Channel Models of the Doppler Effect

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Abstract

We offer a pedagogical study of the Doppler effect as a channel, and derive a mathematical model of the input-output relation in both classical and relativistic settings. We then show for the derived channel models the well known result that the classical model is the limit of the relativistic one at low speeds. We also derive the relativistic channel model in two independent ways and show they are compatible. We infer from these two observations that there is essentially only one Doppler effect in different disguises. Finally, we observe that the channel models we derive correctly demonstrate that, although the Doppler effect is a rescaling, it can be approximated by an additive frequency shift under the usual narrowband assumption in communications. We conclude with a brief survey of the myriad of applications that exploit the Doppler effect in both classical and relativistic form.

Keywords: Doppler effect (classical and relativistic), channel modeling, relativistic electrodynamics

1 Introduction

The Doppler effect, that is the apparent change in the frequency of a signal caused by contractions/dilations of time when the transmitter and the receiver exhibit relative motion, has been studied extensively in the physics literature. Why should then yet another study of this subject be needed? There are two major reasons, in our opinion.

On the one hand, the physics literature insists on studying almost exclusively the frequency change aspect of the Doppler effect (see, for example, [25, 37, 50, 24] among many others). For communications engineering applications, however, it is imperative to know how this frequency change alters the
formula of a signal. To rephrase, we would like to consider the Doppler effect as a channel that transforms an input signal $s(t)$ into the output signal $w(t)$ that is an attenuated, delayed, and rescaled version of the input: $w(t) = cs(at - b)$, where $a, b, c$ can perhaps depend on time. Very little work has been done on that front, and most of it is quite recent (e.g. [42, 11, 18, 41, 31, 30]), while systematic mention of the Doppler effect in the context of wideband channels is very recent [23, 32]. It is clear that the traditional interpretation of the Doppler effect as a simple frequency change is too narrow for our purposes, leaving out of its scope the parameters $b$ (delay) and $c$ (attenuation). Furthermore, there is the question of the nature of the frequency change: while literature recognizes this change to be caused by rescaling, this is almost immediately converted (predominantly in engineering literature) to an additive shift and considered as such ever after. It is clear that the additive frequency shift is an approximation of the multiplicative change induced by rescaling, but under which conditions is the approximation valid?

On the other hand, the Doppler effect can be derived in several different contexts, and, as a result, the formulas produced may look somewhat different. More precisely, for the derivation of the Doppler effect we may use either classical Newtonian mechanics [20], or relativistic mechanics [25, 4, 5, 38]; and, in the latter case, we face anew two alternatives, as we may choose to proceed either kinematically, using, for example, Minkowski diagrams, or by using the full Lorentz transformations of relativistic electrodynamics. Such a plurality of methods begs some questions: Are the results compatible? Is it the case that a particular version of the Doppler effect is suitable for a particular class of signals (acoustic, electromagnetic etc.)?

The main result of this paper is the derivation of the exact mathematical model which describes the input-output relation of the Doppler channel. We perform the derivation of the model in a newtonian setting, a Minkowski diagram setting, and a relativistic electrodynamics setting, and demonstrate they are indeed compatible, in the sense that the relativistic derivation tends to the classical one in the limit of low velocities. Further, although some derivation methods may rely on specific features of the signals (an obvious example being relativistic electrodynamics), the resulting formulas are the same. To put it in a nutshell, there is only one Doppler effect. Further, we show the well known result that the Doppler effect may be interpreted as an additive frequency shift under the usual “narrowband assumption” in communications. To the best of the authors’ knowledge, this is the first presentation of the mathematical channel model of the Doppler effect to appear in the literature, and is therefore presented in an appropriately pedagogical style.

The layout of the paper is depicted in Figure 1. In Section 2 we derive the input-output relationship for the classical Doppler effect (that is, using classical Newtonian mechanics). In Section 3 we derive the input-output relation-
ship for the kinematic relativistic Doppler effect using special relativity, and more precisely the Minkowski diagram. In Section 4 we present an alternative derivation of the relativistic Doppler effect based on relativistic electrodynamics (that is, Maxwell’s equations). In Section 5 we connect the results derived in the previous sections by showing a) that the kinematic and electromagnetic relativistic Doppler effects are essentially the same, b) that the classical and kinematic relativistic Doppler effects are compatible, in the sense that the former is the limit of the latter for non-relativistic velocities of transmitter and receiver. Moreover, we make some observations about whether the roles of the transmitter and the receiver are symmetric or not, we discuss the representability of the multiplicative nature of the Doppler effect by additive shifts, and offer a quantitative example. Finally, in Section 6 we mention applications of the Doppler effect, and we summarize and conclude in Section 7.

2 The classical Doppler effect

In this section we discuss the classical treatment of the Doppler effect [20, 46], as opposed to relativistic. Consider a transmitter $T$ located at the origin, moving along the $x$-axis with constant velocity $v_t$ (positive if moving to the right). The position of $T$ is described by

$$p_x(t) = v_t t$$ (1)
Consider a receiver $R$ located initially at a distance $d_0$ from the origin along the positive $x$-axis, also moving along the $x$-axis with constant velocity $v_r$ (positive if moving to the right). The position of $R$ is described by

$$p_y(t) = v_r t + d_0$$  \hspace{1cm} (2)

We define $\tau$ to be the time delay such that a signal traveling with speed $c_0$ and transmitted from $T$ at time $t - \tau$ reaches $R$ at time $t$; note that we take $c_0$ to be always positive, regardless of the direction of propagation. Clearly, the distance that the signal travels must equal the difference between the current position of $R$ and the position of $T$ $\tau$ seconds ago. This can be described as:

$$c_0 \tau = |p_y(t) - p_x(t - \tau)|$$ \hspace{1cm} (3)

$$= |v_r t + d_0 - v_t (t - \tau)|.$$ \hspace{1cm} (4)

Solving for $\tau$,

**Case 1:** $p_y(t) > p_x(t - \tau)$, $\tau = \frac{(v_r - v_t) t + d_0}{c_0 - v_t}$,  \hspace{1cm} (5)

**Case 2:** $p_y(t) < p_x(t - \tau)$, $\tau = \frac{(-v_r + v_t) t - d_0}{c_0 + v_t}$. \hspace{1cm} (6)

By definition, $\tau$ must be non-negative, otherwise no solution exists. As the two solutions above can be obtained from each other by changing sign to $v_t$, $v_r$, and $d_0$, we need only consider the cases where $d_0 > 0$:

- $c_0 > v_t, (v_r - v_t)t + d_0 > 0$: This implies $p_y(t) > p_x(t)$ and falls under Case 1;
- $c_0 > v_t, (v_r - v_t)t + d_0 < 0$: This implies $p_y(t) < p_x(t)$ and falls under Case 2;
- $c_0 > -v_t, (v_r - v_t)t + d_0 > 0$: This implies $p_y(t) > p_x(t)$ and falls under Case 1;
- $c_0 > -v_t, (v_r - v_t)t + d_0 < 0$: This implies $p_y(t) < p_x(t)$ and falls under Case 2;

It follows that, as long as $|v_t| < c_0$,

**Case 1:** $p_y(t) > p_x(t)$, $\tau = \frac{(v_r - v_t) t + d_0}{c_0 - v_t}$, \hspace{1cm} (7)

**Case 2:** $p_y(t) < p_x(t)$, $\tau = \frac{(-v_r + v_t) t - d_0}{c_0 + v_t}$. \hspace{1cm} (8)

so we only need to compare the position of $T$ and $R$ at the same time instant to decide which expression yields $\tau$. When $|v_t| > c_0$, however, there is a slight
difference (this corresponds to what we call “supersonic” motion in the case of acoustic waves):
\[ c_0 < v_t, (v_r - v_t)t + d_0 > 0 \] This implies \( p_y(t) > p_x(t) \) and falls under Case 2;
\[ c_0 < v_t, (v_r - v_t)t + d_0 < 0 \] This implies \( p_y(t) < p_x(t) \) and falls under Case 1;
\[ c_0 < -v_t, (v_r - v_t)t + d_0 > 0 \] This implies \( p_y(t) > p_x(t) \) and falls under Case 2;
\[ c_0 < -v_t, (v_r - v_t)t + d_0 < 0 \] This implies \( p_y(t) < p_x(t) \) and falls under Case 1;

In other words, when \(|v_t| > c_0\) the role of (7) and (8) is inverted.

In treatises of the Doppler effect in the context of physics it is not customary to consider the decay of the amplitude of the signal at all. An input-output channel description of the Doppler effect, however, could not possibly be complete without it. We assume then here that the amplitude decays with the reciprocal of distance, as is the case for isotropic transmission in a lossless medium. Furthermore, in order to make the decay factor dimensionless, we multiply it with an arbitrary fixed reference/normalization distance, which simply sets the distance at which this factor becomes 1. In the following we will set this distance equal to the initial \( R-T \) distance, here \( d_0 \). The input-output mapping, in the case of subsonic motion (\(|v_t| < c_0\)), becomes:

\[
w(t) = \frac{d_0}{|p_y(t) - p_x(t)|} s(t - \tau) \]

\[
= \begin{cases} 
\frac{d_0(1-v_t/c_0)}{d_0-(v_r-v_t)t} \left( \frac{t-d_0}{c_0-v_r} \right) & : p_y(t) > p_x(t) \\
\frac{d_0(1+v_t/c_0)}{(v_r-v_t)t-d_0} \left( \frac{t+d_0}{c_0+v_r} \right) & : p_y(t) < p_x(t)
\end{cases} \] (10)

where we added \( d_0 \) in the numerator for dimensional reasons. The change in Doppler effect from scale contraction to scale expansion, namely where the apparent frequency switches from being higher to being lower than the actual frequency, occurs when \( T \) and \( R \) are collocated (\( p_y(t) = p_x(t) \)). For the sake of completeness, let us repeat here that, in the case of supersonic motion of \( T \), the two formulas in (10) swap roles. Finally, it is important to note that the Doppler effect appears as a rescaling of the original signal, which, in the frequency domain, corresponds to a multiplicative shift of frequencies.

3 The kinematic relativistic Doppler effect

In the case of electromagnetic waves in the vacuum, where \( c_0 = c, \) or when either \( v_t \) or \( v_r \) is comparable to \( c, \) the speed of light, there may be need to
account for relativistic effects that we ignored in the derivation of (10). This leads us to consider the relativistic Doppler effect [4, 6, 5]. There appears to be, in fact, a general trend for relativity to become increasingly important in modern engineering applications and to be no longer considered as an “exotic” subject; this is partly reflected by the large numbers of tutorial-type papers on relativity published (e.g. [10, 9, 29]).

3.1 Derivation

Consider the same setup as before, which is now depicted in the space-time diagram in Figure 2. We have:

\[ O : (0,0) \] (11)
\[ A : (v_it_0, t_0) \] (12)
\[ B : (d_0,0) \] (13)
\[ D : (d, t) \] (14)

and, as the slope of \( AD \) and \( OC \) is \( 1/c_0 \), the slope of \( OA \) is \( 1/v_i \), and the slope of \( BD \) is \( 1/v_r \),

\[ OA : t = \frac{1}{v_i} d \] (15)
\[ OC : t = \frac{1}{c_0} d \] (16)
\[ BD : t = \frac{1}{v_r} (d - d_0) \] (17)
\[ AD : \sqrt{c^2t_A^2 - d_A^2} = 1 \] (19)

where \( b \) is yet to be determined. In Figure 2 the time dilation factor between \( T \) and \( R \) is given by the ratio \( l_x/l_2 \). Without loss of generality we may set \( l_x = 1 \), and, under this assumption, we proceed to determine \( l_2 \), which equals now the multiplicative inverse of the dilation factor.

For simplicity, we consider here only the case where \( T \) is located to the left of \( R \). The complementing case can be determined by swapping the signs on the velocities and the original separation distance, as was the case in the non-relativistic result. Using (15) and distance as measured in Minkowski space for the space-time diagram (for example, see Chapter 14 in [25]), we obtain

\[ \sqrt{c^2t_A^2 - d_A^2} = 1 \] (19)
Figure 2: Space-time diagram for the derivation of the Doppler effect: note that the vertical axis assumes imaginary values.

and we determine the location of point \( A \),

\[
(d_A, t_A) = \left( \frac{v_t}{\sqrt{c^2 - v_t^2}}, \frac{1}{\sqrt{c^2 - v_t^2}} \right)
\]  
(20)

Now we can determine the \( t \)-intercept in (18),

\[
b = \frac{1 - \frac{v}{c_0}}{\sqrt{c^2 - v_t^2}}
\]  
(21)

The intersection of \( AD \) and \( BD \) yields,

\[
(d_D, t_D) = \left( \frac{(b v_r + d_0)c_0}{c_0 - v_r}, \frac{d_0 + b c_0}{c_0 - v_r} \right)
\]  
(22)

Intersecting \( OC \) and \( BD \),

\[
(d_C, t_C) = \left( \frac{c_0 d_0}{c_0 - v_r}, \frac{d_0}{c_0 - v_r} \right)
\]  
(23)

Then the length from \( C \) to \( D \) is,

\[
l_2 = \sqrt{c^2(t_D - t_C)^2 - (d_D - d_C)^2}
\]  
(24)

\[
= \sqrt\left( \frac{b c_0 c_v}{c_0 - v_r} \right)^2 - \left( \frac{b c_0 v_r}{c_0 - v_r} \right)^2
\]  
(25)
The first term in (27) is the same as the scale factor in (10), the second term is the relativistic correction. The input-output relation takes the form,

$$w(t) = s \left( \frac{t}{l_2} - d_1 \right).$$  \hspace{1cm} (28)

We solve for $d_1$ by extending $AO$ and $BD$ to their point of intersection,

$$(d_I, t_I) = \left( \frac{d_0 v_t}{v_t - v_r}, \frac{d_0}{v_t - v_r} \right)$$  \hspace{1cm} (29)

and note that at that point,

$$w(t_I) = s(t_I/l_2 - d_1) = s(t_I)$$  \hspace{1cm} (30)

thus,

$$t_I = t_I/l_2 - d_1$$  \hspace{1cm} (31)

thus,

$$d_1 = t_I(1/l_2 - 1) = \frac{d_0}{v_t - v_r} \left( \frac{1}{l_2} - 1 \right).$$  \hspace{1cm} (32)

We state (28) in a similar form as (10):

$$w(t) = s \left( t - \frac{d_0}{v_t - v_r} \left( \frac{1}{l_2} - 1 \right) \left( \frac{1 - \frac{v_t}{c}}{1 - \frac{v_r}{c}} \frac{\sqrt{1 - \left( \frac{v_r}{c} \right)^2}}{\sqrt{1 - \left( \frac{v_t}{c} \right)^2}} \right) \right)$$  \hspace{1cm} (33)

Observe here that, contrary to the classical case (10), the relativistic Doppler effect does not depend exclusively on the relative velocity $v_r - v_t$ between $R$ and $T$, unless $c_0 = c$. In particular, this implies the somewhat surprising result that the two scenarios where $(v_t, v_r) = (0, -v)$, and $(v_t, v_r) = (v, 0)$ lead to different Doppler effects, and therefore can be distinguished! This is a well known fact in the literature [6, 7, 16].

When $c = c_0$,

$$l_2 = \sqrt{\frac{1 - \frac{v_t}{c}}{1 + \frac{v_t}{c}} \frac{1 + \frac{v_r}{c}}{1 - \frac{v_r}{c}}}$$  \hspace{1cm} (34)
and defining $v = \frac{v_r - v_t}{1 - v_t v_r/c^2}$

$$l_2 = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} = \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$ \hspace{1cm} (35)$$

the input-output relation takes the form,

$$w(t) = s \left( \frac{t - \frac{d_0}{v_t - v_r} \left( 1 - \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} \right)}{\frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} \left( 1 - \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} \right)} \right)$$ \hspace{1cm} (36)$$

So far we have ignored the $1/d$ decay factor; however, the argument used in the classical analysis applies verbatim and thus the decay factor is the same; note, however, that it can be derived through the Minkowski diagram, as well, should one wish to do so.

Note again that the Doppler effect appears as a rescaling of the original signal, just like in the classical case.

4 The relativistic Doppler effect in electromagnetism

In this section, we work out the derivation of the Doppler effect through relativistic electrodynamics, and show that the result obtained is the same as the one obtained through the Minkowski diagram, as long as the various parameters are chosen to lie in the correct inertial frames. The derivation we are about to present follows similar principles as the derivation based on the Lorentz force [21].

Consider the situation where a transmitter $T$ and a receiver $R$ move in space along the same straight line at constant velocities. Without loss of generality, we are allowed to mount the coordinate system on $T$, so that $T$ is considered stationary at the origin of the coordinate system, while $R$ moves at a constant velocity $v$ along the $x$-axis (positive if moving to the right). We can further assume that $R$ is situated at $d_0$ at $t = 0$, so that its position in time is $x(t) = d_0 + vt$.

Assume now that the transmitter $T$ broadcasts an electromagnetic wave with perpendicular polarization to the line of motion (the $x$-axis), which, in addition, does not depend on $z$, and, without loss of generality, we may take
the electric field to be along the $y$-axis:

$$E(x, t) = E_y(x, t)\hat{y}$$

(37)

Maxwell’s equation \( \frac{\partial B}{\partial t} = -\nabla \times E \) yields readily that

$$B(x, t) = B_z(x, t)\hat{z}, \quad \frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x}$$

(38)

Regarding the propagation velocity of the wave, we take the general case where it is \( v_0 \leq c \), allowing, for example, for the presence of dielectric materials. In this case, we can take the following simple expression for the electric field:

$$E_y(x, t) = \frac{ad_0}{x} \exp[i(\omega t - kx)]$$

(39)

where the factor \( 1/x \) captures the decay of a spherical wave, and \( d_0 \) was inserted for dimensional reasons. Finally, we shall see below that the case of a constant amplitude \( a \) is indeed the only one required. We now get:

$$\frac{\partial B_z(x, t)}{\partial t} = \frac{ad_0}{x^2} \exp[i(\omega t - kx)] + ik\frac{ad_0}{x} \exp[i(\omega t - kx)]$$

(40)

and after a simple integration:

$$B_z(x, t) = \frac{ad_0}{i\omega x^2} \exp[i(\omega t - kx)] + k\frac{ad_0}{\omega x} \exp[i(\omega t - kx)] =$$

$$= \frac{ad_0}{\omega x} \left( \frac{1}{ix} + k \right) \exp[i(\omega t - kx)].$$

(41)

Note that \( k \) and \( v_0 \) are linked through the constitutional wave equation:

$$k = \frac{\omega}{v_0}$$

(42)

At this stage, we would like to re-express the fields in a new inertial coordinate system, the one mounted on \( R \) with axes parallel to the one mounted on \( T \). To achieve this we need to apply the Lorentz transforms. The primed quantities are associated with the new system:
\[ \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \]

\[ t = \gamma \left(t' + \frac{vx'}{c^2}\right) \]

\[ x = \gamma(x' + vt') \]

\[ E_{y'} = \gamma(E_y - vB_z) \]

\[ B_{z'} = \gamma \left(B_z - \frac{vEx}{c^2}\right) \]

\[ x' = \gamma(x - vt) = \gamma d_0 \Rightarrow x = \gamma(d_0 + vt') \]

Using the formulas above we obtain:

\[ \phi := \omega t - kx = \omega \gamma \left(t' + \frac{vx'}{c^2}\right) - \frac{\omega}{v_0} \gamma (d_0 + vt') = \]

\[ = \omega \gamma \left(1 - \frac{v}{v_0}\right) t' + \omega \gamma^2 d_0 \left(\frac{v}{c^2} - \frac{1}{v_0}\right) \]  

\[ (44) \]

while the electric field at \( R \) becomes:

\[ E_{y'}(t') = \frac{d_0 a}{\gamma d_0 + vt'} \exp \left\{ i \omega \gamma \left[ \left(1 - \frac{v}{v_0}\right) t' + \gamma d_0 \left(\frac{v}{c^2} - \frac{1}{v_0}\right) \right] \right\} \]

\[ \left[ 1 - \frac{v}{v_0} + i \frac{v}{\omega \gamma (d_0 + vt')} \right] \]

\[ (45) \]

The corresponding expression for the magnetic field can be obtained but is of no concern to us.

Let now \( s \) be a signal we wish to transmit. The Fourier transforms allows us to express \( s \) as a superposition of waves of constant amplitude:

\[ s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega) \exp(\imath \omega t) d\omega \]  

\[ (46) \]

and each wave propagates according to:

\[ \frac{d_0 \hat{s}(\omega)}{x} \exp[i(\omega t - kx)] \]

\[ (47) \]

where we used our convention about the decay corresponding to a spherical wave (with \( a = \hat{s}(\omega) \)).

What gets reconstructed then at \( R \), in \( R' \)'s frame, is:

\[ w(t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{d_0 \hat{s}(\omega)}{\gamma d_0 + vt'} \left[ 1 - \frac{v}{v_0} + i \frac{v}{\omega \gamma (d_0 + vt')} \right] \]

\[ (48) \]
\[
\exp \left\{ i \omega \gamma \left[ \left( 1 - \frac{v}{v_0} \right) t' + \gamma d_0 \left( \frac{v}{c^2} - \frac{1}{v_0} \right) \right] \right\} \Leftrightarrow \]
\[
w(t') = \frac{d_0 \left( 1 - \frac{v}{v_0} \right)}{\gamma d_0 + vt'} s \left( \gamma \left[ \left( 1 - \frac{v}{v_0} \right) t' + \gamma d_0 \left( \frac{v}{c^2} - \frac{1}{v_0} \right) \right] \right) - \frac{d_0 v}{\gamma (\gamma d_0 + vt')^2} \int_{-\infty}^{\infty} d\omega \frac{\hat{s}(\omega)}{i \omega} \exp \left\{ i \omega \gamma \left[ \left( 1 - \frac{v}{v_0} \right) t' + \gamma d_0 \left( \frac{v}{c^2} - \frac{1}{v_0} \right) \right] \right\} \]
\[
\Leftrightarrow w(t') = \frac{d_0 \left( 1 - \frac{v}{v_0} \right)}{\gamma d_0 + vt'} s \left( \gamma \left[ \left( 1 - \frac{v}{v_0} \right) t' + \gamma d_0 \left( \frac{v}{c^2} - \frac{1}{v_0} \right) \right] \right) - \frac{d_0 v}{\gamma (\gamma d_0 + vt')^2} s_1 \left( \gamma \left[ \left( 1 - \frac{v}{v_0} \right) t' + \gamma d_0 \left( \frac{v}{c^2} - \frac{1}{v_0} \right) \right] \right) \]
\[
= \frac{d_0 \left( 1 - \frac{v}{v_0} \right)}{\gamma d_0 + vt'} s \left( t' + \gamma d_0 \left( \frac{v}{c^2} - \frac{1}{v_0} \right) \right) . \quad (48)
\]

where \( s_1(u) = \int s(u) du \). It is clear, though, that the second term decays much faster than the first. There are two reasons for this: first, \( s_1 \) is, in general, the antiderivative of a vibration, since \( s \) is the transmitted signal, hence \( s_1 \) is bounded and likely to be approximately 0 in practical applications; second, the term containing \( s_1 \) decays much faster than the term containing \( s \). It follows that, when the ratio \( s_1/s \) is bounded and the distance between \( T \) and \( R \) is large (for example, when \( vt' \gg \gamma d_0 \)), we can write:

\[
w(t') \approx \frac{d_0 \left( 1 - \frac{v}{v_0} \right)}{\gamma d_0 + vt'} s \left( \gamma \left[ \left( 1 - \frac{v}{v_0} \right) t' + \gamma d_0 \left( \frac{v}{c^2} - \frac{1}{v_0} \right) \right] \right) =
\]
\[
= \frac{d_0 \left( 1 - \frac{v}{v_0} \right)}{\gamma d_0 + vt'} s \left( t' + \frac{\gamma d_0 \left( \frac{v}{c^2} - \frac{1}{v_0} \right)}{\sqrt{1 - \frac{v^2}{c^2}}} \right) . \quad (49)
\]

When the distance between \( T \) and \( R \) is small, the receiver is, in practice, in what is known as the near-field zone of the transmitter’s antenna, where the electromagnetic field is somewhat irregular; studying the Doppler effect there is outside the scope of this work (but see [17]).

5 Comparison and approximation

In this section, we study the relations between the three Doppler models we have developed so far. We also discuss the effectiveness of the approximation of the multiplicative frequency shift introduced by the Doppler effect by an additive one (commonly known as the “narrowband approximation”).
5.1 Compatibility with the classical Doppler effect

It is easy to see that the kinematic Doppler derivation (33) tends to the classical formula (10) in the limit as $c \to \infty$; since, in that case, $v_r, v_t, v \to 0$, and consequently $\sqrt{1 - \frac{v_r^2}{c^2}}, \sqrt{1 - \frac{v_t^2}{c^2}}, \sqrt{1 - \frac{v^2}{c^2}} \to 1$, (33) becomes

$$w(t) = s \left( t - \frac{d_0}{v_t - v_r} \left( 1 - \frac{1 - \frac{v_r}{c_0}}{1 - \frac{v_t}{c_0}} \right) \right)$$

which is the same as (10), as $\frac{d_0}{v_t - v_r} \left( 1 - \frac{1 - \frac{v_r}{c_0}}{1 - \frac{v_t}{c_0}} \right) = \frac{d_0}{c_0 - v_r}$. The bottom formula in (10) is obtained by repeating the derivation of Section 3.1 but with $T$ located to the right of $R$ this time. Therefore, the classical Doppler effect (10) is the limit of the relativistic Doppler effect as the ratio of the velocities of $T$ and $R$ to $c$ approaches 0, or, equivalently, if we accept that light propagates with infinite velocity ($c = \infty$).

The compatibility between the classical and the relativistic Doppler effects is known in the literature; unification formulas have been proposed from which both individual formulas can be derived [5].

5.2 Compatibility between the Minkowski diagram and the electromagnetic derivation

Formulas (33) and (49) exhibit the same scaling, and this becomes apparent when we express both in the same reference frame. Indeed, assume that $T$ and $R$ move at individual velocities $v_t$ and $v_r$ in the lab inertial frame, and that the wave transmitted by $T$ propagates at velocity $c_0$. The frame we used in the derivation above, mounted on $T$, moves at $v_t$ with respect to the lab frame, so the Lorentz transform establishes the following relations:

$$v_0 = \frac{c_0 - v_t}{1 - \frac{v_t}{c_0} \frac{v_t}{c_0}}$$

$$v = \frac{v_r - v_t}{1 - \frac{v_r v_t}{c^2}}$$

Upon the substitution of these formulas, the dilation factor becomes:
Table 1: A comparison of Doppler effect input-output models.

<table>
<thead>
<tr>
<th>Model</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>classic</td>
<td>( w(t) = \frac{d_0 (1 - v_t/c_0)}{d_0 - (v_t - v_r) l} \times \frac{t - \frac{d_0}{v_t - v_r}}{\frac{c_0 - v_t}{v_t - v_r}} )</td>
</tr>
<tr>
<td>relativistic (Minkowski)</td>
<td>( w(t) = \frac{d_0 (1 - v_t/c_0)}{d_0 - (v_t - v_r) l} \times \frac{t - \frac{d_0}{v_t - v_r} \sqrt{1 - \left(\frac{v_t}{c_0}\right)^2}}{\frac{1 - \frac{v_t}{c_0}}{\sqrt{1 - \left(\frac{v_t}{c_0}\right)^2}} \sqrt{1 - \left(\frac{v_t}{c_0}\right)^2}} )</td>
</tr>
<tr>
<td>relativistic (Minkowski, ( c_0 \rightarrow c ))</td>
<td>( w(t) = \frac{d_0 (1 - v_t/c)}{d_0 - (v_t - v_r) l} \times \frac{t - \frac{d_0}{v_t - v_r} \sqrt{1 + \frac{v_t^2}{c_0}}}{\frac{1 + \frac{v_t}{c_0}}{\sqrt{1 - \left(\frac{v_t}{c_0}\right)^2}} \sqrt{1 - \left(\frac{v_t}{c_0}\right)^2}} )</td>
</tr>
<tr>
<td>relativistic (E/M)</td>
<td>( w(t') = \frac{d_0}{\gamma d_0 + vt'} \times \left( \gamma \left[ (1 - \frac{v}{v_0}) t' + \gamma d_0 \left( \frac{v}{c_0} - \frac{1}{v_0} \right) \right] - \frac{d_0 v}{\gamma (\gamma d_0 + vt')^2} \right) \gamma \left[ (1 - \frac{v}{v_0}) t' + \gamma d_0 \left( \frac{v}{c_0} - \frac{1}{v_0} \right) \right] )</td>
</tr>
<tr>
<td>relativistic (E/M, approximate)</td>
<td>( w(t') = \frac{d_0}{\gamma d_0 + vt'} \times \left( \gamma \left[ (1 - \frac{v}{v_0}) t' + \gamma d_0 \left( \frac{v}{c_0} - \frac{1}{v_0} \right) \right] \right) )</td>
</tr>
</tbody>
</table>

\[
\sqrt{1 - \frac{v_t^2}{c_0^2}} = \frac{1 - \left( \frac{v_t - v}{c_0} \right)^2}{1 - \left( \frac{v_t}{c_0} \right)^2} = \sqrt{1 - \left( \frac{v_t c_0}{c_0^2} - \frac{v}{c_0} \right)^2} = \frac{v_t c_0 + c_0 c_t}{c_0^2} = \frac{\left( c_0 - v_t \right) \sqrt{1 + v_t^2 c_0^2 - v_t^2 c_0^2 + v_t^2 c_0^2}}{c_0 - v_t + \frac{v_t v_t}{c_0} - \frac{v_t v_t}{c_0^2}} = \frac{\left( c_0 - v_t \right) \left( 1 - \frac{v_t}{c_0} \right)}{\left( c_0 - v_t \right) \sqrt{1 - \frac{v_t^2}{c_0^2}}} (52)
\]

which is exactly the same as the result obtained through the Minkowski diagram.

### 5.3 Connection between the models

Table 1 shows the various formulas for the Doppler effect developed so far: the classical model, the relativistic model through the Minkowski diagram, its special case where \( c_0 = c \), and the relativistic model through relativistic electrodynamics. The relation between the exact and the approximate elec-
The electromagnetic relativistic model was explained at the bottom of Section 4; the relation between the electromagnetic and the Minkowski relativistic model was the subject of Section 5.2; further, the compatibility of the classical and the Minkowski relativistic model was proved at the bottom of Section 3.1. The relations between the different models are represented graphically in Figure 1.

5.4 Symmetry and absolute motion issues

As we pointed out earlier, a striking characteristic of the formulas in Table 1 is their lack of symmetry with respect to $v_t$ and $-v_r$: in other words, the effect of $T$ moving towards a stationary $R$ with speed $v_t = v$ is different than the effect of $R$ moving towards a stationary $T$ with absolute speed $v$ ($v_r = -v$). This, however, is not the case when $c_0 = c$: when the speed of signal propagation is the speed of light, it is impossible for either $T$ or $R$ to determine whether $T$ is moving towards a stationary $R$ or $R$ is moving towards a stationary $T$, due to relativistic effects [25, 16, 7, 6]. Furthermore, note that $v_t = v$ and $v_t = -v$ do not shift the frequency by the same amount; this difference remains true even when $c_0 = c$.

5.5 Multiplicative or additive frequency shift?

Consider the case of a single frequency signal $x(t) = \exp(i\omega t)$: ignoring the decay factors, the kinematic relativistic Doppler expression (33) becomes

$$w(t) = \exp\left(i\omega \frac{t - \frac{d_0}{v_t-v_r} \left(1 - \frac{c_0}{c_0 - v_t} \sqrt{\frac{1-(\frac{v_t}{c_0})^2}{\sqrt{1-(\frac{v_r}{c_0})^2}}}ight)}{1 - \frac{c_0}{c_0 - v_t} \sqrt{\frac{1-(\frac{v_t}{c_0})^2}{\sqrt{1-(\frac{v_r}{c_0})^2}}}}\right) = \exp(i\omega \lambda(t - \mu)) \tag{53}$$

while the classical Doppler expression (10) becomes

$$w(t) = \exp\left(i\omega \frac{t - \frac{d_0}{c_0-v_t} \left(1 - \frac{c_0}{c_0 - v_t} \sqrt{\frac{1-(\frac{v_t}{c_0})^2}{\sqrt{1-(\frac{v_r}{c_0})^2}}}ight)}{1 - \frac{c_0}{c_0 - v_t} \sqrt{\frac{1-(\frac{v_t}{c_0})^2}{\sqrt{1-(\frac{v_r}{c_0})^2}}}}\right) = \exp(i\omega \lambda'(t - \mu')) \tag{54}$$

where $\lambda, \lambda', \mu, \mu'$ are defined in the obvious way. In both cases, the Fourier transforms are a single $\delta$-function, “sitting” at $\lambda \omega$ and $\lambda' \omega$, respectively. The change of frequency from $\omega$ to $\lambda \omega$ can be construed as either multiplicative by a factor of $\lambda$ or additive by a frequency increment of $(\lambda - 1)\omega$; similarly for the other case. Hence, for a single frequency the multiplicative and the additive frequency shifts are interchangeable.

When, however, we look at a signal of the form $s(t) = \exp(i\omega_1 t) + \exp(i\omega_2 t)$
with $\omega_1 \neq \omega_2$, the situation is very different: (33) yields
\[ w(t) = \exp(i\omega_1\lambda(t - \mu)) + \exp(i\omega_2\lambda(t - \mu)) \] (55)
while (10) yields
\[ w(t) = \exp(i\omega_1\lambda'(t - \mu')) + \exp(i\omega_2\lambda'(t - \mu')). \] (56)

Let us focus on the former, as the latter is similar: we see that the Fourier transform consists of two $\delta$-functions “sitting” at $\lambda\omega_1$ and $\lambda\omega_2$, respectively. The multiplicative interpretation of the frequency change is still valid, as in both cases frequencies have been multiplied by a factor of $\lambda$; the additive interpretation, however, does not make sense in this case, as the two frequencies suffer different additive frequency shifts $(\lambda - 1)\omega_1$ versus $(\lambda - 1)\omega_2$).

The conclusion is that the Doppler effect is really a scaling phenomenon rather than a frequency translation, unless the signal is a pure tone. We can, however, have signals that are approximately pure tones, in the sense that their frequency content lies in an interval of the form $I = [\omega - \Delta\omega, \omega + \Delta\omega]$, where $\Delta\omega \ll |\omega|$. Then, the frequency content of the Doppler-shifted signal will lie within $[\lambda\omega - \lambda\Delta\omega, \lambda\omega + \lambda\Delta\omega]$, and, as a first approximation, we can write $\lambda\omega' \approx (\omega' - \omega) + \lambda\omega, \omega' \in I$: this is a manifestation of the well known “narrowband approximation” [49] in telecommunications, and we approximate the multiplicative shift of the entire frequency content as an additive shift around the multiplicative shift of a pre-chosen point, e.g. its midpoint or its beginning.

Let us consider the special case of (55) with $v_t = 0$ and $c_0 = c$. We get \[ \lambda = \sqrt{\frac{1 - v_n}{1 + v_n}}, \] where $v_n = \frac{v_r}{c}$ is the normalized receiver velocity, lying in $(-1, 1)$. The additive correction moving $\omega_1$ to $\lambda\omega_1$ is $\Delta\omega_1 = (\lambda - 1)\omega_1$, while, under this correction, $\omega_2$ is moved to $\omega_2^* = \omega_2 + \Delta\omega_1$. Therefore, the error in the approximation by $\omega_2^*$ of the true frequency where $\omega_2$ is moved at, namely $\lambda\omega_2$, is $e_2 = \lambda\omega_2 - \omega_2^* = (\lambda - 1)\Delta\omega$, where $\Delta\omega = \omega_2 - \omega_1$ is the bandwidth of the signal. We are more interested in the absolute relative error $\frac{|e_2|}{\omega_2}$, however:
\[ \epsilon = \frac{|e_2|}{\omega_2} = \left| 1 - \frac{1}{1 + v_n} \frac{|\Delta\omega|}{\omega_2} \right| \] (57)

To give a quantitative example, suppose our signal is wideband with a bandwidth of 8GHz and centered around $f_c = 5GHz$, so that $\omega_1 = 2\pi f_1 = 2\pi \cdot 1GHz$ and $\omega_2 = 2\pi f_2 = 2\pi \cdot 9GHz$, and that the receiver’s velocity is $v_r = 10m/s$: it follows that $\lambda \approx 0.99999996666667$. Considering the Doppler effect to be additive around $f_c$, we obtain the relative errors $e_2 \approx 1.4815 \cdot 10^{-8}$
Channel models of the Doppler effect

Figure 3: Plot of the absolute value of the relative error in the approximation of the multiplicative shift induced by the Doppler effect by an additive shift, as expressed in (57): for non-relativistic velocities of $R$ or when the signal transmitted is narrowband, the approximation is accurate.

and $\epsilon_1 \approx 1.3333 \cdot 10^{-7}$ for the new positions of $f_2$ and $f_1$ after the shift, respectively. While these relative errors appear small, in absolute terms, the upper Doppler frequency error is approximately 133Hz for both $f_1$ and $f_2$ (where the additive correction overestimates $f_2$ and underestimates $f_1$) and must be taken into account when designing a wideband communication system, for example.

Figure 3 plots (57). The contour corresponding to a specific value of $e$ is given by:

$$y = \frac{e}{1 - \sqrt{\frac{1-x}{1+x}}}, \quad y = \frac{\Delta \omega}{\omega_2}, \quad x = v_n$$

Using the approximation $\sqrt{\frac{1-x}{1+x}} \approx 1 - x$ when $x \ll 1$, we obtain $e \approx xy$; in other words, for small velocities, the error $e$ is also small and equal to the product of the velocity and the relative bandwidth. Another direct observation of (57) is that, for small relative bandwidth, $e$ is again small. For example, if we desire less than 1% error ($e \leq 0.01$), we need the product of the velocity and the relative bandwidth to be less than 0.01.

A nice and extensive discussion of the (non-relativistic) narrowband ap-
proximation can be found in [49].

6 Applications

The Doppler effect undeniably has many manifestations and applications in everyday life, and in engineering more generally [51, 1, 26]. To begin with, everyone is familiar with the change of pitch of the siren of an ambulance as it runs by in the street. The consequences of the Doppler effect, however, both classical and relativistic, are much more far reaching, not only on technological applications but also on the self-validation of the physical science itself.

There are several examples of applications of the relativistic Doppler effect:

- The most prominent application of the relativistic Doppler effect today is by far the Global Positioning System (GPS) [28], be it for civil or military use (e.g. weapons guidance [14]). The relativistic Doppler effect is used here for various calibrations [8], but most notably to synchronize the clocks of the satellites of the system [47], calculate the frequency shift of the electromagnetic waves due to the gravitational field of the earth [28], and precise velocity measurement [52]. As gravity is involved, and the motion of the satellites orbiting the earth is accelerated, the special theory of relativity (which is exclusively considered in this paper) is insufficient, and the general theory of relativity must be considered [13].

- The Doppler effect can be used for navigational purposes in space, where GPS is not available, relying on the transmissions by pulsars [2].

- As the Doppler effect is a fundamental consequence of the physical laws, and as it can be measured very accurately [22], it has been used to validate the physical theories themselves, in particular the theory of relativity [3, 39].

- Several astronomical simulation techniques employ the Doppler effect [33].

- Simulations are used to transform images they way they would look like if the observer had the opportunity to pass by them at relativistic speeds; they also rely on the (relativistic) Doppler effect [48].

- Noise present in quantum communication channels, due to the oscillation of electrons, can be reduced through the use of the Doppler effect [36].

- Precise velocity measurement, in an astronomical setting, is possible through the relativistic Doppler effect [35, 43].

Examples of applications of the classical Doppler effect are the following:
• The non-relativistic frequency shift induced by velocity is exploited by RADARs and SONARs for the determination of the velocity of a moving target [45, 43, 44, 19, 40]. Not only average motion but also micromotion can be detected (e.g. vibrations, spin etc.), which can convey information about the type of the vehicle tracked as well as the type of its engine [12]. It is even possible to measure the Doppler dilation of a wideband signal in order to determine the velocity [34].

• As we mentioned previously, a recent application of the Doppler effect in communications is in time-varying channels, both in the narrowband (time-frequency) or the wideband (time-scale) context [42, 18, 41, 23, 32, 30, 36].

• Mapping and surveying RADARs depend on the Doppler effect as well [15].

• Finally, the Doppler effect has biomedical applications, for example in the detection of microemboli in flowing blood [27].

7 Conclusions

We set out in our endeavor to compile the present comprehensive description of the Doppler effect as there seemed to be no unifying document a) laying emphasis on the features of the Doppler effect that are of interest in engineering, and b) deriving the various forms of the Doppler effect from first principles and explaining the relations that connect them. In other words, our intention was to compile a tutorial on the Doppler effect; but, more than that, we viewed the Doppler effect as a channel, namely as an input-output relation, because this is what is mostly needed in engineering, while this aspect is not extensively covered in the various classical treatises in physics.

We derived the input-output model for the Doppler effect in both a classical and a relativistic setting. The relativistic model, in particular, was derived in two independent ways, one kinematic and based on the Minkowski diagram, the other through relativistic electrodynamics. The two derivations yielded consistent mathematical channel models. Furthermore, the relativistic model was shown, as is well known, to give rise to the classical model when the speeds of the transmitter and the receiver are much smaller than the speed of light, or, equivalently, when the light is considered to propagate at infinite speed.

We considered the question whether two moving bodies can determine which is the receiver and which the transmitter, and verified that this is not possible in the non-relativistic setting, while it is, in general, in the relativistic setting, unless the wave propagation speed equals the speed of light in the vacuum $c$. Furthermore, we examined the nature of the frequency shift
the Doppler effect induces, and we reaffirmed that, although in principle it is undoubtedly multiplicative, it can be considered to be additive with a good degree of approximation under the “narrowband assumption”, for most practical purposes.

We ended our study with a list of applications where the Doppler effect plays a role, whereby it is made clear that, in several modern applications, the relativistic Doppler effect is not after all as “exotic” as one might be inclined to think, but is really necessary in order to achieve the desired degree of accuracy. The most characteristic such case, and, at the same time, most important application of the Doppler effect today, is GPS.

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