Vibration Analysis of Visco-Elastic Rectangular Plate with Thickness Varies Linearly in One and Parabolically in Other Direction

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Abstract

In the modern technology, the plates of variable thickness are widely used in engineering applications i.e. nuclear reactor, aeronautical field, naval structure, submarine, earth-quake resistors etc. Thus, in the present work, an analysis of vibration of visco-elastic isotropic rectangular plate with varying thickness in two directions i.e. linearly in one and parabolically in other direction is presented. The thickness variation in two-direction is taken as the Cartesian product of linear variation and parabolic variation along the two concurrent edges of the plate in x-direction and y-direction respectively. Also, it is considered that the plate have clamped boundary conditions on all the four edges. The Rayleigh-Ritz technique has been used to get the required frequency equation. In the present investigation, the effect of various factors i.e. taper constants and aspect ratio are studied to find the values of logarithmic decrement, time period and deflection for the first two modes of vibration.

Keywords: visco-elastic, vibration, rectangular plate, thickness variation, linearly, parabolically

INTRODUCTION

Vibration phenomenon, common in mechanical devices and structures, is undesirable in many cases, such as machine tools. But this phenomenon is not always unwanted; for example, vibration is needed in the operation of vibration screens.

As technology develops new discoveries day by day like in jet engine, field of spacecraft and nuclear power plants etc., the time dependent behavior of materials has become of great importance. Thus, the need of the study of vibration of visco-elastic plates (it may be rectangular, circular, elliptical etc.) of certain aspect ratios with some simple boundary conditions has been increased rapidly.

In the course of time, engineers have become increasingly conscious of the importance of an elastic behavior of many materials and mathematical formulations have been attempted and applied to practical problems. Since plastics and new materials are widely used in the construction of equipment and structures, so the development of the application of visco-elasticity is needed to permit rational design.

Although considerable work has been carried out to examine the vibration of visco-elastic plates whose thickness varies in one direction yet the thickness variation in two directions has been given less attention. The thickness variations can be of any type i.e. uniform or non-uniform. It is difficult to take into account the non-uniform thickness variations in both directions. Studies on vibration of visco-elastic plates of
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linearly and parabolically varying thickness in both directions have already been conducted [3, 15].


Free vibrations of rectangular plates whose thickness varies parabolically have been studied by Jain and Soni [14]. Gupta and Khanna [15] have been worked on the vibration of clamped visco-elastic rectangular plate with parabolic thickness variations. Leissa [16] is studied the vibration of plates. Gutierrez, Laura and Grossi [17] discussed the vibrations of rectangular plates of bi-linearly varying thickness with general boundary conditions. Khanna [18] studied the some vibration problems of visco-elastic plate of variable thickness in two directions.

The main objective of the present paper is to study the effect of taper constants on the linear vibration of clamped visco-elastic isotropic rectangular plate whose thickness varies linearly in one and parabolically in other directions, which is an extension of the author’s previous work [3, 15]. As no authors have been consider the effect of tapers as different in different directions and it has practical importance so authors consider the same. To determine the frequency equation, it is considered that the visco-elastic properties of the plate are of the ‘Kelvin’ or ‘Voigt’ type, which is
combination of the elastic and viscous elements in parallel, and also assumed that
plate is clamped support on all the four edges.

Logarithmic decrement (Λ), time period (K) and deflection (w) for first two
modes of vibration for various values of aspect ratio a/b and taper constants β₁ & β₂ at
different instant of time are calculated. All the above results are also illustrated with
figures.

**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>a</td>
<td>length of the rectangular plate</td>
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<tr>
<td>b</td>
<td>width of the rectangular plate</td>
</tr>
<tr>
<td>x, y</td>
<td>co-ordinates in the plane of the plate</td>
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<tr>
<td>E</td>
<td>young’s modulus</td>
</tr>
<tr>
<td>ν</td>
<td>poisson’s ratio</td>
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<tr>
<td>D₁</td>
<td>flexural rigidity</td>
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<tr>
<td>D</td>
<td>visco-elastic operator</td>
</tr>
<tr>
<td>ρ</td>
<td>mass density per unit volume of the plate material</td>
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<tr>
<td>W(x,y)</td>
<td>deflection function</td>
</tr>
<tr>
<td>Q</td>
<td>deflection function</td>
</tr>
<tr>
<td>β₁</td>
<td>taper constant in X-direction</td>
</tr>
<tr>
<td>β₂</td>
<td>taper constant in Y-direction</td>
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<tr>
<td>p²</td>
<td>a constant</td>
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<tr>
<td>Λ</td>
<td>logarithmic decrement</td>
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<tr>
<td>T</td>
<td>time function</td>
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<tr>
<td>K</td>
<td>time period</td>
</tr>
<tr>
<td>h</td>
<td>thickness of plate at point (x,y)</td>
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<tr>
<td>G</td>
<td>shear modulus</td>
</tr>
<tr>
<td>E₀</td>
<td>constant</td>
</tr>
<tr>
<td>η</td>
<td>visco-elastic constants</td>
</tr>
<tr>
<td>T(t)</td>
<td>time function</td>
</tr>
<tr>
<td>λ²</td>
<td>frequency parameter</td>
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</tbody>
</table>

**ANALYSES AND EQUATION OF MOTION**

The equation of motion of a visco-elastic isotropic plate of variable thickness is [3]:

\[
D₁(\partial^{4}W/\partial x^{4}+2\partial^{4}W/\partial x^{2}\partial y^{2}+\partial^{4}W/\partial y^{4})+2D₁,xx(\partial^{2}W/\partial x^{2}+\partial^{2}W/\partial x\partial y)+2D₁,yy(\partial^{2}W/\partial y^{2}+\partial^{2}W/\partial x\partial y)+D₁,xy(\partial^{2}W/\partial x\partial y)+2(1-\nu)D₁,xx\partial^{2}W/\partial x\partial y\)

and

\[
T+\rho h p² W = 0 \quad \Lambda = 12\rho(1-\nu)^2a²/E₀h², frequency parameter
\]

where equations (1) and (2) are the differential equation of motion for isotropic plate
of variable thickness and time function equation for visco-elastic plate for free
vibration respectively.

Here p² is a constant.
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The expressions for Kinetic energy $T_1$ and Strain energy $V_1$ are [16]

$$ T_1 = \frac{1}{2} \rho p^2 \int_0^a \int_0^b h W^2 dy dx \quad ------ (3) $$

and

$$ V_1 = \frac{1}{2} \rho \int_0^a \int_0^b D_1 \left\{ (\partial^2 W/\partial x^2)^2 + (\partial^2 W/\partial y^2)^2 + 2\nu(\partial^2 W/\partial x^2)(\partial^2 W/\partial y^2) + 2(1-\nu) \right\} dy dx \quad ------ (4) $$

Assuming that the thickness variation of the plate in both directions as

$$ h = h_0 \left( 1 + \beta_1 x/a \right) \left( 1 + \beta_2 y^2/b^2 \right) \quad ------ (5) $$

where $h_0 = h$ at $x=y=0.0$

The flexural rigidity of the plate can now be written as (assuming Poisson’s ratio $\nu$ is constant):

$$ D_1 = \frac{E h_0^3 \left( 1 + \beta_1 x/a \right)^3 \left( 1 + \beta_2 y^2/b^2 \right)^3}{12(1-\nu^2)} \quad ------ (6) $$

**SOLUTION AND FREQUENCY EQUATION**

According to Rayleigh-Ritz technique (which is used to find the solution), maximum strain energy must be equal to the maximum kinetic energy. Therefore, for the problem under consideration, it is necessary that

$$ \delta (V_1 - T_1) = 0 \quad ------ (7) $$

for arbitrary variations of $W$ satisfying relevant geometrical boundary conditions.

Boundary conditions, for a rectangular plate clamped (c) along all the four edges, are

$$ W = W_x = 0 \quad \text{at} \quad x = 0, a $$

and

$$ W = W_y = 0 \quad \text{at} \quad y = 0, b $$

and the corresponding two-term deflection function is taken as [3]

$$ W = [(x/a)(y/b)(1-x/a)(1-y/b)]^2 [A_1 + A_2(x/a)(y/b)(1-x/a)(1-y/b)] \quad ------ (8) $$

which are satisfied equations (8).

Assuming the non-dimensional variables as

$$ X = x/a, \quad Y = y/a, \quad \bar{W} = W/a, \quad \bar{h} = h/a \quad ------ (10) $$

and using equations (5), (6) & (10) in equations (3) & (4), one obtains

$$ T_1 = \frac{1}{2} \rho p^2 h_0 a^4 \int_0^{b/a} \int_0^{1} \left[ (1+\beta_1 X) (1+\beta_2 Y^2 a^2/b^2) \bar{W}^2 \right] dYdX \quad ------ (11) $$
and

\[ V_1 = \frac{Q}{b/a} \int_0^1 \int (1 + \beta_1 X)^3 (1 + \beta_2 Y^2)^3 \left( (\partial^2 W / \partial X^2)^2 + (\partial^2 W / \partial Y^2)^2 + 2\nu (\partial^2 W / \partial X \partial Y)^2 \right) dY dX \] -----(12)

Substituting the values of \( T_1 \) & \( V_1 \) from equations (11) & (12) in equation (7), one obtains

\[ (V_2 - \lambda^2 p^2 T_2) = 0 \] \hspace{1cm} \text{------- (13)}

Equation (13) has the unknowns \( A_1 \) & \( A_2 \) arising due to the substitution of \( W \) from equation (9). These two constants are to be determined from equation (13) as

\[ \frac{\partial (V_2 - \lambda^2 p^2 T_2)}{\partial A_n} = 0, \hspace{1cm} n = 1, 2 \] \hspace{1cm} \text{------- (14)}

by minimizing the Rayleigh quotient with respect to the undetermined coefficients \( A_1 \) & \( A_2 \).

After simplifying equation (14), one gets

\[ b_n A_1 + b_n A_2 = 0, \hspace{1cm} n = 1, 2 \] \hspace{1cm} \text{------- (15)}

where \( b_{11}, b_{21} \) involve parametric constant and the frequency parameter.

Choosing \( A_1 = 1 \), one can get easily from equation (15) \( A_2 \), which is \( -b_{11}/b_{12} \).

For a non-trivial solution, the determinant of the coefficient of equation (15) must be zero. So one gets, the frequency equation as

\[ \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = 0 \] \hspace{1cm} \text{------- (16)}

From equation (16), one can obtains a quadratic equation in \( p^2 \) from which the two values of \( p^2 \) can found. Now with the help of the values of \( A_1 \) and \( A_2 \), one can obtain deflection function \( W \) as

\[ W = [XY(a/b)(1-X)(1-Ya/b)]^2 \left[ 1 + (-b_{11}/b_{12}) X Y(a/b)(1-X)(1-Ya/b) \right] \] \hspace{1cm} \text{-------(17)}

**TIME FUNCTIONS OF VISCO-ELASTIC PLATES**

Equation (2) is define as general differential equation of Time functions of free vibrations

\[ \tilde{\beta} \]

of visco-elastic plate. It depends on visco-elastic operator \( D \).

\[ \tilde{\beta} \]

For Kelvin’s or Voigt’s model, one has \( D \equiv \{ 1 + (\eta/G) (d/dt) \} \)
After using the value of $D$ in equation (2), one obtains

$$\ddot{T} + p^2 \left( \frac{\eta}{G} \right) T + p^2 T = 0$$

Equation (18) is a differential equation of second order for time function $T$.

Solution of equation (18) will be as in [15]:

$$T(t) = e^{a_1 t} \left[ \cos b_1 t + \left( -\frac{a_1}{b_1} \right) \sin b_1 t \right]$$

where

$$a_1 = -\frac{p^2 \eta}{2G} \quad \& \quad b_1 = p \sqrt{1 - \left( \frac{p \eta}{2G} \right)^2}$$

By using equations (19) & (17), deflection $w(x,y,t)$, may be expressed as

$$w = [X Y (a/b) (1-X) (1-Ya/b)]^2 [1 + (-b_1/b_2)XY(a/b)(1-X)(1-Ya/b)] \times$$

$$\left[ e^{a_1 t} \left( \cos b_1 t + \left( -\frac{a_1}{b_1} \right) \sin b_1 t \right) \right]$$

Time period of the vibration of the plate is given by

$$K = \frac{2 \pi}{p}$$

where $p$ is frequency given by equation (16).

Logarithmic decrement of the vibrations given by the standard formula

$$\dot{\gamma} = \log \frac{w_2}{w_1}$$

where $w_1$ is the deflection at any point on the plate at time period $K = K_1$ and $w_2$ is the deflection at same point at the time period succeeding $K_1$.

**RESULTS AND DISCUSSION**

For the calculation of the values of logarithmic decrement ($\dot{\gamma}$), time period ($K$) and deflection ($w$) for an isotropic visco-elastic rectangular plate for different values of aspect ratio $a/b$ and taper constants $\beta_1$ & $\beta_2$ at different points for first two modes of vibrations, the following material parameters are used which is for DURALIUM reported at [3]:

$$E = 7.08 \times 10^{10} \, \text{N/M}^2, \quad G = 2.632 \times 10^{10} \, \text{N/M}^2, \quad v = 0.345, \quad \eta = 14.612 \times 10^5 \, \text{N.S/M}^2, \quad \rho = 2.80 \times 10^3 \, \text{Kg/M}^3$$

The thickness of the plate at the centre is taken as $h_0 = 0.01$ m.

In the present problem, latest computer technology is used to get the numerical results with great accuracy and concentration. Computations have been made for calculating logarithmic decrement, time period and deflection for different values of taper constants $\beta_1$ & $\beta_2$ and aspect ratio $a/b$ for first two modes of vibration. All the results can be predicted from fig.1 to fig.7.

For a fixed value of aspect ratio $a/b (=1.5)$, it is evident from fig.1 and fig.2 that as taper constant $\beta_1$ increases for two fixed values of $\beta_2$ (0.2 & 0.6), logarithmic decrement $\dot{\gamma}$ and time period $K$ decrease continuously for both the modes of vibration respectively.
One can easily analyze that the logarithmic decrement $\Delta$ and time period $K$ also decrease for the increasing value of $\beta_2$ for both the modes of vibration.

Fig. 3 represents the time period $K$ for different values of aspect ratio $a/b$ for both the modes of vibration for uniform and non-uniform thickness having the following cases:

(i) $\beta_1 = \beta_2 = 0.0$  
(ii) $\beta_1 = \beta_2 = 0.6$

After analyzing Fig.3; it can be seen that for the above two values of $\beta_1$ & $\beta_2$, time period $K$ decreases as aspect ratio $a/b$ increases for both the modes of vibration. It is also evident from the fig.3 that time period $K$ gradually decreases as the values of taper constants $\beta_1$ & $\beta_2$ increases.

Figs. 4a and 5a represent the first mode of vibration at time $T=0.K$ and $T=5.K$ respectively for the fixed value of aspect ratio $a/b$ (=1.5) and for the different values of $X$ and $Y$ (i.e. $Y=0.3$ & $Y=0.6$) for the following cases:

(i) $\beta_1 = \beta_2 = 0.0$  
(ii) $\beta_1 = \beta_2 = 0.6$

To analyze the figures 4a & 5a, it is easily seen that deflection w first increases and then decreases as $X$ increases for different values of $Y$ for both the above cases.

In figure 4a, it is clearly seen that the values of first mode of deflection is more for case (ii) i.e. ($\beta_1 = \beta_2 = 0.6$) than case (i) i.e. ($\beta_1 = \beta_2 = 0.0$) for different values of $Y$ while in figure 5a., values of first mode of deflection is less for case (ii) than case(i).

Similarly, figs. 4b and 5b show the second mode of vibration at time $T=0.K$ and $T=5.K$ respectively for the different values of $X$ and $Y$ (i.e. $Y=0.3$ & $Y=0.6$) and for fixed value of aspect ratio $a/b$ (=1.5) and for the following cases:

(i) $\beta_1 = \beta_2 = 0.0$  
(ii) $\beta_1 = \beta_2 = 0.6$

For the second mode of vibration; the nature of the curve of deflection is not same as for the first mode. For the second mode of vibration the deflection w firstly decreases and then increases for different values of $Y$ for both the above cases.

Again it can be seen from fig.4b that the values of second mode of deflection is more for case (ii) i.e. ($\beta_1 = \beta_2 = 0.6$) than case (i) i.e. ($\beta_1 = \beta_2 = 0.0$) for different values of $Y$ while in figure 5b., for $Y=0.3$; values of second mode of deflection is less for case (ii) than case(i).

It is interesting to note that for $Y=0.6$, values of second mode of deflection is less for case (ii) than case (i) for $X=0.2$ & $X=0.8$ but more for case (ii) than case (i) for $X=0.4$ & $X=0.6$. 
CONCLUSION

On comparing the present problem and the author’s published problem [3], there is only the effect of one factor which is to be compared that is the effect of $\beta_2$. Thus, one can easily analyze at once that the values of logarithmic decrement, time period and deflection are increased in the present problem as compared to [3] for the same values of taper’s constant $\beta_1$ & $\beta_2$ and aspect ratio.

As well as on comparing the present problem with author’s accepted paper [15], it is found that the values of logarithmic decrement, time period and the values of first mode of deflection are decreased in the present problem as compared to [15].

The second mode of deflection is increased in the present problem as compared to [15] for time $T=0.K$ & $\beta_1 = \beta_2 =0.6$, fixed value of aspect ratio $a/b (=1.5)$ and for different the values of $X$ & $Y$.

For $Y=0.3$, the second mode of deflection is decreased in present problem as compared to [15] for time $T=5.K$ & $\beta_1 = \beta_2 =0.6$, for different values of $X$ and fixed value of aspect ratio $a/b (=1.5)$. But For $Y=0.6$, the value of second mode of vibration is decreased in the present problem for $X=0.2$ & $X=0.8$ and fixed value of aspect ratio $a/b (=1.5)$ while for $X=0.4$ & $X=0.6$, value of second mode of deflection is increased in the present problem as in [15].

REFERENCES

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Fig. 1

LOGARITHMIC DECREMENT $\Lambda$ Vs TAPER CONSTANT $\beta_1$ FOR A CLAMPED VISCO-ELASTIC RECTANGULAR PLATE

Fig. 2

TIME PERIOD $K$ Vs TAPER CONSTANT $\beta_1$ OF A CLAMPED VISCO-ELASTIC RECTANGULAR PLATE
Fig. 3

TIME PERIOD K Vs ASPECT RATIO a/b OF A CLAMPED VISCO-ELASTIC RECTANGULAR PLATE
Fig. 4a

DEFLECTION $w$ Vs $X$ OF A CLAMPED VISCO-ELASTIC RECTANGULAR PLATE FOR FIRST MODE

$Y=0.6, \beta_1=\beta_2=0.6$

$Y=0.6, \beta_1=\beta_2=0.0$

$Y=0.3, \beta_1=\beta_2=0.6$

$Y=0.3, \beta_1=\beta_2=0.0$

$T=0K, a/b=1.5$
Fig. 4b

DEFLECTION \( w \times 10^{-5} \) Vs \( X \) OF A CLAMPED VISCO-ELASTIC RECTANGULAR PLATE FOR SECOND MODE

T=0. K, \( a/b=1.5 \)

\( Y=0.3, \beta_1=\beta_2=0.6 \)

\( Y=0.3, \beta_1=\beta_2=0.0 \)

\( Y=0.6, \beta_1=\beta_2=0.6 \)

\( Y=0.6, \beta_1=\beta_2=0.0 \)
DEFLECTION $w$ Vs $X$ OF A CLAMPED VISCO-ELASTIC RECTANGULAR PLATE FOR FIRST MODE

Fig.5a.
Fig. 5b.

DEFLECTION $w$ Vs $X$ OF A CLAMPED VISCO-ELASTIC RECTANGULAR PLATE FOR SECOND MODE

- $T=5.0K$, $a/b=1.5$
- $Y=0.3$, $\beta_1=\beta_2=0.0$
- $Y=0.3$, $\beta_1=\beta_2=0.6$
- $Y=0.6$, $\beta_1=\beta_2=0.6$
- $Y=0.6$, $\beta_1=\beta_2=0.0$

Received: May, 2010