Role of General Relativity and Quantum Mechanics in Dynamics of Solar System

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Abstract

Let $m_i$ be the mass of $i$-th planet ($i = 1 \div 9$) and $M_\odot$ be the Solar mass. From astronomical data it is known that ratios $r_i = m_i/(m_i+M_\odot)$ are of order $10^{-6} - 10^{-3}$ for all planets. The same is true for all satellites of heavy planets, with only exception for Moon-Earth for which it is $10^{-2}$ and for Charon-Pluto, for which it is $10^{-1}$. These results suggest that Einstein’s treatment of the Mercury dynamics can be safely extended to almost any object in the Solar System. If this is done, gravitational interactions between planets/satellites can be ignored since they move on geodesics. This fact still does not explain the existing order in the Solar System. Because of it, all planets lie in the same (Suns’s equatorial) plane and move in the same direction coinciding with that for the rotating Sun. The same is true for the regular satellites of heavy planets and for the planetary ring systems associated with these satellites. Such filling pattern is typically explained with help of the hypothesis by which our Solar System is a product of evolutionary dynamics of some pancake-like cloud of dust. This hypothesis would make sense should the order in our planetary system (and that for exoplanets rotating around other stars) be exhausted by the pattern just described. But it is not! In addition to regular satellites there are irregular satellites (and at least one irregular (Saturn) ring associated with such a satellite (Phoebe)) grouped in respective planes (other than equatorial) in which they all move in wrong directions on stable orbits. These are located strictly outside of those for regular satellites. Since this filling pattern is reminiscent of filling patterns in atomic mechanics, following Heisenberg, we develop quantum celestial mechanics explaining this pattern. Such development is nontrivial since in Newton’s mechanics there is no restriction on speed of propagation of gravitational interactions. Reconciliation between the Newtonian and Einsteinian formulations of
gravity is possible only if the Newtonian gravitational potential is modified in such a way that the speed of gravitational interactions becomes restricted. It is demonstrated that such a restriction is equivalent to quantization of motion along geodesics. Under such quantization conditions the Plank’s constant is replaced by another constant, different for each planetary/satellite system. Then, the number of allowed stable orbits for planets and for regular satellites of heavy planets is calculated resulting in good agreement with observational data. In conclusion, the paper briefly discusses quantum mechanical nature of rings of heavy planets and potential usefulness of the obtained results for cosmology.

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1 Introduction

1.1 Background

The impact of quantum mechanics on physics requires no comments. More surprising is the impact of quantum mechanical thinking on mathematics and mathematicians. According to world renown mathematician, Yuri Manin [1], the traditional view ”continuous from discrete” gives (now) a way to the inverted paradigm: ”discrete from continuous”. This inverted paradigm in mathematics brought to life quantum logic, deformation quantization, quantum topology, quantum cohomology, etc. Quantum q-calculus beautifully described in the book by Kac and Cheung [2] provide a nice example of unified view of continuous and discrete using examples from analysis and theory of orthogonal polynomials (used essentially in quantum mechanics). These results are sufficient for, at least formal, looking at results of classical celestial mechanics from such q-deformed (quantum) point of view. Surprisingly, the ”quantization process” had already began in celestial mechanics. Usefulness of quantum mechanics has been recently discussed in [3-5]. This paper is aimed at extension of the emerging trend. We believe, that there are reasons much deeper than just a current fashion for such an extension.

In his ”Les Methodes Nouvelles de la Mecanique Celeste” written between 1892 and 1899 [6] Poincare’ developed theory of celestial mechanics by assuming that all planets, and all satellites of heavy planets are rotating in the same direction coinciding with direction of rotation of the Sun around its axis. To a large degree of accuracy all planets lie in the same (Sun’s equatorial) plane. These assumptions were legitimate in view of the astronomical data available to Poincare’. These data suggest that known to us now Solar System most likely is result of dynamic evolution of the pancake-like rotating
self-gravitating cloud of dust [7]. In 1898 the shocking counter example to the
Poincare’ theory was announced by Pickering who discovered the ninth moon
of Saturn (eventually named Phoebe) rotating in the direction opposite to all
other satellites of Saturn. Since that time the satellites rotating in the ”nor-
mal” direction are called ”regular” (or ”prograde”) while those rotating in the
opposite direction called ”irregular” (or ”retrograde”). At the time of writing
of this paper 103 irregular satellites were discovered (out of those, 93 were dis-
covered after 1997 thanks to space exploration by rockets)\(^1\). Furthermore, in
the late 2009 Phoebe had brought yet another surprise to astronomers. Two
articles in Nature [8, 9] are describing the largest new ring of Saturn. This new
ring lies in the same plane as Phoebe’s orbit and, in fact, the Phoebe’s tra-
jectory is located inside of the ring. The same arrangement is true for regular
satellites and the associated with them rings.

From the point of view of Newton’s laws of gravity there is no reason to re-
strict trajectories of planets to the same plane or to expect that all planets will
rotate in the same direction. Their observed locations thus far are attributed
to the conditions at which the Solar System was born. This assumption is
plausible and is in accord with the model (and its interpretation) of Solar
System used by Poincare’. But, as stated above, the major exception occurred
already in 1898 when Poincare’ was about to finish his treatise. To repair the
existing theory one has to make an assumption that all irregular satellites are
”strangers”. That is that they were captured by the already existing and
fully developed Solar System. Such an explanation would make perfect sense
should the orbits of these strangers be arranged in a completely arbitrary fash-
ion. But they are not! Without an exception it is known that: a) all retrograde
satellite orbits are lying strictly outside of the orbits of prograde satellites, b)
the inclinations of their orbits is noticeably different from those for prograde
satellites, however, c) by analogy with prograde satellites they tend to group
(with few exceptions) in orbits-all having the same inclination so that different
groups of retrogrades are having differently inclined orbits in such a way
that these orbits do not overlap if the retrograde plane of satellites with one
inclination is superimposed with that for another inclination\(^2\). In addition,
all objects lying outside the sphere made by the rotating plane in which all
planets lie are arranged in a similar fashion[10]. Furthermore, the orbits of
prograde satellites of all heavy planets lie in the respective equatorial planes—
just like the Sun and the planets—thus forming miniature Solar-like systems.
These equatorial planes are tilted with respect to the Solar equatorial plane

\(^1\)E.g. read ”Irregular moon” in Wikipedia
\(^2\)E.g. see http://nssdc.gsfc.nasa.gov/planetary/ then, go to the respective planet and,
then-to the ”fact sheet” link for this planet.
since all axes of rotation of heavy planets are tilted\(^3\) with different angles for different Solar-like systems. These "orderly" facts make nebular origin of our Solar System questionable. To strengthen the doubt further we would like to mention that for the exoplanets\(^4\) it is not uncommon to observe planets rotating in the "wrong" direction around the respective stars\(^5\). This trend goes even further to objects such as galaxies. In spiral galaxies the central bulge typically co-rotates with the disc. But for the galaxy NGC7331 the bulge and the disc are rotating in the opposite directions. These facts bring us to the following subsection.

1.2 Statements of problems to be solved

From the discussion above it looks like there is some pattern of filling of the orbits in Solar and Solar-like systems. First, the prograde orbits are being filled-all in the same equatorial plane. Second, the retrograde orbits start to fill in-also in respective planes tilted with respect to equatorial (prograde) plane. These tilted planes can be orderly arranged by the observed typical distance for retrogrades satellites: those lying in different planes will have different typical distances to the planet around which they rotate. All these retrograde orbits (without an exception!) are more distant from the respective planets than the prograde orbits. Inclusion of rings into this picture does not change the pattern just described. All heavy planets do have system of rings. These are located in the respective equatorial planes. The newly discovered Phoebe ring fits this pattern perfectly. While the rings associated with the prograde satellites all live in the respective equatorial planes which are "correctly" rotating, the Phoebe ring is rotating in "wrong" direction and is tilted (along with Phoebe’s trajectory) by 27° with respect to Saturn’s equatorial plane [8,9]. Given all these facts, we would like to pose the following

Main Question: What all these just noticed filling patterns have to do with general relativity?

In this paper we would like to argue that, in fact, to a large extent the observed patterns are manifestations of effects of general relativity at the length scales of our Solar System. Indeed, let \(M_\odot\) be the mass of the Sun (or, respectively, heavy planet such as Jupiter, Saturn, etc.) and \(m_i\) be the mass of an i-th planet(respectively, the i-th satellite of heavy planet). Make a ratio \(r_i = \frac{m_i}{m_i + M_\odot}\). The analogous ratios can be constructed for respective heavy planets (Jupiter, Saturn, Uranus, Neptune) and for any of their satellites. The

\(^3\)That is the respective axes of rotation of heavy planets are not perpendicular to the Solar equatorial plane.

\(^4\)E.g. see http://exoplanets.org/

\(^5\)E.g. read "Retrograde motion" in Wikipedia
observational data indicate that with only two exceptions: Earth-Moon (for which the ratio $r \sim 10^{-2}$), and Pluto-Charon (for which the ratio $r \sim 10^{-1}$), all other ratios in the Solar System are of order $10^{-6} - 10^{-3}$[11]. Everybody familiar with classical mechanics knows that under such circumstances the center of mass of such a binary system practically coincides with that for $M_\odot$. And if this is so, then the respective trajectories can be treated as geodesics. Hence, not only motion of the Mercury can be treated in this way, as it was done by Einstein, but also motion of almost any satellite$^6$ in the Solar System! Such a replacement of Newton’s mechanics by Einsteinian mechanics of general relativity even though plausible but is still not providing us with the answer to the Main Question. Evidently, the observed mass ratios and the observed filling patterns must have something in common. If we accept the point of view that the observed filling patterns are possible if and only if the observed mass ratios allow us to use the Einstein’s geodesics, then we inevitably arrive at quantization of Solar System dynamics. Such a statement looks rather bizarre since the traditional quantum mechanics is dealing with microscopic objects. Nevertheless, as results of Refs.[3-5] indicate, the formalism of quantum mechanics can indeed be adopted to problems emerging in celestial mechanics. Thus, we arrive at the statements of problems to be studied in the main text.

First, we need to prove that Einsteinian relativity favors quantum mechanical description of Solar System dynamics. Second, we need to prove that such quantum mechanical description is capable of explaining the observed in Solar System filling patterns. Evidently, the combined solutions of the first and second problems provide an answer to the Main Question.

1.3 Organization of the rest of the paper

This paper contains 5 sections and 3 appendices. In Section 2 we discuss historical, mathematical and physical reasons for quantization of the Solar System. In particular, by using some excerpts from works by Laplace and Poincare$^\prime$ we demonstrate that Laplace can be rightfully considered as founding father of both general relativity and quantum mechanics. He used basics of both of these disciplines in his study of dynamics of known at that time satellites of Jupiter. Specifically, in his calculations masses of satellites were ignored and when they were made nonzero but small the (Einsteinian) orbits were replaced by those which form standing waves around the Einsteinian orbits. Mathematical comments made by Poincare$^\prime$ on Laplace’s work are essentially same as were later unknowingly used by Heisenberg in his formulation of quantum

$^6$Regrettably, not our Moon! Description of dynamics of Moon is similar to that for rings of heavy planets (to be discussed in Section 4) and, as such, is also quantizable.
mechanics. In Section 3 we discuss some changes in the existing apparatus of quantum mechanics needed for development of physically meaningful quantization of Solar System dynamics. The SO(2,1) symmetry typical for planar configurations is investigated in detail so that amendment to traditional quantization scheme remain compatible with this symmetry group. In Section 4 we use this amended formalism for description of Solar System dynamics and explanation of the empirically observed filling patterns. The main results are summarized in Table 2 and in Subsection 4.2. In Table 2 we compare our calculation of available orbits for planets and for regular satellites of heavy planets with empirically observed. Obtained theoretical results are in reasonable accord with empirically observed and with the quantum mechanical rules for the filling of orbits discussed in Subsection 4.2. These results are extended in Subsection 4.3 describing rationale for quantization of dynamics of rings around heavy planets. In Section 5 we discuss the problem of embedding of the Lorentzian group SO(2,1) into larger groups such as SO(3,1), SO(4,1), SO(4,2), etc. This is done with assumption that these larger symmetries should be taken into account in anticipation that quantum dynamics of Solar System could be eventually used in testing some cosmological models/theories. Paper concludes with Section 6 in which we discuss the reasons why the developed formalism fits the combinatorial theory of group representations recently discussed by mathematicians Knutson and Tao [12-14] and applied to quantum mechanical problems by Kholodenko [15,16]. Appendices A-C supplement some results of Sections 3 and 4.

2 Harmonious coexistence of general relativity and quantum mechanics in the Solar System

2.1 From Laplace to Einstein and via Poincare’

Everybody knows that Einstein considered the Copenhagen version of quantum mechanics as incomplete/temporary. He was hoping for a deeper quantum theory in which God is not playing dice. His objections, in part, had been caused by the fact that the ”new” quantum mechanics and the ”old” general relativity have nothing in common. In this section we would like to argue, that such an attitude by Einstein is caused, most likely, by circumstances of his life and that these two disciplines actually have the same historical origin. It is known that Einstein was not too excited about the works of Henry

\[\text{See also } \text{www.math.ucla.edu/~tao/java/Honeycomb.html}\]
Poincare’ and, in return, Poincare’ never quoted works by Einstein\textsuperscript{8}. As result of this historical peculiarity, in his seminal works on general relativity Einstein never quoted Poincare’s revolutionary results on celestial mechanics. Thus, the celebrated shift of Mercury’s perihelium was obtained entirely independent of Poincare’s results! Correctness of Einsteinian relativity had been tested many times, including results obtained in 2008 and 2010 [18,19]. These latest results are the most accurate to date. They unambiguously support general relativity in its canonical form at least at the scales of our Solar System. Einstein’s victory over Poincare’ is mysterious in view the following facts from celestial mechanics. To discuss these facts, we need to provide some background from classical mechanics first. In particular, even though classical Hamiltonians for Coulombic and Newtonian potentials look almost the same, they are far from being exactly the same. In the classical Hamiltonians for multielectron atoms all electron masses are the same, while for the Solar-like planetary system the masses of all satellites are different. In some instances to be discussed below such difference can be made non existent. In such cases formal quantization for both systems can proceed in the same way. To explain how this happens, we begin with two-body Kepler problem treated in representative physics textbooks [20]. Such treatments tend to ignore the equivalence principle- essential for the gravitational Kepler problem and nonexistent for the Coulomb-type problems. Specifically, the description of general relativity in Vol.2 of the world-famous Landau-Lifshitz course in theoretical physics [21] begins with the Lagrangian for the particle in gravitational field $\varphi$:

$$L = \frac{m v^2}{2} - m \varphi.$$  

The Newton’s equation for such a Lagrangian reads:

$$\dot{v} = -\nabla \varphi.$$  \hspace{1cm} (2.1)

Since the mass drops out of this equation, it is possible to think about such an equation as an equation for a geodesic in (pseudo)Riemannian space. This observation, indeed, had lead Einstein to full development of theory of general relativity and to his calculation of the Mercury’s perihelion shift. The above example is misleading though. Indeed, let us discuss the 2-body Kepler problem for particles with masses $m_1$ and $m_2$ interacting gravitationally. The Lagrangian for this problem is given by

$$L = \frac{m_1}{2} \dot{r}_1^2 + \frac{m_2}{2} \dot{r}_2^2 + \gamma \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}.$$  \hspace{1cm} (2.2)

\textsuperscript{8}Only late in his life Einstein did acknowledged Poinancare’s contributions to science [17]
Introducing, as usual, the center of mass and relative coordinates via \( m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = 0 \) and \( \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \), the above Lagrangian acquires the following form:

\[
\mathcal{L} = \frac{\mu}{2} \dot{\mathbf{r}}^2 + \gamma \frac{m_1 m_2}{|\mathbf{r}|} = \frac{m_1 m_2}{m_1 + m_2} \left( \frac{\dot{\mathbf{r}}^2}{2} + \gamma \frac{(m_1 + m_2)}{|\mathbf{r}|} \right),
\]  

(2.3)

where, as usual, we set \( \mu = \frac{m_1 m_2}{m_1 + m_2} \). The constant \( \frac{m_1 m_2}{m_1 + m_2} \) can be dropped and, after that, instead of the geodesic, Eq.(2.1), we obtain the equation for a fictitious point-like object of unit mass moving in the field of gravity produced by the point-like body of mass \( m_1 + m_2 \). Clearly, in general, one cannot talk about geodesics in this case even though Infeld and Schild had attempted to do just this already in 1949 [22]. The case is far from being closed even in 2010 [23]. These efforts look to us mainly as academic (unless dynamics of binary stars is considered) for the following reasons. If, say, \( m_1 \gg m_2 \) as for the electron in Hydrogen atom or for the Mercury rotating around Sun, one can (to a very good accuracy) discard mass \( m_2 \) thus obtaining the equation for a geodesic coinciding with (2.1). In the Introduction we defined the ratio \( r = \frac{m_2}{m_1 + m_2} \). If we do not consult reality for guidance, the ratio \( r \) can have any nonnegative value. However, what is observed in the sky (and in atomic systems as well) leads us to the conclusion that (ignoring our Moon) all satellites of heavy planets as well as all planets of our Solar System move along geodesics described by (2.1), provided that we can ignore interaction between the planets/satellites. We shall call such an approximation the Einstein’s limit. It is analogous to the mean field Hartree-type approximation in atomic mechanics. If we believe Einstein, then such Hartree-type approximation does not require any corrections. This looks like "too good to be true". Indeed, the first who actually used Einstein’s limit (more than 100 years before Einstein!) in his calculations was Laplace [24], Vol.4. In his book [6], Vol.1, art 50, Poincare’ discusses Laplace’s work on dynamics of satellites of Jupiter. Incidentally, Laplace also studied motion of the Moon, of satellites of Saturn and Uranus and of Saturn’s ring system [24], Vol.s 2,4.

Quoting from Poincare’:

"(Following Laplace) consider the central body of large mass (Jupiter) and three other small bodies (satellites Io, Europe and Ganymede), whose masses can be taken to be zero, rotating around a large body in accordance with Kepler’s law. Assume further that the eccentricities and inclinations of the orbits of these (zero mass) bodies are equal to zero, so that the motion is going to be circular. Assume further that the frequencies of their rotation \( \omega_1, \omega_2 \) and \( \omega_3 \) are such that there is a linear relationship

\[
\alpha \omega_1 + \beta \omega_2 + \gamma \omega_3 = 0 
\]  

(2.4)
with $\alpha, \beta$ and $\gamma$ being three mutually simple integers such that

$$\alpha + \beta + \gamma = 0. \quad (2.5)$$

Given this, it is possible to find another three integers $\lambda, \lambda'$ and $\lambda''$ such that $\alpha \lambda + \beta \lambda' + \gamma \lambda'' = 0$ implying that $\omega_1 = \lambda A + B, \omega_2 = \lambda' A + B, \omega_3 = \lambda'' A + B$ with $A$ and $B$ being some constants. After some time $T$ it is useful to construct the angles $T(\lambda A + B), T(\lambda' A + B)$ and $T(\lambda'' A + B)$ describing current location of respective satellites (along their circular orbits) and, their differences: $(\lambda - \lambda')AT$ and $(\lambda - \lambda'')AT$. If now we choose $T$ in such a way that $AT$ is proportional to $2\pi$, then the angles made by the radius-vectors (from central body to the location of the planet) will coincide with those for $T = 0$. Naturally, such a motion (with zero satellite masses) is periodic with period $T$.

The question remains: Will the motion remain periodic in the case if masses are small but not exactly zero? That is, if one allows the satellites to interact with each other?.....

Laplace had demonstrated that the orbits of these three satellites of Jupiter will differ only slightly from truly periodic. In fact, the locations of these satellites are oscillating around the zero mass trajectory$^\text{v}$

Translation of this last paragraph into language of modern quantum mechanics reads: Laplace demonstrated that only the Einsteinian trajectories are subject to the Bohr-Sommerfelf’d- type quantization condition. That is at the scales of Solar System correctness of Einsteinian general relativity is assured by correctness of quantum mechanics (closure of the Laplace-Lagrange oscillating orbits $^{[25]}$) and vice-versa so that these two theories are inseparably linked together.

The attentive reader of this excerpt from Poincare’ could already realized that Laplace came to such a conclusion based on (2.4) as starting point. Thus, the condition (2.4) can be called quantization condition (since eventually it leads to the Bohr-Sommerfelf’d condition). Interestingly enough, this condition was chosen by Heisenberg $^{[26]}$ as fundamental quantization condition from which all machinery of quantum mechanics can be deduced! This topic is further discussed in the next subsection. Before doing so we notice that extension of work by Laplace to the full $n + 1$ body planar problem was made only in 20th century and can be found in the monograph by Charlier $^{[27]}$. More rigorous mathematical proofs involving KAM theory have been obtained just recently by Fejoz $^{[28]}$ and Biasco et al $^{[29]}$. The difficulty, of course, is caused by proper accounting of the effects of finite but nonzero masses of satellites and by showing that, when these masses are very small, the Einsteinian limit makes perfect sense and is stable. A sketch of these calculations for planar
four-body problem (incidentally studied by de Sitter in 1909) can be found in nicely written lecture notes by Moser and Zehnder [30].

2.2 From Laplace to Heisenberg and beyond

Very much like Einstein, who without reading of works by Poincare on celestial mechanics arrived at correct result for dynamics of Mercury, Heisenberg had arrived at correct formulation of quantum mechanics without reading works by both Poincare’ and Laplace’. In retrospect, this is not too surprising: correctly posed problems should lead to correct solutions. In the case of Einstein, his earlier obtained result $E = mc^2$ caused him to think about both dynamics of planets/satellites and light in the gravitational field of heavy mass on equal footing [31]. Very likely, this equal footing requirement was sufficient for developing of his relativity theory without consulting the works by Poincare’ on celestial mechanics in which dynamics of light was not discussed. Analogously, for Heisenberg the main question was: To what extent can one restore the underlying microscopic dynamic system using combinatorial analysis of the observed spectral data? Surprisingly, the full answer to this question compatible with Heisenberg’s original ideas had been obtained only quite recently. Details and references can be found in our work, Ref.[15]. For the sake of space, in this paper we only provide absolute minimum of results needed for correct modern understanding of Heisenberg’s ideas.

We begin with observation that the Schrödinger equation cannot be reduced to something else which is related to our macroscopic experience. It has to be postulated.\(^9\) On the contrary, Heisenberg’s basic equation from which all quantum mechanics can be recovered is directly connected with experimental data and looks almost trivial. Indeed, following Bohr, Heisenberg looked at the famous equations for energy levels difference

$$\omega(n, n - \alpha) = \frac{1}{\hbar}(E(n) - E(n - \alpha)),$$

where both $n$ and $n - \alpha$ are some integers. He noticed [26] that this definition leads to the following fundamental composition law:

$$\omega(n - \beta, n - \alpha - \beta) + \omega(n, n - \beta) = \omega(n, n - \alpha - \beta). \quad (2.7a)$$

Since by design $\omega(k, n) = -\omega(n, k)$, the above equation can be rewritten in a symmetric form as

$$\omega(n, m) + \omega(m, k) + \omega(k, n) = 0. \quad (2.7b)$$

\(^9\) Usually used appeal to the DeBroigle wave-particle duality is of no help since the wave function in the Schrödinger’s equation plays an auxiliary role.
In such a form it is known as the honeycomb equation (condition) in current mathematics literature [12-14] where it was rediscovered totally independently of Heisenberg’s key quantum mechanical paper and, apparently, with different purposes in mind. Connections between mathematical results of Knutson and Tao [12-14] and those of Heisenberg were noticed and discussed in recent paper by Kholodenko[15,16]. We would like to use some results from this work now.

We begin by noticing that (2.7b) due to its purely combinatorial origin does not contain the Plank’s constant \( \hbar \). Such fact is of major importance for this work since the condition (2.4) can be equivalently rewritten in the form of (2.7b), where \( \omega(n, m) = \omega_n - \omega_m \). It would be quite unnatural to think of the Planck’s constant in this case\(^{10}\). Equation (2.7b) is essentially of the same type as (2.4). It looks almost trivial and yet, it is sufficient for restoration of all quantum mechanics. Indeed, in his paper of October 7th of 1925, Dirac\(^{11}\), being aware of Heisenberg’s key paper, streamlined Heisenberg’s results and introduced notations which are in use up to this day. He noticed that the combinatorial law given by (2.7a) for frequencies, when used in the Fourier expansions for composition of observables, leads to the multiplication rule \( a(nm)b(mk) = ab(nk) \) for the Fourier amplitudes for these observables. In general, in accord with Heisenberg’s assumptions, one expects that \( ab(nk) \neq ba(nk) \). Such a multiplication rule is typical for matrices. In the modern quantum mechanical language such matrix elements are written as \( <n | \hat{O} | m> \exp(i\omega(n,m)t) \) so that (2.7.b) is equivalent to the matrix statement

\[
\sum_m <n | \hat{O}_1 | m><m | \hat{O}_2 | k> \exp(i\omega(n,m)t) \exp(i\omega(m,k)t)
= <n | \hat{O}_1 \hat{O}_2 | k> \exp(i\omega(n,k)t). \tag{2.8}
\]

for some operator (observables) \( \hat{O}_1 \) and \( \hat{O}_2 \) evolving according to the rule: \( \hat{O}_k(t) = U \hat{O}_k U^{-1}, k = 1,2 \), provided that \( U^{-1} = \exp(-i \hat{H}t) \). From here it follows that \( U^{-1} | m >= \exp(-\frac{E_m}{\hbar}t) | m > \) if one identifies \( \hat{H} \) with the Hamiltonian operator. Clearly, upon such an identification the Schrödinger equation can be obtained at once as is well known [35] and with it, the rest of quantum mechanics. In view of Ref.s[12-16] it is possible to extend the traditional pathway: from classical to quantum mechanics and back. This topic is discussed in the next section.

\(^{10}\)Planck’s constant is normally being used for objects at the atomic scales interacting with light. However, there are systems other than atomic, e.g. polymers, in which Schrödinger (or even Dirac-type)-type equations are being used with Planck’s constant being replaced by the stiffness parameter-different for different polymers [32]. Incidentally, conformational properties of very stiff (helix-type) polymers are described by the neutrino-type equation, etc. [33].

\(^{11}\)This paper was sent to Dirac by Heisenberg himself.
3 Space, time and space-time in classical and quantum mechanics

3.1 General comments

If one contemplates quantization of dynamics of celestial objects using traditional textbook prescriptions, one will immediately run into myriad of small and large problems. Unlike atomic systems in which all electrons repel each other, have the same masses and are indistinguishable, in the case of, say, Solar System all planets (and satellites) attract each other, have different masses and visibly distinguishable. Besides, in the case of atomic systems the Planck constant $\hbar$ plays prominent role while no such a role can be given to the Planck constant in the sky.

In the previous section it was demonstrated that in the Einsteinian limit it is possible to remove the above objections so that, apparently, the only difference between the atomic and celestial quantum mechanics lies in replacement of the Planck constant by another constant to be determined in Section 4.

3.2 Space and time in classical and quantum mechanics

Although celestial mechanics based on Newton’s law of gravity is considered to be classical (i.e. non quantum), with such an assumption one easily runs into serious problems. Indeed, such an assumption implies that the speed with which the interaction propagates is infinite and that time is the same everywhere. Whether this is true or false can be decided only experimentally. Since at the scales of our Solar System one has to use radio signals to check correctness of Newton’s celestial mechanics, one is faced immediately with all kind of wave mechanics effects such as retardation, the Doppler effect, etc. Because of this, measurements are necessarily having some error margins. The error margins naturally will be larger for more distant objects. Accordingly, even at the level of classical mechanics applied to the motion of celestial bodies we have to deal with certain inaccuracies similar in nature to those in atomic mechanics. To make formalisms of both atomic and celestial quantum mechanics look the same we have to think carefully about the space, time and space-time transformations already at the level of classical mechanics.

We begin with observation that in traditional precursor of quantum mechanics—the Hamiltonian mechanics—the Hamiltonian equations by design remain invariant with respect to the canonical transformations. That is if sets $\{q_i\}$ and $\{p_i\}$ represent the ”old” canonical coordinates and momenta while $Q_i = Q_i(\{q_i\},\{p_i\})$ and $P_i = P_i(\{q_i\},\{p_i\})$, $i = 1 - N$, represent the ”new” set of canonical coordinates and momenta, the Hamiltonian equations in the old
variables given by
\[ \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \]  \tag{3.1}
and those rewritten in "new" variables will have the same form. Here we used the commonly accepted notations, e.g. \( \dot{q}_i = \frac{d}{dt} q_i \), etc. Quantum mechanics uses this form-invariance essentially as is well known.

We would like to complicate this traditional picture by investigating the "canonical" time changes in classical mechanics. Fortunately, such task was accomplished to a large extent in the monograph by Pars [36]. For the sake of space, we refer our readers to pages 535-540 of this monograph for more details. Following Dirac [37], we notice that good quantization procedure should always begin with the Lagrangian formulation of mechanics since it is not always possible to make a transition from the Lagrangian to Hamiltonian form of mechanics (and, thus, to quantum mechanics) due to presence of some essential constraints (typical for mechanics of gauge fields, etc.). Hence, we also begin with the Lagrangian functional \( \mathcal{L} = \mathcal{L} \{ q_i, \dot{q}_i \} \). The Lagrangian equations of motion can be written in the form of Newton’s equations \( \dot{p}_i = F_i \), where the generalized momenta \( p_i \) are given by \( p_i = \frac{\delta \mathcal{L}}{\delta \dot{q}_i} \) and the generalized forces \( F_i \) by \( F_i = -\frac{\delta \mathcal{L}}{\delta q_i} \) as usual. In the case if the total energy \( E \) is conserved, it is possible instead of "real" time \( t \) to introduce the fictitious time \( \theta \) via relation \( dt = u(q_i) d\theta \) where the function \( u(q_i) \) is assumed to be nonnegative and is sufficiently differentiable with respect to its arguments. At this point we can enquire if Newton’s equations can be written in terms of new time variable so that they remain form- invariant. To do so, following Pars, we must: a) to replace \( \mathcal{L} \) by \( u \mathcal{L} \), b) to replace \( \dot{q}_i \) by \( q'_i / u \), where \( q'_i = \frac{dq_i}{d\theta} \), c) to rewrite new Lagrangian in terms of such defined new time variables and, finally, d) to obtain Newton’s equations according to the described rules, provided that now we have to use \( p'_i \) instead of \( \dot{p}_i \). In the case if the total energy of the system is conserved, we shall obtain back the same form of Newton’s equations rewritten in terms of new variables. This means that by going from the Lagrangian to Hamiltonian formalism of classical mechanics we can write the Hamilton’s equations (3.1) in which the dotted variables are replaced by primed. Furthermore, (3.1) will remain the same if we replace the Hamiltonian \( H \) by some nonnegative function \( f(H) \) while changing time \( t \) to time \( \theta \) according to the rule \( d\theta / dt = df(H) / dH \big|_{H=E} \). Such a change while leaving classical mechanics form-invariant will affect quantum mechanics where now the Schrödinger’s equation
\[ i\hbar \frac{\partial}{\partial t} \Psi = \hat{H} \Psi \]  \tag{3.2}
is replaced by

\[ i\hbar \frac{\partial}{\partial \theta} \Psi = f(\hat{H}) \Psi. \]  

(3.3)

With such information at our hands, we would like to discuss the extent to which symmetries of our (empty) space-time affect dynamics of particles "living" in it.

### 3.3 Space-time in quantum mechanics

#### 3.3.1 General comments

Use of group-theoretic methods in quantum mechanics had began almost immediately after its birth. It was initiated by Pauli in 1926. He obtained a complete quantum mechanical solution for the Hydrogen atom employing symmetry arguments. His efforts were not left without appreciation. Our readers can find many historically important references in two comprehensive review papers by Bander and Itzykson [38]. In this subsection we pose and solve the following problem: Provided that the symmetry of (classical or quantum) system is known, will this information be sufficient for determination of this system uniquely?

Below, we shall provide simple and concrete examples illustrating meaning of the word "determination". In the case of quantum mechanics this problem is known as the problem about hearing of the "shape of the drum". It was formulated by Mark Kac [39]. The problem can be formulated as follows. Suppose that the sound spectrum of the drum is known, will such an information determine the shape of the drum uniquely? The answer is "No" [40]. Our readers may argue at this point that non uniqueness could come as result of our incomplete knowledge of symmetry or, may be, as result of the actual lack of true symmetry (e.g. the Jahn-Teller effect in molecules, etc. in the case of quantum mechanics). These factors do play some role but they cannot be considered as decisive as the basic example below demonstrates.

#### 3.3.2 Difficulties with the correspondence principle for Hydrogen atom

In this subsection we do not use arguments by Kac since our arguments are much more straightforward. We choose the most studied case of Hydrogen atom as an example.

As it is well known, the Keplerian motion of a particle in the centrally symmetric field is planar and is exactly solvable for both the scattering and
bound states at the classical level \[36\]. The result of such a solution depends on two parameters: the energy and the angular momentum. The correspondence principle formulated by Bohr is expected to provide the bridge between the classical and quantum realities by requiring that in the limit of large quantum numbers the results of quantum and classical calculations for observables should coincide. However, this requirement may or may not be possible to implement. It is violated already for the Hydrogen atom! Indeed, according to the naive canonical quantization prescriptions, one should begin with the \textit{classical} Hamiltonian in which one has to replace the momenta and coordinates by their operator analogs. Next, one uses such constructed quantum Hamiltonian in the Schrödinger’s equation, etc. Such a procedure breaks down at once for the Hamiltonian of Hydrogen atom since the intrinsic planarity of the classical Kepler’s problem is entirely ignored thus leaving the projection of the angular momentum without its classical analog. Accordingly, the \textit{scattering differential crossection for Hydrogen atom obtained quantum mechanically (within the 1st Born approximation) uses essentially 3-dimensional formalism and coincides with the classical result by Rutherford obtained for planar configurations!} Thus, even for the Hydrogen atom classical and quantum (or, better, \textit{pre quantum}) Hamiltonians do \textit{not} match thus formally violating the correspondence principle. Evidently, semiclassically we can only think of energy and the angular momentum thus leaving the angular momentum projection undetermined. Such a ”sacrifice” is justified by the agreement between the observed and predicted Hydrogen atom spectra and by use of Hydrogen-like atomic orbitals for multielectron atoms, etc. Although, to our knowledge, such a mismatch is not mentioned in any of the students textbooks on quantum mechanics, its existence is essential if we are interested in extension of quantum mechanical ideas to dynamics of Solar System. In view of such an interest, we would like to reconsider traditional treatments of Hydrogen atom, this time being guided only by the symmetry considerations. This is accomplished in the next subsection.

### 3.3.3 Emergence of the SO(2,1) symmetry group

In April of 1940 Jauch and Hill \[41\] published a paper in which they studied the planar Kepler problem quantum mechanically. Their work was stimulated by earlier works by Fock of 1935 and by Bargmann of 1936 in which it was shown that the spectrum of bound states for the Hydrogen atom can be obtained by using representation theory of SO(4) group of rigid rotations of 4-dimensional Euclidean space while the spectrum of scattering states can be obtained by using the Lorentzian group SO(3,1). By adopting results of Fock and Bargmann to the planar configuration Jauch and Hill obtained the antic-
ipated result: In the planar case one should use SO(3) group for the bound states and SO(2,1) group for the scattering states. Although this result will be reconsidered almost entirely, we mention about it now having several purposes in mind.

First, we would like to reverse arguments leading to the final results of Jauch and Hill in order to return to the problem posed at the beginning of this section. That is, we want to use the fact that the Kepler problem is planar (due to central symmetry of the force field) and the fact that the motion takes place in (locally) Lorentzian space-time in order to argue that the theory of group representations for Lorentzian SO(2,1) symmetry group-intrinsic for this Kepler problem-correctly reproduces the Jauch-Hill spectrum. Nevertheless, the question remains: Is Kepler’s problem the only one exactly solvable classical and quantum mechanical problem associated with the SO(2,1) group? Below we demonstrate that this is not the case! In anticipation of such negative result, we would like to develop our intuition by using some known results from quantum mechanics.

3.3.4 Classical-quantum correspondence allowed by SO(2,1) symmetry: a gentle introduction

For the sake of space, we consider here only the most generic (for this work) example in some detail: the radial Schrödinger equation for the planar Kepler problem with the Coulombic potential. It is given by

$$\frac{-\hbar^2}{2\mu} \left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2}\right) \Psi(\rho) - \frac{Ze^2}{\rho} = E \Psi(\rho). \quad (3.4)$$

Here $|m| = 0, 1, 2, ...$ is the angular momentum quantum number as required. For $E < 0$ it is convenient to introduce the dimensionless variable $x$ via $\rho = ax$ and to introduce the new wave function: $\psi(\rho) = \sqrt{\rho} \Psi(\rho)$. Next, by the appropriate choice of constant $a$ and by redefining $\psi(\rho)$ as $\psi(\rho) = \gamma x^{\frac{1}{2}+|m|} \exp(-y) \varphi(y)$, where $y = \gamma x$, $-\gamma^2 = \frac{2\mu E}{\hbar^2 a^2}$, $a = \frac{\hbar^2}{mZe^2}$, the following hypergeometric equation can be eventually obtained:

$$\left\{ y \frac{d^2}{dy^2} + 2|m| + \frac{1}{2} - y \frac{d}{dy} + 2\left[ \frac{1}{y} - |m| - \frac{1}{2}\right]\right\} \varphi(y) = 0. \quad (3.5)$$

Formal solution of such an equation can be written as $\varphi(y) = \mathcal{F}(-A(m), B(m), y)$, where $\mathcal{F}$ is the confluent hypergeometric function. Physical requirements imposed on this function reduce it to a polynomial leading to the spectrum of the

---

12 The rationale for discussing the Coulombic potential instead of gravitational will be fully explained in the next section.
planar Kepler problem. Furthermore, by looking into standard textbooks on quantum mechanics, one can easily find that exactly the same type of hypergeometric equation is obtained for problems such as one-dimensional Schrödinger’s equation with the Morse-type potential,\textsuperscript{13} three dimensional radial Schrödinger equation for the harmonic oscillator\textsuperscript{14} and even three dimensional radial equation for the Hydrogen atom\textsuperscript{15}. Since the two-dimensional Kepler problem is solvable with help of representations of SO(2,1) Lorentz group, the same should be true for all quantum problems just listed. That this is the case is demonstrated, for example, in the book by Wybourne \cite{Wybourne}. A sketch of the proof is provided in Appendix A. This proof indicates that, actually, the discrete spectrum of all problems just listed is obtainable with help of SO(2,1) group. The question remains: If the method outlined in Appendix A provides the spectra of several quantum mechanical problems listed above, can we be sure that these are the only exactly solvable quantum mechanical problems associated with the SO(2,1) Lorentz group? Unfortunately, the answer is "No"! More details are given below.

### 3.3.5 Common properties of quantum mechanical problems related to SO(2,1) Lorentz group

In Appendix A a sketch of the so called spectrum-generating algebras (SGA) method is provided. It is aimed at producing the exactly solvable one-variable quantum mechanical problems. In this subsection we would like to put these results in a broader perspective. In particular, in our works\cite{Saxena1, Saxena2} we demonstrated that all exactly solvable quantum mechanical problem should involve hypergeometric functions of single or multiple arguments. We argued that the difference between different problems can be understood topologically in view of the known relationship between hypergeometric functions and braid groups. These results, even though quite rigorous, are not well adapted for immediate practical use. In this regard more useful would be to solve the following problem: For a given set of orthogonal polynomials find the corresponding many-body operator for which such a set of orthogonal polynomials forms the complete set of eigenfunctions. At the level of orthogonal polynomials of one variable relevant for all exactly solvable two-body problems of quantum mechanics, one can think about the related problem of finding all potentials in one-dimensional radial Schrödinger equation, e.g. equation (A.1),\textsuperscript{13}

\begin{align*}
\text{That is, } V(x) &= A(e^x - 2e^{-x}).
\end{align*}

\begin{align*}
\text{That is, } V(r) &= \frac{A}{r^2} + Br^2.
\end{align*}

\begin{align*}
\text{That is, } V(r) &= \frac{A}{r^2} - \frac{B}{r}.
\end{align*}
leading to the hypergeometric-type solutions. Very fortunately, such a task was accomplished already by Natanzon [43]. Subsequently, his results were re-investigated by many authors with help of different methods, including SGA. To our knowledge, the most complete recent summary of the results, including potentials and spectra can be found in the paper by Levai [44]. Even this (very comprehensive) paper does not cover all aspects of the problem. For instance, it does not mention the fact that these results had been extended to relativistic equations such as Dirac and Klein-Gordon for which similar analysis was made by Cordero with collaborators [45]. In all cited cases (relativistic and non relativistic) the underlying symmetry group was SO(2,1). The results of Appendix A as well as of all other already listed references can be traced back to the classically written papers by Bargmann [46] and Barut and Fronsdal [47] on representations of SO(2,1) Lorentz group. Furthermore, the discovered connection of this problematic with supersymmetric quantum mechanics [48,49] can be traced back to the 19th century works by Gaston Darboux. The fact that representations of the planar SO(2,1) Lorentz group are sufficient to cover all known exactly solvable two-body problems (instead of the full SO(3,1) Lorentz group!) is quite remarkable. It is also sufficient for accomplishing the purposes of this work-to quantize the dynamics of Solar System- but leaves open the question: Will use of the full Lorentz group lead to the exactly solvable quantum mechanical problems not accounted by the SO(2,1) group symmetry? This topic will be briefly discussed in Section 5. In the meantime, we would like to address the problem of quantizing the Solar System dynamics using the obtained results This is accomplished in the next section.

4 Quantum celestial mechanics of Solar System

4.1 General remarks

We begin this subsection by returning back to (2.4). Based on previous discussions, this equation provides us with opportunity to think seriously about quantum nature of dynamics of our Solar System dynamics. Nevertheless, such an equation reveals only one aspect of quantization and, as such, provides only sufficient condition for quantization. The necessary condition in atomic and celestial mechanics lies in the non dissipativity of dynamical systems in both cases. Recall that Bohr introduced his quantization prescription to avoid dissipation caused by the emission of radiation by electrons in orbits in general position. New quantum mechanics have not shed much light on absence of dissipation for stationary Bohr's orbits. At the level of old Bohr theory absence
of dissipation at the stationary Bohr orbit was explained by Boyer [50]. Subsequently, his result was refined by Puthoff [51]. In the case of Solar System absence of dissipation for motion on stable orbits was discussed by Goldreich [52] who conjectured that the dissipative (tidal) effects adjust the initial motion of planets/satellites in such a way that eventually the orbits become stable. More on this is discussed in Subsection 5.2. Notice that, dynamics of Solar System as considered by Poincare’ and by those who developed his ideas does not involve treatment of tidal effects! Treatment of tidal effects in general relativity represents one of the serious challenges for this theory [53]. Thus, very much by analogy with Bohr, we have to postulate that in the case of Solar System (Hamiltonian) dynamics of stable orbits is non dissipative. This assumption then leads us to the following Table:

Table 1

<table>
<thead>
<tr>
<th>Type of mechanics Properties</th>
<th>Quantum atomic mechanics</th>
<th>Quantum celestial mechanics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dissipation (type of)(\text{yes}\ (\text{no})) on stable orbits</td>
<td>electromagnetic friction(\text{no}) Bohr orbits</td>
<td>tidal friction (\text{no}) Einstein’s geodesics</td>
</tr>
<tr>
<td>Accidental degeneracy(\text{yes}\ (\text{no})) origin</td>
<td>yes\Bohr-Sommerfeld condition</td>
<td>yes\Laplace condition</td>
</tr>
<tr>
<td>Charge neutrality</td>
<td>yes</td>
<td>no (but see below)</td>
</tr>
<tr>
<td>Masses</td>
<td>electrons are having the same masses</td>
<td>(up to validity of the equivalence principle) masses are the same</td>
</tr>
<tr>
<td>Minimal symmetry group</td>
<td>(\text{SO}(2,1))</td>
<td>(\text{SO}(2,1))</td>
</tr>
<tr>
<td>Correspondence principle</td>
<td>occasionally violated</td>
<td>occasionally violated</td>
</tr>
<tr>
<td>Discrete spectrum: finite or infinite(\text{reason}) Pauli principle(\text{yes}\ (\text{no}))</td>
<td>finite and infinite\charge neutrality\yes</td>
<td>finite\no charge neutrality\yes</td>
</tr>
</tbody>
</table>

4.1.1 Celestial spectroscopy and the Titius-Bode law of planetary distances

The atomic spectroscopy was inaugurated by Newton in the second half of 17th century. The celestial spectroscopy was inaugurated by Titius in the second half of 18th century and become more famous after it was advertised by Johann Bode, the Editor of the ”Berlin Astronomical Year-book”. The book by Nieto [54] provides extensive bibliography related to uses and interpretations of the
Titius-Bode (T-B) law up to second half of 20th century. Unlike the atomic spectroscopy, where the observed atomic and molecular spectra were expressed using simple empirical formulas which were (to our knowledge) never elevated to the status of "law", in celestial mechanics the empirical T-B formula

\[ r_n = 0.4 + 0.3 \cdot 2^n, \quad n = -\infty, 0, 1, 2, 3, ... \]  \hspace{1cm} (4.1)

for the orbital radii (semimajor axes) of planets acquired the status of a law in the following sense. In the case of atomic spectroscopy the empirical formulas used for description of atomic/molecular spectra have not been used (to our knowledge) for making predictions. Their purpose was just to describe in mathematical terms what had been already observed. Since the T-B empirical formula for planetary distances was used as the law, it was used in search for planets not yet discovered. In such a way Ceres, Uranus, Neptune and Pluto were found \[10\]. However, the discrepancies for Neptune and Pluto were much larger than the error margins allowed by the T-B law. This fact divided the astronomical community into "believers" and "atheists" (or non believers) regarding to the meaning and uses of this law. Without going into historical details, we would like to jump to the very end of the Titius-Bode story in order to use its latest version which we found in the paper by Neslušan \[55\] who, in turn, was motivated by the work of Lynch \[56\]. Instead of (4.1) these authors use another empirical power law dependence

\[ r_n = r_0 B^n, \quad n = 1, 2, 3, ..., 9. \]  \hspace{1cm} (4.2)

For planets (except Pluto and including the asteroid belt) Neslušan obtained\[16\] \[r_0(\text{au}) = 0.203\] and \[B = 1.773\] with the rms deviation accuracy of 0.0534\[17\]. Analogous power law dependencies were obtained previously in the work by Dermott \[57\] for both planets and satellites of heavy planets such as Jupiter, Saturn and Uranus.

It should be noted that because of noticed discrepancies the attempts were made to prove or disprove the Titius-Bode law by using statistical analysis, e.g. see papers by Lynch \[56\] and Hayes and Tremaine \[58\], with purpose of finding out to which extent the observed dependencies can be considered as non accidental. Following logic of Bohr we would like to use the observed empirical radial dependencies as a guide to our calculations to be discussed below. We leave with astronomers to resolve the semantic aspects of these observed dependencies.

\[16\]In astronomical units (to be defined below).

\[17\]This result gives for the Earth in astronomical (au) units the result \(r_3 \simeq 1.13\). Much better result is obtained in case if we choose \(B = 1.7\). In this case we obtain: \(r_3 \simeq 0.97339\). Lynch provides \(B = 1.706\) and \(r_0 = 0.2139\).
4.1.2 An attempt at quantization of celestial (Solar System) dynamics

Being guided by the Table 1 at the beginning of this section we will be assuming that planets do not interact since they move along geodesics independently. In the case of atomic mechanics it was clear from the beginning that such an approximation should sooner or later fail. The nonexisting electroneutrality in the sky provides strong hint that the T-B law must be of very limited use since the number of discrete levels for gravitating systems should be always finite. Otherwise, we would observe the countable infinity of satellites around Sun or of any of heavy planets. This is not observed and is physically wrong. It is wrong because such a system would tend to capture all matter in the Universe.

In the literature one can find many attempts at quantization of Solar System using standard prescriptions of quantum mechanics. Since this work is not a review, we do not provide references to papers whose results do not affect ours\(^{18}\). Blind uses of standard rules of quantum mechanics for quantization of our Solar System do not contain any provisions for finite number of energy levels/orbits for gravitating systems.

To facilitate matters in the present case, we would like to make several additional observations. First, we have to find an analog of the Planck constant. Second, we have to have some mechanical model in mind to make our search for physically correct answer successful. To accomplish the first task, we have to take into account the 3-rd Kepler’s law. In accord with (2.3), it can be written as

\[
\frac{r_n^3}{T_n^2} = \frac{4\pi^2}{\gamma(M_\odot + m)}.
\]

In view of arguments presented in the Subsection 2.1, we can safely approximate this result by \(4\pi^2/\gamma M_\odot\), where \(M_\odot\) is mass of the Sun. For the purposes of this work, it is convenient to restate this law as

\[
3mr_n - 2\ln T_n = \ln 4\pi^2/\gamma M_\odot = \text{const}\]

Below, we choose the astronomical system of units in which \(4\pi^2/\gamma M_\odot = 1\). By definition, in this system of units we have for the Earth: \(r_3 = T_3 = 1\).

Consider now the Bohr result (2.6) and take into account that

\[
E = \hbar \omega = \hbar \frac{2\pi}{2\pi T} = \omega(n) - \omega(m).
\]

Taking into account equations (2.6), (3.3), (4.2) and the third Kepler’s law, we formally obtain:

\[
\omega(n, m) = \frac{1}{c \ln A} (nc \ln \tilde{A} - mc \ln \tilde{A}),
\]

(4.3)

where the role of Planck’s constant is played now by \(c \ln \tilde{A}\), \(\tilde{A} = B^2\) and \(c\) is

\(^{18}\)With one exception to be mentioned in Section 5.
some constant which will be determined selfconsistently below\textsuperscript{19}.

At first, one may think that what we obtained is just a simple harmonic oscillator spectrum. After all, this should come as not too big a surprise since in terms of the action-angle variables all exactly integrable systems are reducible to the sets of harmonic oscillators. This result is also compatible with the results of Appendix A. The harmonic oscillator option is physically undesirable in the present case though since the harmonic oscillator has countable infinity of energy levels. Evidently, such a spectrum is equivalent to the T-B law. But it is well known that this law is not working well for larger numbers. In fact, it would be very strange should it be working in this regime in view of arguments already presented.

To make a progress, we have to use the 3rd Kepler’s law once again, i.e. we have to take into account that in the astronomical system of units $3lnr_n = 2 ln T_n$. A quick look at equations (A.11), (A.12) suggests that the underlying mechanical system is likely to be associated with that for the Morse potential. This is so because the low lying states of such a system cannot be distinguished from those for the harmonic oscillator. However, this system does have only a finite number of energy levels which makes sense physically. The task remains to connect this system with the planar Kepler’s problem. Although in view of results of Appendix A such a connection does indeed exist, we want to demonstrate it explicitly at the level of classical mechanics.

Before doing so we have to make several comments. First, according to the Table 1, the planets/satellites should move along the geodesics. Second, the geodesics which were used by Einstein for bending of light and for motion of Mercury are obtainable with help of the metric coming from the Schwarzschild’s solution of Einstein’s equations for pure gravity in the vacuum [31]. Clearly, one can think about uses of Kerr solutions, Weyl solutions, etc. as well for the same purposes. Such thinking is perfectly permissible but is not of much help for a particular case of our Solar System. In it, Kepler’s laws describe reality sufficiently well so that Einsteinian geodesics can be safely approximated by the Newtonian orbits. This conclusion is perfectly compatible with the original Einstein’s derivation of his equations for gravity. Thus, we are adopting his strategy in our paper. Clearly, once the results are obtained, they can be recalculated if needed. Fortunately, as far as we can see, there is no need for doing this as we shall demonstrate shortly below.

Following Pars [36], the motion of a point of unit mass in the field of Newtonian gravity is described by the following equation

$$\dot{r}^2 = \frac{(2Er^2 + 2\gamma Mr - \alpha^2)/r^2}{r^2}, \quad (4.4)$$

\textsuperscript{19}Not to be confused with the speed of light!
where $\alpha$ is the angular momentum integral (e.g. see equation (5.2.55) of Pars book). We would like now to replace $r(t)$ by $r(\theta)$ in such a way that $dt = u(r(\theta))d\theta$. Let therefore $r(\theta) = r_0 \exp(x(\theta))$, $-\infty < x < \infty$. Unless otherwise specified, we shall write $r_0 = 1$. In such (astronomical) system of units we obtain, $\dot{r} = x' \frac{d\theta}{dt} \exp(x(\theta))$. This result can be further simplified by choosing $\frac{d\theta}{dt} = \exp(-x(\theta))$. With this choice (4.4) acquires the following form:

$$(x')^2 = 2E + 2\gamma M \exp(-x) - \alpha^2 \exp(-2x). \quad (4.5)$$

Consider points of equilibria for the potential $U(r) = -2\gamma Mr^{-1} + \alpha^2 r^{-2}$. Using it, we obtain: $r^* = \frac{\alpha^2}{\gamma M}$. According to Goldstein et al [20] such defined $r^*$ coincides with the major elliptic semiaxis. It can be also shown, e.g. Pars, equation (5.4.14), that for the Kepler problem the following relation holds: $E = -\frac{\gamma M}{2r^*}$. Accordingly, $r^* = -\frac{\gamma M}{2E}$, and, furthermore, using the condition $\frac{dU}{dr} = 0$ we obtain: $\frac{\alpha^2}{\gamma M} = -\frac{\gamma M}{2E}$ or, $\alpha^2 = -\frac{(\gamma M)^2}{2E}$. Since in the chosen system of units $r(\theta) = \exp(x(\theta))$, we obtain as well: $\frac{\alpha^2}{\gamma M} = \exp(x^*(\theta))$. It is convenient to choose $x^*(\theta) = 0$. This requirement makes the point $x^*(\theta) = 0$ as the origin and implies that with respect to such chosen origin $\alpha^2 = \gamma M$. In doing so some caution should be exercised since upon quantization equation $r^* = \frac{\alpha^2}{\gamma M}$ becomes $r^*_n = \frac{\alpha^2_n}{\gamma M}$. By selecting the astronomical scale $r^*_3 = 1$ as the unit of length implies then that we can write the angular momentum $\alpha^2_n$ as $\propto \frac{r^*_n}{r^*_3}$ and to define $\zeta$ as $\alpha^2_3 \equiv \alpha^2$. Using this fact (4.5) can then be conveniently rewritten as

$$\frac{1}{2}(x')^2 - \gamma M(\exp(-x) - \frac{1}{2} \exp(-2x)) = E \quad (4.6a)$$

or, equivalently, as

$$\frac{p^2}{2} + A(\exp(-2x) - 2 \exp(-x)) = E, \quad (4.6b)$$

where $A = \frac{\gamma M}{2}$. Since this result is exact classical analog of the quantum Morse potential problem, transition to quantum mechanics can be done straightforwardly at this stage. By doing so we have to replace the Planck’s constant $\hbar$ by $c \ln \tilde{A}$. After that, we can write the answer for the spectrum at
once [59]:

\[ -\tilde{E}_n = \frac{\gamma M}{2} \left[ 1 - \frac{c \ln \tilde{A}}{\sqrt{\gamma M}} (n + \frac{1}{2}) \right]^2. \]  

(4.7)

This result contains an unknown parameter \( c \) to be determined now. To do so it is sufficient to expand the potential in (4.6b) and to keep terms up to quadratic. Such a procedure produces the anticipated harmonic oscillator result

\[ \frac{p^2}{2} + A x^2 = \tilde{E} \]  

(4.8)

with the quantum spectrum given by \( \tilde{E}_n = (n + \frac{1}{2}) c \sqrt{2A} \ln \tilde{A} \). In the astronomical system of units the spectrum reads: \( \tilde{E}_n = (n + \frac{1}{2}) c 2\pi \ln \tilde{A} \). This result is in agreement with (4.3). To proceed, we notice that in (4.3) the actual sign of the Planck-type constant is undetermined. Specifically, in our case (up to a constant) the energy \( \tilde{E}_n \) is determined by \( \ln \left( \frac{1}{T_n} \right) = -\ln \tilde{A} \) so that it makes sense to write \( -\tilde{E}_n \sim n \ln \tilde{A} \). To relate the classical energy defined by the Kepler-type equation \( E = -\frac{\gamma M}{2r_*} \) to the energy we just have defined, we have to replace the Kepler-type equation by \( -\tilde{E}_n \equiv -\ln |E| = -2 \ln \sqrt{2\pi} + \ln r_n \). This is done in view of the 3rd Kepler’s law and the fact that the new coordinate \( x \) is related to the old coordinate \( r \) via \( r = e^x \). Using (4.2) (for \( n = 1 \)) in the previous equation and comparing it with the already obtained spectrum of the harmonic oscillator we obtain:

\[ -2 \ln \sqrt{2\pi} + \ln r_0 B = -c 2\pi \ln \tilde{A}, \]  

(4.9)

where in arriving at this result we had subtracted the nonphysical ground state energy. Thus, we obtain:

\[ c = \frac{1}{2\pi \ln \tilde{A}} \ln \frac{2\pi^2}{r_0 B}. \]  

(4.10)

Substitution of this result back into (4.7) produces

\[ -\tilde{E}_n = 2\pi^2 [1 - \left( n + \frac{1}{2} \right) \ln \left( \frac{2\pi^2}{r_0 B} \right)]^2 \simeq 2\pi^2 [1 - \frac{1}{9.87} (n + \frac{1}{2})]^2 \]

\[ \simeq 2\pi^2 - 4(n + \frac{1}{2}) + 0.2(n + \frac{1}{2})^2. \]  

(4.11)

To determine the number of bound states, we follow the same procedure as was developed long ago in chemistry for the Morse potential. For this purpose we introduce the energy difference \( \Delta \tilde{E}_n = \tilde{E}_{n+1} - \tilde{E}_n = 4 - 0.4(n + 1) \)

\[ ^{20} \text{Recall, that in chemistry the Morse potential is being routinely used for description of the vibrational spectra of diatomic molecules.} \]
first. Next, the maximum number of bound states is determined by requiring \( \Delta \tilde{E}_n = 0 \). In our case, we obtain: \( n_{\text{max}} = 9 \). This number is in perfect accord with observable data for planets of our Solar System (with Pluto being excluded and the asteroid belt included). In spite of such a good accord, some caution must be still exercised while analyzing the obtained result. Should we not insist on physical grounds that the discrete spectrum must contain only finite number of levels, the obtained spectrum for the harmonic oscillator would be sufficient (that is to say, that the validity of the T-B law would be confirmed). Formally, it also solves the quantization problem completely and even is in accord with the numerical data [55]. The problem lies however in the fact that these data were fitted to the power law (4.2) in accord with the original T-B empirical guess. Heisenberg's honeycomb rule (2.7b) does not rely on specific \( n \)-dependence. In fact, we have to consider the observed (the Titius-Bode-type) \( n \)-dependence only as a hint, especially because in this work we intentionally avoid use of any adjustable parameters. The developed procedure, when supplied with correctly interpreted numerical data, is sufficient for obtaining results without any adjustable parameters as we just demonstrated. In turn, this allows to replace the T-B law in which the power \( n \) is unrestricted by more accurate result working especially well for larger values of \( n \). For instance, the constant \( c \) was determined using the harmonic approximation for the Morse-type potential. This approximation is expected to fail very quickly as the following arguments indicate. Although \( r'_n s \) can calculated using the T-B law given by (4.2), the arguments following this equation cause us to look also at the equation \( -\tilde{E}_n \equiv -\ln |E| = -2 \ln \sqrt{2\pi} + \ln r_n \) for this purpose. This means that we have to use (4.11) (with ground state energy subtracted) in this equation in order to obtain the result for \( r_n \). If we ignore the quadratic correction in (4.11) (which is equivalent of calculating the constant \( c \) using harmonic oscillator approximation to the Morse potential) then, by construction, we recover the T-B result (4.2). If, however, we do not resort to such an approximation, calculations will become much more elaborate. The final result will indeed replace the T-B law but its analytical form is going to be too cumbersome for practitioners. Since corrections to the harmonic oscillator potential in the case of the Morse potential are typically small, they do not change things qualitatively. Hence, we do not account for these complications in our paper. Nevertheless, accounting for these (anharmonic) corrections readily explains why the empirical T-B law works well for small \( n \)'s and becomes increasingly unreliable for larger \( n \)'s [54].

In support of our way of doing quantum calculations, we would like to discuss now similar calculations for satellite systems of Jupiter, Saturn, Uranus and Neptune. To do such calculations the astronomical system of units is
not immediately useful since in the case of heavy planets one cannot use the relation $4\pi^2/\gamma M_\odot = 1$. This is so because we have to replace the mass of the Sun $M_\odot$ by the mass of respective heavy planet. For this purpose we write $4\pi^2 = \gamma M_\odot$, multiply both sides by $M_j$ (where $j$ stands for the $j$-th heavy planet) and divide both sides by $M_\odot$. Thus, we obtain: $4\pi^2 q_j = \gamma M_j$, where $q_j = \frac{M_j}{M_\odot}$. Since the number $q_j$ is of order $10^{-3} - 10^{-5}$, it causes some inconveniences in actual calculations. To avoid this difficulty, we need to readjust (4.6a) by rescaling $x$ coordinate as $x = \delta \tilde{x}$ and, by choosing $\delta^2 = q_j$. After transition to quantum mechanics such a rescaling results in replacing (4.7) for the spectrum by the following result:

$$-\tilde{E}_n = \frac{\gamma M_j}{2} [1 - \frac{c\delta \ln \tilde{A}_j}{\sqrt{\gamma M_j}}(n + \frac{1}{2})]^2. \quad (4.12)$$

Since the constant $c$ is initially undetermined, we can replace it by $\tilde{c} = c\delta$. This replacement allows us to reobtain back equation almost identical to (4.11). That is

$$-\tilde{E}_n = 2\pi^2 [1 - \frac{(n + \frac{1}{2})}{4\pi^2} \ln \left(\frac{\gamma M_j}{(r_j)_1}\right)]^2 \quad (4.13)$$

In this equation $\gamma M_j = 4\pi^2 q_j$ and $(r_j)_1$ is the semimajor axis of the satellite lying in the equatorial plane and closest to the $j$-th planet. Our calculations are summarized in the Table 2 below. Appendix B contains the input data used in calculations of $n_{\text{theory}}^*$. Observational data are taken from the web link given in the 1st footnote.

<table>
<thead>
<tr>
<th>Satellite system</th>
<th>$n_{\text{max}}$</th>
<th>$n_{\text{theory}}^*$</th>
<th>$n_{\text{obs}}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solar system</td>
<td>9</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>Jupiter system</td>
<td>11-12</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Saturn system</td>
<td>20</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Uranus system</td>
<td>40</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>Neptune system</td>
<td>33</td>
<td>6-7</td>
<td></td>
</tr>
</tbody>
</table>

Since the discrepancies for Uranus and Neptune systems may be genuine or not we come up with the following general filling pattern which is being compared with that discussed in the Introduction.

### 4.2 Filling patterns in Solar System: similarities and differences with atomic mechanics

From atomic mechanics we know that the approximation of independent electrons used by Bohr fails rather quickly with increased number of electrons.
Already for this reason to expect that the T-B law is going to hold for satellites of heavy planets is naive. At the same time, for planets rotating around the Sun such an approximation is seemingly good but also not without flaws. The SO(2,1) symmetry explains why motion of all planets should be planar but it does not explain why motion of all planets is taking place in the plane coinciding with the equatorial plane of the Sun or why all planets are moving in the same direction. The same is true for the regular satellites of all heavy planets as discussed by Dermott [57]. Such a configuration can be explained by a plausible hypothesis [7] that all planets of Solar System and regular satellites of heavy planets are originated from evolution of the pancake-like cloud. This assumption is not without problems though. For instance, all irregular satellites and Saturn’s Phoebe ring are rotating in the ”wrong” direction. Under conditions of such a hypothesis all these objects were randomly captured by the already existing Solar System at later times. The exoplanets rotating in the wrong directions\(^{21}\) apparently also had been captured so that the origins of some planetary systems are quite different from that for ours if we believe that ours originated from the pancake-like cloud. This hypothesis would make sense should irregular satellites be arranged around respective planets at random. But they are not! This is discussed in the Introduction.

In view of these facts, in this work we tend to provide a quantum mechanical explanation of the observed filling patterns summarized in Table 1. This Table requires some extension, for instance, to account for the fact that all planets and regular satellites are moving in the respective equatorial planes. This fact can be accounted for by the effects of spin-orbital interactions. Surprisingly, these effects exist both at the classical, newtonian, level [11] and at the level of general relativity [60]. At the classical level the most famous example of spin-orbital resonance (but of a different kind) is exhibited by the motion of the Moon whose orbital period coincides with its rotational period so that it always keeps only one face towards the Earth. Most of the major natural satellites are locked in analogous 1:1 spin-orbit resonance with respect to the planets around which they rotate. Mercury represents an exception since it is locked into 3:2 resonance around the Sun (that is Mercury completes 3/2 rotation around its axis while making one full rotation along its orbit). Goldreich [52] explains such resonances as results of influence of dissipative (tidal) processes on evolutionary dynamics of Solar System. The resonance structures observed in the sky are stable equilibria in the appropriately chosen reference frames [11]. Clearly, the spin-orbital resonances just described are not explaining many things. For instance, while nicely explaining why our Moon is always facing us with the same side the same pattern is not observed for Earth

\(^{21}\)E.g. read the Introduction
rotating around the Sun with exception of Mercury. Mercury is treated as pointlike object in general relativity. Decisive attempt to describe the motion of extended objects in general relativity was made in seminal paper by Papa-petrou [61] and continues up to the present day. From his papers it is known that, strictly speaking, motion of the extended bodies is not taking place on geodesics. And yet, for the Mercury such an approximation made originally by Einstein works extremely nicely. The spin-orbital interaction [60] causing planets and satellites of heavy planets to lie in the equatorial plane is different from that causing 1:1, etc. resonances. It is analogous to the NMR-type resonances in atoms and molecules where in the simplest case we are dealing, say, with the Hydrogen atom. In it, the proton having spin 1/2 is affected by the magnetic field created by the orbital s-electron. In the atomic case due to symmetry of electron s-orbital this effect is negligible but nonzero! This effect is known in chemical literature as ”chemical shift”. In celestial case the situation is similar but the effect is expected to be much stronger since the orbit is planar (not spherical as for the s-electron in hydrogen atom, e.g. see Subsection 3.3.2). Hence, the equatorial location of planetary orbits and regular satellites is likely the result of such spin-orbital interaction. The equatorial plane in which planets (satellites) move can be considered as some kind of an orbital (in terminology of atomic physics). It is being filled in accordance with the equivalent of the Pauli principle: each orbit can be occupied by no more than one planet. Once the orbital is filled, other orbitals (planes) will begin to be filled out. Incidentally, such a requirement automatically excludes Pluto from the status of a planet. Indeed, on one hand, the T-B-type law, can easily accommodate Pluto, on another, not only this would contradict the data summarized in Table 1 and the results of previous subsection but also, and more importantly, it would be in contradiction with the astronomical data for Pluto. According to these data the orbit inclination for Pluto is 17° as compared to the rest of planets whose inclination is within boundary margins of ±2° (except for Mercury for which it is 7°). Some of the orbitals can be empty and not all orbits belonging to the same orbital (a plane) must be filled (as it is also the case in atomic physics). This is indeed observed in the sky[10] and is consistent with results of Table 2. It should be said though that it appears (according to available data, that not all of the observed satellites are moving on stable orbits. It appears also as if and when the ”inner shell” is completely filled, it acts as some kind of an s-type spherical orbital since the orbits of other (irregular) satellites lie strictly outside the sphere whose di-

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22The meteorite belt can be looked upon as some kind of a ring. We shall discuss the rings below, in the next subsection.

23E.g. see footnote 1 and [10].
ameter is greater or equal to that corresponding to the last allowed energy level in the first shell. In accord with results of previous subsection, the location of secondary planes appears to be quite arbitrary as well as filling of their stable orbits. Furthermore, without account of spin-orbital interactions, one can say nothing about the direction of orbital rotation. Evidently, the "chemical shift" created by the orbits of regular satellites lying in the s-shell is such that it should be more energetically advantageous to rotate in the opposite direction. This proposition requires further study. In addition to planets and satellites on stable orbits there are many strangers in the Solar System: comets, meteorites, etc. These are moving not on stable orbits and, as result, should either leave the Solar System or eventually collide with those which move on "legitimate" orbits.

It is tempting to extend the picture just sketched beyond the scope of our Solar System. If for a moment we would ignore relativistic effects (they will be discussed in the next section), we can then find out that our Sun is moving along almost circular orbit around our galaxy center with the period \( T = 185 \cdot 10^6 \) years [62]. Our galaxy is also flat as our Solar System and the major mass is concentrated in the galaxy center. Hence, again, if we believe that stable stellar motion is taking place along the geodesics around the galaxy center in accord with laws of Einstein’s general relativity, then we have to accept that our galaxy is also a quantum object. It would be very interesting to estimate the number of allowed energy levels (stable orbits) for our galaxy and to check if the Pauli-like principle works for our and other galaxies as well.

### 4.3 The restricted 3-body problem and planetary rings

Although the literature on the restricted 3-body problem is huge, we would like to discuss this problem from the point of view of its connection with general relativity and quantization of planetary orbits along the lines advocated in this paper. We begin with several remarks. First, the existence of ring systems for all heavy planets is well documented [10]. Second, these ring systems are interspersed with satellites of these planets. Third, both rings and satellites lie in the respective equatorial planes (with exception of Phoebe’s ring) so that satellites move on stable orbits. From these observations it follows that:

a) While each of heavy planets is moving along the geodesics around the Sun, the respective satellites are moving along the geodesics around respective planets;

b) The motion of these satellites is almost circular (the condition which Laplace took into account while studying Jupiter’s regular satellites).
The restricted 3-body problem can be formulated now as follows.

Given that the rings are made of some kind of small objects whose masses can be neglected\(^{24}\) as compared to masses of both satellite(s) and the respective heavy planet, we can ignore mutual gravitational interaction between these objects (as Laplace did). Under such conditions we end up with the motion of a given piece of a ring (of zero mass) in the presence of two bodies of masses \(m_1\) and \(m_2\) respectively (the planet and one of the regular satellites). To simplify matters, it is usually being assumed that the motion of these two masses takes place on a circular orbit with respect to their center of mass. Complications associated with the eccentricity of such a motion are discussed in the book by Szebehely [63] and can be taken into account if needed. They will be ignored nevertheless in our discussion since we shall assume that satellites of heavy planets move on geodesics so that the center of mass coincides with the position of a heavy planet anyway thus making our computational scheme compatible with Einsteinian relativity. By assuming that ring pieces are massless we also are making their motion compatible with requirements of general relativity, since whatever orbits they may have-these are geodesics anyway.

Thus far only the motion of regular satellites in the equatorial planes (of respective planets) was considered as stable (and, hence, quantizable). The motion of ring pieces was not accounted by these stable orbits. The task now lies in showing that satellites lying inside the respective rings of heavy planets are essential for stability of these rings motion thus making it quantizable. For the sake of space, we would like only to provide a sketch of arguments leading to such a conclusion. Our task is greatly simplified by the fact that very similar situation exists for 3-body system such as Moon, Earth and Sun. Dynamics of such a system was studied thoroughly by Hill whose work played pivotal role in Poincare' studies of celestial mechanics [6]. Avron and Simon [64] adopted Hill’s ideas in order to develop formal quantum mechanical treatment of the Saturn rings. In this work we follow instead the original Hill’s ideas of dynamics of the Earth-Moon-Sun system. When these ideas are looked upon from the point of view of modern mathematics of exactly integrable systems, they enable us to describe not only the Earth-Moon-Sun system but also the dynamics of rings of heavy planets. These modern mathematical methods allow us to find a place for the Hill’s theory within general quantization scheme discussed in previous sections.

4.3.1 Basics of the Hill’s equation

\(^{24}\)This approximation is known as Hill’s problem/approximation in the restricted 3-body problem [25, 36].
To avoid repetitions, we refer our readers to the books of Pars [36] and Cheb-otarev [62] for detailed and clear account of the restricted 3-body problem and Hill’s contributions to Lunar theory. Here we only summarize the ideas behind Hill’s ground breaking work.

In a nutshell his method of studying the Lunar problem can be considered as extremely sophisticated improvement of previously mentioned Laplace method. Unlike Laplace, Hill realized that both Sun and Earth are surrounded by the rings of influence\footnote{Related to the so called Roche limit [10,36].}. The same goes for all heavy planets. Each of these planets and each satellite of such a planet will have its own domain of influence whose actual width is controlled by the Jacobi integral of motion. For the sake of argument, consider the Saturn as an example. It has Pan as its the innermost satellite. Both the Saturn and Pan have their respective domains of influence. Naturally, we have to look first at the domain of influence for the Saturn. Within such a domain let us consider a hypothetical closed Kepler-like trajectory. Stability of such a trajectory is described by the Hill equation\footnote{In fact, there will be the system of Hill’s equations in general [63]. This is so since the disturbance of trajectory is normally decomposed into that which is perpendicular and that which is parallel to the Kepler’s trajectory at a given point. We shall avoid these complications in our work.}.

Since such an equation describes a wavy-type oscillations around the presumably stable trajectory, the parameters describing such a trajectory are used as an input (perhaps, with subsequent adjustment) in the Hill equation given by

\[
\frac{d^2x}{dt^2} + (q_0 + 2q_1 \cos 2t + 2q_2 \cos 4t + \cdots)x = 0. \tag{4.14}
\]

If we would ignore all terms except \(q_0\) first, we would naively obtain: \(x_0(t) = A_0 \cos(t\sqrt{q_0} + \varepsilon)\). This result describes oscillations around the equilibrium position along the trajectory with the constant \(q_0\) carrying information about this trajectory. The amplitude \(A\) is expected to be larger or equal to the average distance between the pieces of the ring. This naive picture gets very complicated at once should we use the obtained result as an input into (4.14). In this case the following equation is obtained

\[
\frac{d^2x}{dt^2} + q_0x + A_0q_1 \{\cos[t(\sqrt{q_0} + 2) + \varepsilon] + \cos[t(\sqrt{q_0} - 2) - \varepsilon]\} = 0 \tag{4.15}
\]

whose solution will enable us to determine \(q_1\) and \(A_1\) using the appropriate boundary conditions. Unfortunately, since such a procedure should be repeated infinitely many times, it is obviously impractical. Hill was able to design a much better method. Before discussing Hill’s equation from the perspective of modern mathematics, it is useful to recall the very basic classical facts.
about this equation summarized in the book by Ince [65]. For this purpose, we shall assume that the solution of (4.14) can be presented in the form

$$x(t) = e^{\alpha t} \sum_{r=-\infty}^{\infty} b_r e^{irt}.$$ \hspace{1cm} (4.16)

Substitution of this result into (4.14) leads to the following infinite system of linear equations

$$(\alpha + 2ir)^2 b_r + \sum_{k=-\infty}^{\infty} q_k b_{r-k} = 0, \; r \in \mathbb{Z}.$$ \hspace{1cm} (4.17)

As in finite case, obtaining of the nontrivial solution requires the infinite determinant $\Delta(\alpha)$ to be equal to zero. This problem can be looked upon from two directions: either all constants $q_k$ are assigned and one is interested in the bounded solution of (4.16) for $t \to \infty$ or, one is interested in the relationship between constants made in such a way that $\alpha = 0$. In the last case it is important to know whether there is one or more than one of such solutions available. Although answers can be found in the book by Magnus and Winkler [66], we follow McKean and Moerbeke [67], Trubowitz [68] and Moser [69].

For this purpose, we need to bring our notations in accord with those used in these references. Thus, the Hill operator is defined now as $Q(q) = -\frac{d^2}{dt^2} + q(t)$ with periodic potential $q(t) = q(t+1)$. Equation (4.14) can now be rewritten as

$$Q(q)x = \lambda x.$$ \hspace{1cm} (4.18)

This representation makes sense since $q_0$ in (4.14) plays the role of $\lambda$ in (4.18). Since this is the second order differential equation, it has formally 2 solutions. These solutions depend upon the boundary conditions. For instance, for periodic solutions such that $x(t) = x(t+2)$ the "spectrum" of (4.18) is discrete and is given by

$$-\infty < \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots \uparrow +\infty.$$

We wrote the word spectrum in quotation marks because of the following. Equation (4.18) does have a normalizable solution only if $\lambda$ belongs to the (preassigned) intervals $(\lambda_0, \lambda_1), (\lambda_2, \lambda_3), \ldots, (\lambda_{2i}, \lambda_{2i+1}), \ldots$. In such a case the eigenfunctions $x_i$ are normalizable in the usual sense of quantum mechanics and form the orthogonal set. Periodic solutions make sense only for vertical displacement from the reference trajectory. For the horizontal displacement the boundary condition should be chosen as $x(0) = x(1) = 0$. For such chosen boundary condition the discrete spectrum also exists but it lies exactly in the gaps between
the intervals just described, i.e. \( \lambda_1 \leq \mu_1 \leq \lambda_2 < \lambda_3 \leq \mu_2 \leq \lambda_4 < \cdots \). For such a spectrum there is also set of normalized mutually orthogonal eigenfunctions. Thus in both cases quantum mechanical description is assured. One can do much more however. In particular, Trubowitz [69] designed an explicit procedure for recovering the potential \( q(t) \) from the \( \mu \)-spectrum supplemented by information about normalization constants.

It is quite remarkable that the Hill’s equation can be interpreted in terms of the auxiliary dynamical (Neumann) problem. Such an interpretation is very helpful for us since it allows us to include the quantum mechanics of Hill’s equation into general formalism developed in this work.

### 4.3.2 Connection with the dynamical Neumann problem and the Korteweg -de Vries equation

Before describing such connections, we would like to add few details to the results of previous subsection. First, as in the planetary case, the number of pre-assigned intervals is always finite. This means that, beginning with some pre-assigned \( \hat{i} \), we would be left with \( \lambda_{2i} = \lambda_{2i+1} \forall i > \hat{i} \). These double eigenvalues do not have independent physical significance since they can be determined by the set of single eigenvalues (for which \( \lambda_{2i} \neq \lambda_{2i+1} \)) as demonstrated by Hochstadt [70]. Because of this, potentials \( q(t) \) in the Hill’s equation are called the finite gap potentials\(^{27}\). Hence, physically, it is sufficient to discuss only potentials which possess finite single spectrum. The auxiliary \( \mu \)-spectrum is then determined by the gaps of the single spectrum as explained above. With this information in our hands, we are ready to discuss the exactly solvable Neumann dynamical problem. It is the problem about dynamics of a particle moving on \( n \)-dimensional sphere \( <\xi, \xi> = \xi_1^2 + \cdots + \xi_n^2 = 1 \) under the influence of a quadratic potential \( \phi(\xi) = <\xi, A\xi> \). Equations of motion describing the motion on \( n \)-sphere are given by

\[
\ddot{\xi} = -A\xi + u(\xi)\xi \quad \text{with} \quad u(\xi) = \phi(\xi) - <\dot{\xi}, \dot{\xi}> .
\]

Without loss of generality, we assume that the matrix \( A \) is already in the diagonal form: \( A := diag(\alpha_1, ..., \alpha_n) \). With such an assumption we can equivalently rewrite Eq.(4.19) in the following suggestive form

\[
\left( -\frac{d^2}{dt^2} + u(\xi(t)) \right) \xi_k = \alpha_k \xi_k ; \quad k = 1, ..., n .
\]

Thus, in the case if we can prove that \( u(\xi(t)) \) in (4.19) is the same as \( q(t) \) in (4.18), the connection between the Hill and Neumann’s problems will be

\(^{27}\)Since there is only finite number of gaps \([\lambda_1, \lambda_2], [\lambda_3, \lambda_4], ...\) where the spectrum is forbidden.
established. The proof is presented in Appendix C. It is different from that
given in the lectures by Moser [69] since it is more direct and much shorter.

This proof brought us the unexpected connection with hydrodynamics
through the static version of Korteweg-de Vries equation. Attempts to de-
scribe the Saturnian rings using equations of hydrodynamic are described in
the recent monograph by Esposito [71]. This time, however, we can accom-
plish more using just obtained information. This is the subject of the next
subsection.

4.3.3 Connections with SO(2,1) group and the K-Z equations

Following Kirillov [72], we introduce the commutator for the fields (operators)
\(\xi\) and \(\eta\) as follows: \([\xi, \eta] = \xi \partial \eta - \eta \partial \xi\). Using the KdV equation (C.10), let us
consider 3 of its independent solutions: \(\xi_0, \xi_{-1}\) and \(\xi_1\). All these solutions can
be obtained from general result: \(\xi_k = t^{k+1} + O(t^2)\), valid near zero. Consider
now a commutator \([\xi_0, \xi_1]\). Straightforwardly, we obtain, \([\xi_0, \xi_1] = \xi_1\). Analo-
gously, we obtain, \([\xi_0, \xi_{-1}] = -\xi_{-1}\) and, finally, \([\xi_1, \xi_{-1}] = -2\xi_0\). According to
Kirillov, such a Lie algebra is isomorphic to that for the group \(SL(2, R)\) which
is the center for the Virasoro algebra\(^{28}\). Vilenkin [74] demonstrated that the
group \(SL(2, R)\) is isomorphic to \(SU(1, 1)\). Indeed, by means of transformation:
\(w = \frac{z - i}{z + i}\), it is possible to transform the upper half plane (on which \(SL(2, R)\)
acts) into the interior of unit circle on which \(SU(1, 1)\) acts. Since, according
to Appendix A, the group \(SU(1, 1)\) is the connected component of \(SO(2, 1)\),
the anticipated connection with \(SO(2, 1)\) group is established.

In Appendix C we noticed connections between the Picard-Fuchs, Hill and
Neumann-type equations. In a recent paper by Veselov et al [75] such a con-
nection was developed much further resulting in the Knizhnik-Zamolodchikov-
type equations for the Neumann-type dynamical systems. We refer our readers
to original literature, especially to the well written lecture notes by Moser [69].
These notes as well and his notes in collaboration with Zehnder [30] provide
an excellent background for study the whole circle of ideas ranging from Hill’s
equation and integrable models to string theory, etc.

\(^{28}\)Since connections between the KdV and the Virasoro algebra are well documented [73],
it is possible in principle to reinterpret fine structure of the Saturn’s rings string-theoretically.
5 Solar System at larger scales: de Sitter, anti-de Sitter and conformal symmetries compatible with orbital quantization

Results obtained in previous section demonstrate remarkable interplay between the Newtonian and Einsteinian mechanics at the scale of our Solar System. Quantization of stable (Laplace-Einstein) orbits makes sense only with account of observational/empirical facts unequivocally supporting general relativity. It is only natural to reverse this statement and to say that the observed filling patterns of stable (quantum) orbits is yet another manifestation of general relativity.

Since quantum mechanics can be developed group-theoretically, the same should be true for relativity. Quoting Einstein, Infel’d and Hoffmann [76]: ”Actually, the only equations of gravitation which follow without ambiguity from the fundamental assumptions of the general theory of relativity are the equations for empty space, and it is important to know whether they alone are capable of determining the motion of bodies”. The results of this work strongly support such a conclusion. Given this, we would like to discuss how such locally Lorentzian space-time embeds into space-times of general relativity possessing larger symmetry groups. Since this topic is extremely large, we shall discuss only the most basic facts from the point of view of results obtained in this paper.

To our knowledge, Dirac [77] was the first who recognized the role of space-time symmetry in quantum mechanics. In his paper he wrote: ”The equations of atomic physics are usually formulated in terms of space-time of special relativity. They then have to form a scheme which remains invariant under all transformations which carry the space-time over into itself. These transformations consist of the Lorentz rotations about a point combined with arbitrary translations, and form a group.... Nearly all of more general spaces have only trivial groups\(^{29}\) of operations which carry the spaces into themselves....There is one exception, however, namely the de Sitter space (with no local gravitational fields). This space is associated with a very interesting group, and so the study of the equations of atomic physics in this space is of special interest, from mathematical point of view.” Subsequent studies indicated that the symmetry of space-time could be important even at the atomic scale [78,79]. This fact suggests that quantum mechanics of Solar System can be potentially useful for studies in cosmology, e.g. for studies of the cosmological constant problem

\(^{29}\)This statement of Dirac is not correct. However, it was correct based on mathematical knowledge at the time of writing of his paper.
A. L. Kholodenko

[78,80], of cold dark energy (CDE) [81], of cold dark matter (CDM) [82] and of the modified Newtonian dynamics (MOND) [83]. Clearly, we are unable to discuss these issues within the scope of this paper. Nevertheless, we would like to notice that, for instance, the MOND presupposes use of Newtonian and the modified Newtonian mechanics at the galactic scales which, strictly speaking, is not permissible. As we had argued, it is not permissible even at the scales of our Solar System. Mathematical rationale behind what is called in literature as ”dark energy and dark matter” is explained in our recent paper [84]. In it we discussed some physical applications of mathematical results by Grisha Perelman used in his proof of the Poincare' and geometrization conjectures.

As by-product of the results discussed in [84], we would like to discuss briefly a simple construction of the de Sitter and anti-de Sitter spaces. We begin with the Hilbert-Einstein functional

\[ S^c(g) = \int_M d^d x \sqrt{|g|} R + \Lambda \int_M d^d x \sqrt{|g|} \]  \hspace{1cm} (5.1)

defined for some (pseudo) Riemannian manifold \( M \) of total space-time dimension \( d \). The (cosmological) constant \( \Lambda \) is just the Lagrangian multiplier assuring volume conservation. It is determined as follows\(^{30}\). On one hand, with help of the Ricci curvature tensor \( R_{ij} \), the Einstein space is defined as solution of the equation

\[ R_{ij} = \lambda g_{ij} \]  \hspace{1cm} (5.2)

with \( \lambda \) being a constant. From this definition it follows that \( R = d \lambda \). On another hand, variation of the action \( S^c(g) \) produces

\[ G_{ij} + \frac{1}{2} \Lambda g_{ij} = 0, \]  \hspace{1cm} (5.3)

where the Einstein tensor \( G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R \) with \( R \) being the scalar curvature determined by the metric tensor \( g_{ij} \)\(^{31}\). The combined use of (5.2) and (5.3) produces: \( \Lambda = \lambda (d - 2) \). Substitution of this result back into (5.3) produces:

\[ G^i_j = (\frac{1}{d} - \frac{1}{2}) \delta^i_j R. \]  \hspace{1cm} (5.4)

\(^{30}\)It should be noted though that mathematicians study related but not identical problem of minimization of the Yamabe functional, given by \( Y(g) = (\int_M d^d x R \sqrt{|g|}) / (\int_M d^d x \sqrt{|g|})^2 \) with \( p = 2d/(2 - d) \), e.g. see our papers [85]. It is conformal invariant -different for different manifolds. Only at the mean field level results of minimization of \( S^c(g) \) coincide with those obtainable by minimization of \( Y(g) \). In this work this approximation is sufficient.

\(^{31}\)Eq.(5.3) illustrates the meaning of the term ”dark matter”. The constant \( \Lambda \) enters into the stress-energy tensor (typically associated with matter). In the present case it is given by \( - \frac{1}{2} \Lambda g_{ij} \).
Since by design $G_j,i = 0$, we obtain our major result:

$$
\left( \frac{1}{d} - \frac{1}{2} \right) R,j = 0,
$$

(5.5)

implying that the scalar curvature $R$ for Einsteinian spaces is a constant.

For isotropic homogenous spaces the Riemann curvature tensor can be presented in the form [86]:

$$
R_{ijkl} = k(x) (g_{ik}g_{jl} - g_{il}g_{jk}).
$$

(5.6)

Accordingly, the Ricci tensor is obtained as: $R_{ij} = k(x)g_{ij}(d - 1)$. Schur’s theorem [86] guarantees that for $d \geq 3$ we must have $k(x) = k = \text{const}$ for the entire space. Therefore, we obtain: $\lambda = (d - 1)k$ and, furthermore, $R = d(d - 1)k$. The spatial coordinates can always be rescaled so that $R = k$ or, alternatively, the constant $k$ can be normalized to unity. For $k > 0$, $k = 0$ and $k < 0$ we obtain respectively de Sitter, flat and anti-de Sitter spaces. Thus, we just demonstrated that the homogeneity and isotropy of space-time is synonymous with spaces being de Sitter, flat and anti-de Sitter very much like in Riemannian geometry there are spaces of positive, negative and zero curvature. This observation can be used for obtaining simple description of just obtained results.

We begin by noticing that the surface of constant positive curvature is conformally equivalent to a sphere embedded into flat Euclidean space [77,84]. In particular, let us consider a 3-sphere embedded into 4d Euclidean space. It is described by the equation

$$
S^3 = \{ x \in E_4, \; x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2 \}.
$$

(5.7)

$S^3$ is homogenous isotropic space with positive scalar curvature whose value is $6/R^2$. The group of motions associated with this homogenous space is the rotation group $SO(4)$. The space of constant negative curvature $H^3$ is obtained analogously. For this purpose it is sufficient, following Dirac [77], to make $x_1$ purely imaginary and to replace $R^2$ by $-R^2$ in (5.7). Such replacements produce:

$$
H^3 = \{ x \in M_4, \; x_1^2 - x_2^2 - x_3^2 - x_4^2 = R^2 \}.
$$

(5.8)

In writing this result we have replaced the Euclidean space $E_4$ by Minkowski space $M_4$ so that the rotation group $SO(4)$ is now replaced by the Lorentz group $SO(3,1)$. The de Sitter space can now be obtained according to Dirac as follows. In (5.7) we replace $E_4$ by $E_5$ and make $x_1$ purely imaginary thus
converting $E_5$ into $M_5$. The obtained space is the de Sitter space whose group of symmetry is $SO(4,1)$

$$dS_4 = \{ x \in M_5, \ x^2_1 - x^2_2 - x^2_3 - x^2_4 - x^2_5 = R^2 \}.$$ \hspace{1cm} (5.9)

It has a constant positive scalar curvature whose value is $12/R^2$. Very nice description of such a space is contained in the book by Hawking and Ellis [87]. The connection between parameter $R$ and the cosmological constant $\Lambda$ is given by $R = \sqrt{\frac{3}{\Lambda}}$. The anti-de Sitter space of constant negative curvature is determined analogously. Specifically, it is given by

$$adS_4 = \{ x \in E_{3,2}, \ x^2_1 - x^2_2 - x^2_3 - x^2_4 + x^2_5 = R^2 \}, \hspace{1cm} (5.10)$$

where the five dimensional space $E_{3,2}$ is constructed by adding the time-like direction to $M_4$. Hence, the symmetry group of $adS_4$ is $SO(3,2)$. All these groups can be described simultaneously if, following Dirac [77], we introduce the quadratic form

$$\sum_{\mu=1}^{5} x_{\mu}x_{\mu} = R^2$$ \hspace{1cm} (5.11)

in which some of the arguments are allowed to be purely imaginary. Transformations preserving such a quadratic form are appropriate respectively for groups $SO(5)$, $SO(4,1)$ and $SO(3,2)$. We can embed all these groups into still a larger (conformal) group $SO(4,2)$ by increasing summation from 5 to 6 in (5.11). In such a case all groups discussed in this work, starting from $SO(2,1)$, can be embedded into this conformal group as subgroups as discussed in great detail by Wybourne [42]. Incidentally, the work by Graner and Dubrulle entitled “Titius-Bode laws in the solar system I. Scale invariance explains everything” [88], when interpreted group-theoretically, becomes just a corollary of such an embedding. The Titius-Bode law which these authors reproduce by requiring the underlying system of equations to be conformally invariant contains no restrictions on number of allowed orbit discussed in Subsection 4.1.2. Furthermore, their work requires many ad hoc fitting assumptions. When using equations of fluid dynamics in their subsequent work [89] to model evolution of the protoplanetary cloud of dust, the obtained results contain only orbits of regular planets/satellites and, hence suffers from the same type of problems as mentioned in the Introduction. Uses of conformal symmetry in both gravity and conformal field theories has been recently extended in our works [84,85]. The task still remains to find out if representations of these larger groups can produce exact solutions of the radial Schrödinger equations not listed in the Natanzon-style classification given in Ref.[44] for $SO(2,1)$. If such solutions do exist, one might be able to find those of them which are of relevance to celestial quantum mechanics and, hence, to cosmology.
Concluding remarks

Although Einstein was not happy with the existing formulation of quantum mechanics, the results presented in this work demonstrate harmonious coexistence of general relativity and quantum mechanics to the extent that existence of one implies existence of the other at the scales of our Solar System. It should be noted though that such harmony had been achieved at the expense of partial sacrificing of the correspondence principle. This principle is not fully working anyway, even for such well studied system as Hydrogen atom as discussed in Subsection 3.3.2. This fact is not too worrisome to us as it was to Einstein. Indeed, as Heisenberg correctly pointed out: all what we know about microscopic system is its spectrum (in the very best of cases). The results of our recent works \cite{15,16} as well as by mathematicians Knutson and Tao [12-14] indicate that there are numerous ways to develop quantum mechanics-all based on systematically analyzing combinatorics of the observed spectral data. Such an approach is not intrinsic to quantum mechanics. In works by Knutson and Tao quantum mechanics was not discussed at all! Quantum mechanical significance of their work(s) is discussed in detail in our recent papers \cite{15} While in \cite{16} we developed quantum mechanical formalism based on the theory of Poisson-Dirichlet-type processes. These stochastic processes are not necessarily microscopic. Mathematically rigorous detailed exposition of these processes is given in [90]. The combinatorial formalism developed in [16] works equally well for quantum field and string theories. Not surprisingly such formalism can be successfully applied to objects as big as involved in the Solar System dynamics. Much more surprising is the unifying role of gravity at the microscopic scales as discussed in our latest work on gravity assisted solution of the mass gap problem\cite{32} for Yang-Mills fields \cite{91}. Role of gravity in solution of the mass gap problem for Yang-Mills fields is striking and unexpected. Although Einstein did not like quantum mechanics because, as he believed, it is incompatible with his general relativity, results of this work and those of reference \cite{91} underscore the profoundly deep connections between gravity and quantum mechanics/quantum field theory at all scales. It is being hoped that our work will stimulate development of more detailed expositions in the future, especially those involving detailed study of spin-orbital interactions.

Appendix A. Some quantum mechanical problems associated with the Lie algebra of SO(2,1) group

\footnote{This is one of the Millennium prize problems proposed by the Clay Mathematics Institute, e.g. see \url{http://www.claymath.org/millennium/Yang-Mills_Theory/}}
Following Wybourne [42] consider the second order differential equation of the type

\[ \frac{d^2 Y}{dx^2} + V(x)Y(x) = 0 \]  

(A.1)

where \( V(x) = a/x^2 + bx^2 + c \). Consider as well the Lie algebra of the noncompact group SO(2,1) or, better, its connected component SU(1,1). It is given by the following commutation relations

\[
[X_1, X_2] = -iX_3;\quad [X_2, X_3] = iX_1;\quad [X_3, X_1] = iX_2
\]  

(A.2)

We shall seek the realization of this Lie algebra in terms of the following generators

\[
X_1 := \frac{d^2}{dx^2} + a_1(x);\quad X_2 := i[k(x)\frac{d}{dx} + a_2(x)];\quad X_3 := \frac{d^2}{dx^2} + a_3(x).
\]  

(A.3)

The unknown functions \( a_1(x), a_2(x), a_3(x) \) and \( k(x) \) are determined upon substitution of (A.3) into (A.2). After some calculations, the following result is obtained

\[
X_1 := \frac{d^2}{dx^2} + a + \frac{x^2}{16};\quad X_2 := -i\left[ x\frac{d}{dx} + \frac{1}{2}\right];\quad X_3 := \frac{d^2}{dx^2} + a - \frac{x^2}{16}.
\]  

(A.4)

In view of this, (A.1) can be rewritten as follows

\[
\left[ (\frac{1}{2} + 8b)X_1 + (\frac{1}{2} - 8b)X_3 + c\right] Y(x) = 0.
\]  

(A.5)

This expression can be further simplified by the unitary transformation \( UX_1U^{-1} = X_1 \cosh \theta + X_3 \sinh \theta;\ UX_3U^{-1} = X_1 \sinh \theta + X_3 \cosh \theta \) with \( U = \exp(-i\theta X_2) \). By choosing \( \tanh \theta = -(1/2 + 8b)/(1/2 - 8b) \) (A.5) is reduced to

\[
X_3\tilde{Y}(x) = \frac{c}{4\sqrt{-b}} \tilde{Y}(x),
\]  

(A.6)

where the eigenfunction \( \tilde{Y}(x) = UY(x) \) is an eigenfunction of both \( X_3 \) and the Casimir operator \( X^2 = X_3^2 - X_2^2 - X_1^2 \) so that by analogy with the Lie algebra of the angular momentum we obtain

\[
X_3^2 \tilde{Y}_{jn}(x) = J(J + 1)\tilde{Y}_{jn}(x) \quad \text{and}
\]

\[
X_3 \tilde{Y}_{jn}(x) = \frac{c}{4\sqrt{-b}} \tilde{Y}_{jn}(x) \equiv (-J + n)\tilde{Y}_{jn}(x);\quad n = 0, 1, 2, \ldots
\]  

(A.7a, b)

It can be shown that \( J(J + 1) = -a/4 - 3/16 \). From here we obtain : \( J = -\frac{1}{2}(1 \pm \sqrt{\frac{1}{4} - a}); \frac{1}{4} - a \geq 0 \). In the case of discrete spectrum one should choose
the plus sign in the expression for $J$. Using this result in (A.7) we obtain the
following result of major importance

$$4n + 2 + \sqrt{1 - 4a} = \frac{c}{\sqrt{-b}}.$$  \hspace{1cm} (A.8)

Indeed, consider the planar Kepler problem. In this case, in view of (3.5), the
radial Schrödinger equation can be written in the following symbolic form

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{v}{r} + \frac{u}{r^2} + g \right] R(r) = 0$$ \hspace{1cm} (A.9)

By writing $r = x^2$ and $R(r) = x^{-\frac{1}{2}} R(x)$ this equation is reduced to the
canonical form given by (A.1), e.g. to

$$\left( \frac{d^2}{dx^2} + \frac{4u + 1/4}{x^2} + 4gx^2 + 4v \right) R(x) = 0$$ \hspace{1cm} (A.10)

so that the rest of arguments go through. Analogously, in the case of Morse-
type potential we have the following Schrödinger-type equation initially:

$$\left[ \frac{d^2}{dz^2} + p e^{2az} + q e^{az} + k \right] R(z) = 0$$ \hspace{1cm} (A.11)

By choosing $z = \ln x^2$ and $R(z) = x^{-\frac{1}{2}} R(x)$ (A.11) is reduced to the canonical
form

$$\left( \frac{d^2}{dx^2} + \frac{16k + \alpha^2}{4\alpha^2 x^2} + \frac{4p}{\alpha^2} x^2 + \frac{4q}{\alpha^2} \right) R(x) = 0.$$ \hspace{1cm} (A.12)

By analogous manipulations one can reduce to the canonical form the radial
equation for Hydrogen atom and for 3-dimensional harmonic oscillator.

**Appendix B. Numerical data used for calculations of $n_{\text{theory}}^*$**  
( Supplement to Table 2).

1 au=149.598·10^6 km
Masses (in kg): Sun 1.988·10^{30}, Jupiter 1.8986·10^{27}, Saturn 5.6846·10^{26},
Uranus 8.6832·10^{25}, Neptune 10.243·10^{25}.
$q_j$ : Jupiter 0.955·10^{-3}, Saturn 2.86·10^{-4}, Uranus 4.37·10^{-5}, Neptune 5.15·10^{-5}.
$(r_j)_1$ (km) : Jupiter 127.69·10^3, Saturn 133.58·10^3, Uranus 49.77·10^3,
Neptune 48.23·10^3.
$\ln \left( \frac{\gamma M}{2r_1} \right)$ : Earth 4.0062, Jupiter 3.095, Saturn 1.844, Uranus 0.9513,
Neptune 1.15.

Appendix C. Connections between the Hill and Neumann’s dynamical problems.

Following our paper [92], let us consider the Fuchsian-type equation given by
\[ y'' + \frac{1}{2} \phi y = 0, \tag{C.1} \]
where the potential \( \phi \) is determined by the equation \( \phi = [f] \) with \( f = y_1/y_2 \) and \( y_1, y_2 \) being two independent solutions of (C.1) normalized by the requirement \( y_1' y_2 - y_2' y_1 = 1 \). The symbol \([f]\) denotes the Schwarzian derivative of \( f \). Such a derivative is defined as follows
\[ [f] = \frac{f'' f''' - \frac{3}{2} (f'')^2}{(f')^2}. \tag{C.2} \]
Consider (C.1) on the circle \( S^1 \) and consider some map of the circle given by
\[ F(t+1) = F(t)+1. \]
Let \( t = F(\xi) \) so that \( y(t) = Y(\xi) \sqrt{F'}(\xi) \) leaves (C.1) form-invariant, i.e. in the form \( Y'' + \frac{1}{2} \Phi Y = 0 \) with potential \( \Phi \) being defined now as \( \Phi(\xi) = \phi(F(\xi))[F'(\xi)]^2 + [F(\xi)] \). Consider next the infinitesimal transformation \( F(\xi) = \xi + \delta \varphi(\xi) \) with \( \delta \) being some small parameter and \( \varphi(\xi) \) being some function to be determined. Then, \( \Phi(\xi + \delta \varphi(\xi)) = \phi(\xi) + \delta \hat{T} \varphi(\xi) + O(\delta^2) \). Here \( \hat{T} \varphi(\xi) = \phi(\xi) \varphi'(\xi) + \frac{1}{2} \varphi''(\xi) + 2 \phi'(\xi) \varphi(\xi) \). Next, we assume that the parameter \( \delta \) plays the same role as time. Then, we obtain
\[ \lim_{t \to 0} \frac{\Phi - \phi}{t} = \frac{\partial \phi}{\partial t} = \frac{1}{2} \varphi''(\xi) + \phi(\xi) \varphi'(\xi) + 2 \phi'(\xi) \varphi(\xi) \tag{C.3} \]
Since thus far the perturbing function \( \varphi(\xi) \) was left undetermined, we can choose it now as \( \varphi(\xi) = \phi(\xi) \). Then, we obtain the Korteweg-de Vries (KdV) equation
\[ \frac{\partial \phi}{\partial t} = \frac{1}{2} \varphi''(\xi) + 3 \phi(\xi) \varphi'(\xi) \tag{C.4} \]
determining the potential \( \phi(\xi) \). For reasons which are explained in the text, it is sufficient to consider only the static case of KdV, i.e.
\[ \phi''(\xi) + 6 \phi(\xi) \varphi'(\xi) = 0. \tag{C.5} \]
We shall use this result as a reference for our main task of connecting the Hill and the Neumann’s problems. Using (4.19) we write
\[ u(\xi) = \phi(\xi) - < \dot{\xi}, \ddot{\xi} >. \tag{C.6} \]
Consider an auxiliary functional $\varphi(\xi) = \langle \xi, A^{-1}\xi \rangle$. Suppose that $\varphi(\xi) = u(\xi)$. Then,

$$\frac{du}{dt} = 2 < \dot{\xi}, A\xi > - 2 < \ddot{\xi}, \dot{\xi} > .$$  \hspace{1cm} (C.7)

But $< \ddot{\xi}, \dot{\xi} > = 0$ because of the normalization constraint $< \xi, \xi > = 1$. Hence, $\frac{du}{dt} = 2 < \dot{\xi}, A\xi >$. Consider as well $\frac{d\varphi}{dt}$. By using (4.19) it is straightforward to show that $\frac{d\varphi}{dt} = 2 < \dot{\xi}, A^{-1}\xi >$. Because by assumption $\varphi(\xi) = u(\xi)$, we have to demand that $< \ddot{\xi}, A^{-1}\xi > = < \dot{\xi}, A\xi >$ as well. If this is the case, consider

$$\frac{d^2u}{dt^2} = 2 < \ddot{\xi}, A^{-1}\xi > + 2 < \dot{\xi}, A^{-1}\dot{\xi} > .$$  \hspace{1cm} (C.8)

Using (4.19) once again we obtain,

$$\frac{d^2u}{dt^2} = -2 + 2u\varphi + 2 < \dot{\xi}, A^{-1}\dot{\xi} > .$$  \hspace{1cm} (C.9)

Finally, consider as well $\frac{d^3u}{dt^3}$. Using (C.9) as well as (4.19) and (C.7) we obtain,

$$\frac{d^3u}{dt^3} = 2 \frac{du}{dt} \varphi + 4u \frac{du}{dt} = 6u \frac{du}{dt} .$$  \hspace{1cm} (C.10)

By noticing that in (C.5) we can always make a rescaling $\phi(\xi) \rightarrow \lambda \phi(\xi)$, we can always choose $\lambda = -1$ so that (C.5) and (C.10) coincide. This result establishes correspondence between the Neumann and Hill-type problems.

QED

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