes
\[
\frac{(u^2 + v^2)^2}{(u^3 - 2uv)^2} = \frac{1}{r^2 \sin^2 \theta \sin^2 \varphi (\sin^2 \varphi - 3 \cos^2 \varphi)^2} = \frac{1}{r^2 \sin^2 \theta \sin^2 (3\varphi)},
\]
using \( \sin^2 (3\varphi) = (1 - 4 \cos^2 \varphi) \sin^2 \varphi \).

The stationary Schrödinger equation based on (1) in terms of spherical coordinates (5) takes the form
\[
\left\{-\frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} + \frac{\omega^2 r^2}{v(r, \theta)} + \frac{\mu}{r^2} + \frac{1}{r^2} \left[-\frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left(-\frac{\partial^2}{\partial \varphi^2} + \frac{9 \lambda}{2 \sin^2 (3\varphi)}\right)\right]\right\} \Psi(r, \theta, \varphi) = E \Psi(r, \theta, \varphi).
\]
(6)

### 3. Solution of the Schrödinger Equation.

To start to obtain a solution in separated form, the wavefunction is written as
\[
\Psi(r, \theta, \varphi) = v(r, \theta) \Phi_n(\varphi).
\]
(7)

Substituting (7) into (6), the Schrödinger equation takes the form
\[
\frac{r^2 \sin^2 \theta}{v(r, \theta)} \left\{-\frac{d^2}{d\varphi^2} - \frac{2}{r} \frac{d}{dr} + \frac{\omega^2 r^2}{v(r, \theta)} + \frac{\mu}{r^2} + \frac{1}{r^2} \left[-\frac{d^2}{d\theta^2} - \cot \theta \frac{d}{d\theta} + \frac{1}{\sin^2 \theta} \left(-\frac{d^2}{d\varphi^2} + \frac{9 \lambda}{2 \sin^2 (3\varphi)}\right)\right]\right\} \Phi_n(\varphi) = E \Phi_n(\varphi).
\]
(8)

In this form, (8) is clearly separable, and so introducing a separation constant \( B_n \), this separates into
\[
\frac{d^2}{d\varphi^2} + \frac{9 \lambda}{2 \sin^2 (3\varphi)} \Phi_n(\varphi) = B_n \Phi_n(\varphi),
\]
(9)

\[
r^2 \left(-\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\omega^2 r^2}{v(r, \theta)} - E\right)v(r, \theta) + \left(-\frac{d^2}{d\theta^2} - \cot \theta \frac{d}{d\theta} + \frac{B_n}{\sin^2 \theta}\right)v(r, \theta) = 0.
\]

Hence, this results in an ordinary differential equation and a partial differential equation, which will be expressed in a separable form, that is, the product of a radial function and an angular function of \( \theta \). Suppose that \( v(r, \theta) \) is taken to have the form,
\[
v(r, \theta) = \frac{F_n(r)}{r} \frac{P_l(\theta)}{\sqrt{\sin \theta}}.
\]
(10)

Under differentiation, we obtain,
\[
\left(-\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr}\right) \frac{F_k}{r} = -\frac{F_k''}{r} + 2 \frac{F_k'}{r} - \frac{2}{r^3} F_k - 2 \frac{F_k'}{r^2} + \frac{2}{r^3} F_k = -\frac{1}{r} F_k''.
\]
(11)
Substituting (11)-(14) into the second equation of (9), it takes the form
\[
\frac{\partial}{\partial \theta} \left( \frac{P_l}{\sqrt{\sin \theta}} \right) = \frac{1}{\sqrt{\sin \theta}} \left( \frac{dP_l}{d\theta} - \frac{1}{2} \cot \theta P_l \right),
\]
(12)
\[
\frac{\partial^2}{\partial \theta^2} \left( \frac{P_l}{\sqrt{\sin \theta}} \right) = \frac{1}{\sqrt{\sin \theta}} \left( \frac{d^2P_l}{d\theta^2} - \cot \theta \frac{dP_l}{d\theta} + \frac{\cos^2 \theta + 2}{4 \sin^2 \theta} P_l \right).
\]
(13)
Hence, the right-hand side of the second equation in (9) takes the form
\[
\left( - \frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} + \frac{B_n}{\sin^2 \theta} \right) \frac{P_l}{\sqrt{\sin \theta}} = \frac{1}{\sqrt{\sin \theta}} \left( - \frac{d^2P_l}{d\theta^2} + \frac{B_n - \frac{1}{4}}{\sin^2 \theta} P_l - \frac{1}{4} P_l \right).
\]
(14)
Substituting (11)-(14) into the second equation of (9), it takes the form
\[
\frac{r^2}{F_k} \left( - \frac{d^2F_k}{dr^2} + \omega^2 r^2 F_k + \frac{\mu}{r^2} F_k - EF_k \right) + \frac{1}{P_l} \left( - \frac{d^2P_l}{d\theta^2} + \frac{B_n - \frac{1}{4}}{\sin^2 \theta} P_l - \frac{1}{4} P_l \right) = 0.
\]
(15)
This partial differential equation is clearly separable, and upon introducing the separation constant \(D_{l,n} - 1/4\), it admits the following pair of ordinary equations
\[
\frac{1}{P_l} \left( - \frac{d^2P_l}{d\theta^2} + \frac{B_n - \frac{1}{4}}{\sin^2 \theta} P_l - \frac{1}{4} P_l \right) = D_{l,n} - \frac{1}{4},
\]
(16)
\[
\frac{r^2}{F_k} \left( - \frac{d^2F_k}{dr^2} + \omega^2 r^2 F_k + \frac{\mu}{r^2} F_k - EF_k \right) = -(D_{l,n} - \frac{1}{4}).
\]
(17)
In (17), \(D_{l,n} - \frac{1}{4}\) is a separation constant. To summarize the three independent equations, we have
\[
\left( - \frac{d^2}{d\varphi^2} + \frac{9\lambda}{2 \sin^2(3\varphi)} \right) \Phi_n(\varphi) = B_n \Phi_n(\varphi),
\]
(18)
\[
\left( - \frac{d^2}{d\theta^2} + \frac{B_n - \frac{1}{4}}{\sin^2 \theta} \right) P_l(\theta) = D_{l,n} P_l(\theta),
\]
(19)
\[
\left( - \frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\mu + D_{l,n} - \frac{1}{4}}{r^2} \right) F_{kln}(r) = E_{kln} F_{kln}(r).
\]
(20)
Consider (18) first, then on the interval \(0 \leq \varphi \leq 2\pi\), the potential in (18) has periodicity of \(\pi/3\) and singularities at \(\varphi = k \frac{\pi}{3}\), for \(k = 0, \ldots, 5\). This equation has been solved by Calogero. To do so, the interval \([0, 2\pi]\) is divided into six different sectors \([p \frac{\pi}{3}, (p + 1) \frac{\pi}{3}]\) for \(p = 0, 1, \ldots, 5\). Each sector corresponds to a particular ordering among the positions of the three particles. To begin with, the equation is solved for \(\varphi\) in the interval \((0, \frac{\pi}{3})\), which corresponds to the ordering \(x_1 > x_2 > x_3\). To extend this to the whole interval \([0, 2\pi]\), the prescription given originally by Calogero is adopted. This entails the use of symmetry arguments according to the statistics obeyed by the particles. In a neighborhood of zero and \(\frac{\pi}{3}\), the singularity resembles that of a centrifugal barrier, since the potential assumes the form \(\frac{\lambda}{2 \varphi^2}\) and \(\frac{\lambda}{2(\varphi - \pi/3)^2}\), respectively. As usual, the aim is to express solutions of (18) in the interval \([0, \pi/3]\) with Dirichlet boundary conditions at the boundaries, in terms of orthogonal polynomials. Orthogonal polynomials
can be obtained by imposing constraints on the parameters \( \lambda, B_n \). To do this, an additional transformation is introduced,

\[
\Phi_n(\varphi) = (\sin 3\varphi)^\nu f_n(z), \quad z = \cos(3\varphi).
\]

The quadratic equation requiring only square integrability of the solution, then acceptable for attractive potentials when \(-\lambda > 1 + 2\lambda \), is

\[
(1 - z^2)\frac{d^2 f_n}{dz^2} - (2\nu + 1)z \frac{df_n}{dz} + (\frac{B_n}{9} - \nu - \frac{\lambda - 2\nu(\nu - 1)z^2}{2(1 - z^2)})f_n(z) = 0. \tag{21}
\]

Equation (21) has polynomial solutions when each of the following constraints is satisfied,

\[
\lambda = 2\nu(\nu - 1), \quad B_n = 9(n + \nu)^2, \quad n = 0, 1, 2, \cdots . \tag{22}
\]

Imposing constraints (22) in (21), it becomes

\[
(1 - z^2)\frac{d^2 f_n}{dz^2} - (2\nu + 1)z \frac{df_n}{dz} + ((n + \nu)^2 - \nu^2) f_n(z) = 0. \tag{23}
\]

This is exactly the differential equation for Gegenbauer polynomials, namely \( f_n(z) = C^{(\nu)}_n(z) \).

The quadratic equation \( \lambda = 2\nu(\nu - 1) \) from (22) has two solutions for \( \nu \) given by

\[
\nu_+ = \frac{1}{2}(1 + \sqrt{1 + 2\lambda}) = \frac{1}{2} + a, \quad \nu_- = \frac{1}{2}(1 - \sqrt{1 + 2\lambda}) = \frac{1}{2} - a, \quad a = \frac{1}{2}\sqrt{1 + 2\lambda}. \tag{24}
\]

These solutions for \( \nu \) are real and distinct in the case \( \lambda > -\frac{1}{2} \), which is the condition of physically acceptable solutions near the singularities. The regular solution \( \nu_+ \) is usually retained, but with the constraint of the Dirichlet condition for irregular solutions corresponding to \( \nu_- \), it is also acceptable for attractive potentials when \(-1/2 < \lambda < 0 \). Relaxing the Dirichlet condition and requiring only square integrability of the solution, then \( \nu_- \) can be retained when \(-1/2 < \lambda < 3/2 \).

It is interesting to note that for \( \lambda = 0 \), corresponding to \( \nu_+ = 1 \) and \( \nu_- = 0 \), there is no interaction between the particles. To summarize, the regular eigensolution reads

\[
\Phi_n(\varphi) = (\sin 3\varphi)^\varphi C^{(\varphi + \frac{1}{2})}_n(\cos 3\varphi), \quad 0 \leq \varphi \leq \frac{\pi}{3}, \quad n = 0, 1, 2, \cdots . \tag{25}
\]

The corresponding eigenvalues using (24) are,

\[
B_n = 9(n + \nu_+)^2 = 9(n + a + \frac{1}{2})^2, \quad n = 0, 1, 2, \cdots . \tag{26}
\]

The angular equation (19) for polar angle \( \theta \) can be put in the form

\[
\left(-\frac{d^2}{d\theta^2} + \frac{(b_n^2 - \frac{1}{2})}{\sin^2 \theta} - D_{l,n}\right)P_{l,n}(\theta) = 0, \tag{27}
\]

where the auxiliary constant \( b_n \) is defined to be

\[
b_n = \pm \sqrt{B_n} = \pm (3n + 3a + \frac{3}{2}). \tag{28}
\]
Since \( b_n \neq 0 \), the Hamiltonian for (27) is a self-adjoint operator on the domain \( \mathcal{D} = \{ P \in L^2[0, \pi] | P(0) = P(\pi) = 0 \} \). An equation which produces Gegenbauer polynomials as solutions can be obtained from (27) upon using the following form for \( P_{l,n}(\theta) \),

\[
P_{l,n}(\theta) = (\sin \theta)^2 h_{l,n}(y), \quad y = \cos \theta.
\]  

(29)

Differentiating both sides of (29) with respect to \( \theta \), the derivatives can be transformed into the \( y \) variable, and a differential equation for \( h_{l,n} \) is obtained,

\[
(1 - y^2) \frac{d^2 h_{l,n}(y)}{dy^2} - (2\beta + 1)y \frac{dh_{l,n}(y)}{dy} + (D_{l,n} - \beta + \frac{1 - 4b_n^2 + 4\beta(\beta - 1)y^2}{4(1 - y^2)}) h_{l,n}(y) = 0,
\]  

(30)

This will produce physically acceptable solutions if the constants \( b_n \) and \( D_{l,n} \) satisfy,

\[
b_n^2 = (\beta - \frac{1}{2})^2, \quad D_{l,n} = (l + \beta)^2, \quad l = 0, 1, 2, \ldots.
\]  

(31)

Using these, we conclude that \( 1 - (4\beta^2 - 4\beta + 1) + 4\beta(\beta - 1)y^2 = -4\beta(\beta - 1)(1 - y^2) \), hence the last term in (30) can be compressed to read,

\[
D_{l,n} - \beta + \frac{1 - 4b_n^2 + 4\beta(\beta - 1)y^2}{4(1 - y^2)} = (l + \beta)^2 - \beta - 4\beta(\beta - 1) \frac{1 - y^2}{4(1 - y^2)} = l(l + 2\beta).
\]

In this instance, (30) becomes,

\[
(1 - y^2) \frac{d^2 h_{l,n}(y)}{dy^2} - (2\beta + 1)y \frac{dh_{l,n}(y)}{dy} + l(l + 2\beta) h_{l,n}(y) = 0.
\]  

(32)

Equation (32) has Gegenbauer polynomial solutions, that is \( h_{l,n}(y) = C_{l}^{(\beta)}(y) \). The quadratic equation \( b_n^2 = (\beta - \frac{1}{2})^2 \) has two solutions, which are

\[
\beta_+ = \frac{1}{2} + b_n, \quad \beta_- = \frac{1}{2} - b_n,
\]  

(33)

considering only the positive root of \( b_n^2 = B_n \), that is \( b_n > 0 \). The regular solution \( P_{l,n}(\theta) \) corresponds to \( \beta_+ \), whereas the irregular solution corresponds to \( \beta_- \). The irregular solution can be discarded as being non-square integrable for most values of \( n \). To summarize, the regular eigensolutions and the corresponding eigenvalues for the angular equation giving \( P_{l,n}(\theta) \) on \([0, \pi]\) are

\[
P_{l,n}(\theta) = (\sin \theta)^{b_n + \frac{1}{2}} C_{l}^{(b_n + \frac{1}{2})}(\cos \theta), \quad l = 0, 1, 2, \ldots, \quad D_{l,n} = (l + b_n + \frac{1}{2})^2.
\]  

(34)

The choice of \( b_n = 3n + 3a + \frac{3}{2} > 0 \) implies that for every value of \( n \), the function \( P_{l,n}(\theta) \) has to vanish at the boundaries of the interval \([0, \pi]\).

It remains to solve the reduced radial equation, which is,

\[
(- \frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\mu + D_{l,n} - \frac{1}{2}}{r^2} = E_{kn}) F_{kn}(r) = 0.
\]  

(35)
Equation (35) is to be solved over $0 \leq r < \infty$ with the additional condition of square integrability for the solutions, which implies that $F_{kln}(r) \to 0$ as $r \to \infty$. Moreover, the condition $\mu + D_{l,n} > 0$ must be imposed in order to treat the centrifugal barrier in the region of $r = 0$. If $\mu + D_{l,n} = 0$ is imposed, this leads to several self-adjoint extensions parametrized by a phase. In the case of an attractive centrifugal barrier $\mu + D_{l,n} < 0$, the energy spectrum is not bounded below.

From the definition of $D_{l,n}$ already presented, it follows that

$$\mu + D_{l,n} = \mu + (l + b_n + \frac{1}{2})^2 = \mu + (l + 3n + 3a + 2)^2 > 0, \quad l \geq 0. \quad (36)$$

The quantity in (36) is minimized when $n = 0$, $l = 0$ and $a = 0$, where $a \geq 0$. At this point $\mu + D_{l,n} = \mu + 4$, so for (36) to be consistent, the constraint $\mu > -4$ is imposed. This means that an auxiliary parameter $\alpha_{l,n}$ can be introduced. It is defined by

$$\alpha_{l,n}^2 = \mu + D_{l,n}, \quad \alpha_{l,n} = \sqrt{\mu + D_{l,n}}. \quad (37)$$

To solve radial equation (35), assume a form which is consistent with the above properties,

$$F_{kln}(r) = r^{\alpha_{l,n} + \frac{1}{2}} \exp\left(-\frac{\alpha r^2}{2}\right) g_{kln}(s), \quad s = \omega r^2. \quad (38)$$

Differentiating both sides of (38) twice with respect to $r$, substituting the resulting derivatives in terms of $s$ into (35), the radial equation then simplifies into the following form

$$s \frac{d^2 g_{kln}(s)}{ds^2} + (\alpha_{l,n} + 1 - s) \frac{dg_{kln}(s)}{ds} + \left(\frac{E}{4\omega} - \frac{1}{2} - \frac{\alpha_{l,n}}{2}\right) g_{kln}(s) = 0. \quad (39)$$

Equation (39) is exactly the differential equation for the generalized Laguerre polynomials $L_k^{(\alpha_{l,n})}(s)$ provided the coefficient of the last term multiplying $g_{kln}$ is equal to a non-negative integer value $k$,

$$\frac{E_{kln}}{4\omega} - \frac{1}{2} - \frac{\alpha_{l,n}}{2} = k, \quad k = 0, 1, 2, \ldots \quad (40)$$

With this in mind, the regular solutions of the reduced radial equation are written

$$F_{kln}(r) = r^{\alpha_{l,n} + \frac{1}{2}} \exp\left(-\frac{\omega r^2}{2}\right) L_k^{(\alpha_{l,n})}(\omega r^2), \quad k = 0, 1, 2, \ldots \quad (41)$$

These have the associated eigenvalues,

$$E_{kln} = 2\omega (2k + \alpha_{l,n} + 1), \quad k = 0, 1, 2, \ldots \quad (42)$$

The above choice of the positive root for $\alpha_{l,n}$ implies that $F_{kln}(r)$ will vanish at $r = 0$, and the exponential Gaussian term in (41) ensures the square integrability of the solutions. The negative root would lead to non-square integrable solutions for high values of $l$.

Therefore, the physically acceptable solutions of Schrödinger equation (6) are given by

$$\Psi_{kln}(r, \theta, \varphi) = r^{\alpha - \frac{1}{2}} \exp\left(-\frac{\omega r^2}{2}\right) L_k^{(\alpha)}(\omega r^2) \cdot (\sin \theta)^{3a+3\alpha+\frac{1}{2}} C_l^{(3a+3\alpha+2)}(\cos \theta)$$
With regard to the radial equation, the constant \( \mu \) leads to square integrable solutions for every value of \( n \).

This end, in (43) constitute a basis.

In (44), \( \epsilon \) where use has been made of the orthogonality properties of Gegenbauer and Laguerre polynomials. To continue solution (43) to larger \( \varphi \) angular regions the following following prescription is followed,

\[
\Psi_{kln}(r, \theta, \varphi + \frac{1}{3} p \pi) = (-1)^p n \sigma^{1/2} \Psi_{kln}(r, \theta, \varphi), \quad 0 < \varphi < \frac{\pi}{3}, \quad p = 1, 2, 3, 4, 5. \tag{44}
\]

The normalization constants \( N_{kln} \) for (43) are determined by the integrals,

\[
\int_0^\infty r^2 \, dr \int_0^\pi \sin \theta \, d\theta \int_0^{\pi/3} d\varphi \Psi_{kln}(r, \theta, \varphi) \Psi_{k'l'n'}(r, \theta, \varphi) = \delta_{kk'} \delta_{ll'} \delta_{nn'} N_{kln}, \tag{45}
\]

where use has been made of the orthogonality properties of Gegenbauer and Laguerre polynomials. In (44), \( \epsilon = 1 \) for bosons and \( \epsilon = -1 \) for fermions. Note that the Gegenbauer polynomials in (43) constitute a basis.

The final full expression of the eigenenergies for (6) is expressed by

\[
E_{kln} = 2 \omega (2k + \sqrt{\mu + (l + 3n + 3a + 2)^2 + 1}), \tag{46}
\]

where \( k = 0, 1, 2, \ldots, l = 0, 1, 2, \ldots \) and \( n = 0, 1, 2, \ldots \).

There are irregular solutions we should briefly consider corresponding to \( \nu_- = \frac{1}{2} - a \). To this end, \( a \) must be replaced by \( -a \) in all equations. Provided that \(-\frac{1}{2} < \lambda < \frac{3}{2}\), these irregular solutions are square integrable. Moreover, the requirement of self-adjointness of the Sturm-Liouville operator in \( P_{ln}(\theta) \) suggests that the case \( \lambda = 0 \) be disregarded in order to have \( b_n \neq 0 \).

Since for all \( n \geq 0 \), \( b_n = 3n + \frac{3}{2} - 3a \geq \frac{3}{2} - 3a \), it can be claimed that the function \( P_{l,n}(\theta) \) leads to square integrable solutions for every value of \( n \) if \( a < \frac{3}{8} \), which happens when \( \lambda < \frac{3}{2} \). With regard to the radial equation, the constant \( \mu + D_{l,n} > 0 \) allows for the treatment of the centrifugal barrier in the vicinity of \( r = 0 \). This will be satisfied for every \((l, n)\) such that \( \mu + (2 - 3a)^2 > 0 \). This constraint specifies a domain in which the permissible values of \( \mu \) reside, depending on the values of \( \lambda \in (-\frac{1}{2}, 0) \cup (0, \frac{3}{2}) \). The radial solutions under these conditions are then square integrable since \( \alpha_{l,n} > 0 \). The variables \((r, \theta, \varphi)\) introduced in (5) are linked to the coordinates of the three particles \( x_1, x_2 \) and \( x_3 \) by means of the relations

\[
r^2 = l^2 + u^2 + v^2 = x_1^2 + x_2^2 + x_3^2, \\
\theta = \arccos \left( \frac{l}{r} \right) = \arccos \left( \frac{x_1 + x_2 + x_3}{\sqrt{3(x_1^2 + x_2^2 + x_3^2)}} \right), \quad 0 \leq \theta \leq \pi, \tag{47} \]

\[
\varphi = \arctan \left( \frac{u}{v} \right) = \arctan \left( \frac{\sqrt{3(x_1 - x_2)}}{x_1 + x_2 - 2x_3} \right), \quad 0 \leq \varphi \leq \frac{\pi}{3}.
\]
There is clearly a great deal of new physics associated with wavefunction (44) and there should be a band structure associated with the model. For identical particles, the triplet \( k, l, n \) determines a symmetrized, unique wavefunction. Once it is determined in the \( \varphi \) angular sector \( 0 \leq \varphi \leq \frac{\pi}{2} \), it is extended to the entire interval \([0, 2\pi]\) by considering the symmetry property implied by the statistics.


One can continue to develop solvable potentials based on this model. Consider from (2) the differences \( x_i - x_j \),

\[
x_1 - x_2 = \sqrt{2}u, \quad x_1 - x_3 = \frac{1}{\sqrt{2}}u + \sqrt{\frac{3}{2}}v, \quad x_2 - x_3 = -\frac{1}{\sqrt{2}}u + \sqrt{\frac{3}{2}}v.
\]  

(48)

If \( u \) and \( v \) on the right-hand side of these differences are permuted, and then the results are transformed back into the \( x_j \) coordinates,

\[
\sqrt{2}v = \frac{1}{\sqrt{3}}(x_1 + x_2 - 2x_3), \quad \frac{1}{\sqrt{2}}v + \frac{\sqrt{3}}{2}u = \frac{1}{\sqrt{3}}(2x_1 - x_2 - x_3), \quad \frac{1}{\sqrt{2}}v - \frac{\sqrt{3}}{2}u = \frac{1}{\sqrt{3}}(x_1 + x_3 - 2x_2).
\]

Based on these expressions in \( x_i \), a new potential function can be constructed, namely,

\[
V = 3f\left(\frac{1}{(x_1 + x_2 - 2x_3)^2} + \frac{1}{(x_2 + x_3 - 2x_1)^2} + \frac{1}{(x_1 + x_3 - 2x_2)^2}\right).
\]  

(49)

In terms of \( u \) and \( v \), (49) takes a concise form

\[
V = 9f\frac{(u^2 + v^2)^2}{(v^3 - 3u^2v)^2}.
\]

The Schrödinger equation in terms of \( t, u \) and \( v \) is

\[
\left[-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} + \omega^2(t^2 + u^2 + v^2) + \frac{\mu}{t^2 + u^2 + v^2} + \frac{9}{2}\left(\frac{u^2 + v^2}{(u^3 - 3uv^2)^2}\right) + \frac{9}{2}\left(\frac{v^2}{(v^3 - 3uv^2)^2}\right) - E\right] \Psi(t, u, v) = 0.
\]  

(50)

Expression (50) can be transformed into spherical coordinates by using transformation (5),

\[
\left\{-\frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\mu}{r^2} \left[ -\frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left( -\frac{\partial^2}{\partial \varphi^2} + 2 \frac{g}{\sin^2(3\varphi)} + \frac{9f}{2 \cos^2(3\varphi)} \right) \right] \right\} \Psi(r, \theta, \varphi) = E \Psi(r, \theta, \varphi).
\]  

(51)

Assume the wavefunction breaks up as in the previous example (7), so it has the form,

\[
\Psi_{kln}(r, \theta, \varphi) = \frac{F_k(r)}{r} \frac{\Theta_l(\theta)}{\sqrt{\sin \theta}} \Phi_n(\varphi),
\]
Substituting $\Psi_{kln}$ into (51) results in three decoupled ordinary differential equations,

$$
(-\frac{d^2}{d\varphi^2} + \frac{9g}{2\sin^2(3\varphi)} + \frac{9f}{2\cos^2(3\varphi)})\Phi_n(\varphi) = \tilde{B}_n\Phi_n(\varphi),
$$

(52)

$$
(-\frac{d^2}{d\theta^2} + \frac{\tilde{b}_n^2 - 1/4}{\sin^2\theta})\Theta_{l,n}(\theta) = \tilde{D}_{l,n}\Theta_{l,n}(\theta),
$$

(53)

$$
(-\frac{d^2}{dr^2} + \omega^2r^2 + \frac{\mu + \tilde{D}_{l,n} - 1/4}{r^2})F_{kln}(r) = E_{kln}F_{kln}(r).\n$$

(54)

Clearly, the potential in the $\varphi$ variable has singularities when $\varphi = k\frac{\pi}{6}$, for $k = 0, 1, 2, \cdots, 11$ and defines twelve sectors: $q\frac{\pi}{6} < \varphi < (q + 1)\frac{\pi}{6}$, $q = 0, \cdots, 11$. In each sector, there is an order between the positions of the three particles as well as a polarization in the sense that the middle particle is closer to one or other of the outside particles. This is important as far as the statistics of the particles is concerned. The solution studies is restricted to the sector $q = 0$, $0 < \varphi < \frac{\pi}{6}$, which corresponds to $x_1 > x_2 > x_3$, $x_1 - x_2 < x_2 - x_3$. The regular eigensolutions which satisfy Dirichlet conditions at the interval’s boundaries are

$$
\Phi_n(\varphi) = (\sin 3\varphi)^{a+\frac{1}{2}}(\cos 3\varphi)^{b+\frac{1}{2}}P_n^{(a,b)}(\cos 6\varphi), \quad 0 \leq \varphi \leq \frac{\pi}{6}, \quad a = \frac{1}{2}\sqrt{1 + 2g}, \quad b = \frac{1}{2}\sqrt{1 + 2f},
$$

(55)

where $P_n^{(a,b)}(z)$ denotes the Jacobi polynomials [14].

In (53), $\tilde{b}_n^2 = \sqrt{B_n} = 3(2n + a + b + 1)$, and it has the same structure as equation (19). The physically acceptable solutions on $0 \leq \theta \leq \pi$ are written

$$
\Theta(\theta) = (\sin \theta)^{\tilde{b}_n+\frac{1}{2}}C^{(\tilde{b}_n+\frac{1}{2})}(\cos \theta), \quad \tilde{D}_{l,n} = (l + \tilde{b}_n + \frac{1}{2})^2, \quad l = 0, 1, 2, \cdots.
$$

(56)

Finally, the radial equation is identical to the previous $r$ equation (20) with $D_{l,n}$ replaced by $\tilde{D}_{l,n}$. Introducing the constant $\tilde{\alpha}_{l,n} = \sqrt{\mu + \tilde{D}_{l,n}}$, the radial solutions are given by

$$
F_{kln}(r) = r^{\tilde{\alpha}_{l,n}+\frac{1}{2}}\exp(-\frac{\omega r^2}{2})L_{k}^{(\tilde{\alpha}_{l,n})} E_{kln} = 2\omega(2k + \tilde{\alpha}_{l,n} + 1), \quad k = 0, 1, 2, \cdots.
$$

(57)

The regular solutions of this three body problem are found by multiplying these results together. The eigenenergies of the equation are given by

$$
E_{kln} = 2\omega(2k + \sqrt{\mu + (l + 6n + 3a + 3b + \frac{7}{2})^2 + 1}).
$$

(58)

In (58), $k, l, n$ entend over zero and the positive integers.

The utility of this example in terms of physically motivated solution of partial differential equations is clear from this. To conclude, we just mention some applications with regard to statistics displayed here. The interchange of particles is governed by the $\varphi$ variable. The first cell is dictated by $0 < \varphi < \pi/3$, which is the ordering $x_1 > x_2 > x_3$. A change in the particle ordering is brought about by the transformation $\varphi \rightarrow \varphi + p\pi/3$, and as $p$ goes through the values 1, 2, 3, 4, 5 one translates from one cell to the next. This means the wavefunction will pick up a phase which may be integral or fractional. This will decide the statistics of the system.
References.


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