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Phase Synchronization of Van der Pol-Duffing Oscillators Using Decomposition Method

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Abstract

In this paper the phase of two Duffing-Vanderpol oscillators with different initial conditions are synchronized using an alternative force added to the slave system. The master and slave systems are solved by applying Adomian's decomposition methodphas and the solutions and their phase differences are plotted to show the effectiveness of the method.

Keywords: Duffing-Vanderpol oscillator, Master, Slave, Phase synchronization, Adomian's decomposition method

1 Introduction

Chaos synchronization has received increasing attention as an interesting phenomenon of practical importance in coupled chaotic oscillators, due to its theoretical challenge and its great potential applications in secure communication, chemical reaction, biological systems and so on [1]. Various types of synchronization including complete synchronization (CS), [2, 3] generalized ynchronization (GS), [4, 5, 6] and phase synchronization (PS), [7, 8, 9, 10, 11] have been studied.

Recently many researchers worked on chaos synchronization by considering Duffing-Van der Pol (DVP) oscillator problem. A.N. Njah and U.E. Vincent [12] presented chaos synchronization between single and double wells DVP oscillators with U4 potential based on the active control technique. They used numerical simulations too present and verify the analytical results. Dibakar

Ghosh, et. al presented a detailed investigation performed about the various zones of stability for delayed DVP system, which in term shows the specific role of delay in the formation of the attractor [13]. In the study of phase synchronization the machinery of empirical mode decomposition (EMD) analysis is adapted and lastly maximal Lyapunov exponent is computed as a verifying criterion. Rene Yamapi and Giovanni Filatrella [14] considered the synchronization dynamics of coupled chaotic DVP systems.

There are many other recent reports on the synchronization of DVP oscillators. H.G. Enjieu Kadjia and R. Yamapi [15] considered the general synchronization dynamics of coupled DVP oscillators and examined the linear and nonlinear stability analysis on the synchronization process through the Whittaker method and the Floquet theory in addition to the multiple time scales method. Our work mainly focused on the systemes considered in [15].

In this paper we consider the phase synchronization dynamics of coupled DVP oscillators applying an alternative excitation force to the slave system. The results are examined by Adomian's decomposition method(ADM) that is a numerical technique for solving functional equations developed at the beginning of 1980s, by George Adomian [16, 17]. ADM gives the solution of equations as an infinite series usually converging to an accurate solution. Concrete application to different functional equations are given by G. Adomian and his collaborators [18, 19]. There are also some papers by Cherruault and his collaborators [20, 21] proving the convergence of the method for different functional equations.

2 The problem

The classical DVP oscillator appears in many physical problems and is governed by the nonlinear differential equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x + \alpha x^3 = 0; \quad x(0) = x_0, \ \dot{x}(0) = \dot{x}_0, \tag{1}$$

where the overdot represents the derivative compared to time, μ and α are two positive coefficients. It describes electrical circuits and has many applications in science, engineering and also displays a rich variety of nonlinear dynamical behaviors. It generates the limit cycle for small values of μ , developing into relaxation oscillations when μ becomes large which can be evaluated through the Lindsted's perturbation method [11]. One particular characteristic in DVP model is that its phase depends on initial conditions. Therefore, if two DVP oscillators run with different initial conditions, their trajectory will finally circulate on the same limit cycle with different phases ϕ_1 and ϕ_2.

Basically, chaos synchronization problem can be formulated as follows. Given a chaotic system, which is considered as the master (or driving) system, and another identical system, which is considered as the slave (or response) system, the aim is to force the response of the slave system to synchronize the master system in such a way that the dynamical behaviors of these two systems be identical after a transient time.

The objective of the synchronization in this paper is to phase-lock the oscillators (phase synchronization) so that $\phi_2 - \phi_1 = 0$. We show the master and slave systems by the variables x and y respectively, and choose $\alpha = 0.01, \mu = 0.1$ to obtain

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x + \alpha x^3 = 0; \quad x(0) = 2,\ \dot{x}(0) = 0 \tag{2}$$

$$\ddot{y} - \mu(1 - y^2)\dot{y} + y + \alpha y^3 = 0; \quad y(0) = 2.5,\ \dot{y}(0) = 0.5 \tag{3}$$

The objective is coupling the slave to the master system in such a way that $\lim_{t\to\infty} \phi_2 - \phi_1 = 0$.

3 Solution methods

3.1 Approximate periodic solution

H. G. Enjieu Kadji and R. Yampani considered an approximate periodic solution of (1), by the form $x(t) = x_0(\tau) + \mu x_1(\tau) + \mu^2 x_2(\tau) + \ldots$ where $x_i(\tau)$s are periodic functions of period 2π [18] and obtained the solution as

$$x(t) = Acos\omega t + \frac{\alpha}{4}cos3\omega t + \mu(\frac{3}{4}sin\omega t - \frac{1}{4}sin3\omega t) + O(\mu^2) \tag{4}$$

with

$$A = 2 - \frac{1}{2}\alpha, \quad \omega = 1 + \frac{3}{2}\alpha - \frac{27}{16}\alpha^2 - \frac{1}{16}\mu^2 + O(\mu^2) \tag{5}$$

The plot of this solution for $0 \leq t \leq 3\pi$ and $\alpha = 0.01,\ \mu = 0.1$ is presented in figure 1.

3.2 ADM to a system of ODE

Most applied problems are described by second-order or higher-order differential equations. A differential equation of order n, can be written as

$$x^{(n)}(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)); \qquad n \geq 2 \tag{6}$$

with $x^{(n)}(0) = x_{n0}$ as initial conditions. Using $x^{(i)}(t) = y_{i+1}(t)$ this equation converts to a system of first-order ordinary differential equations as

$$\begin{cases} y_1'(t) = f_1(t, y_1(t), \ldots, y_n(t)) \\ y_2'(t) = f_2(t, y_1(t), \ldots, y_n(t)) \\ \vdots \\ y_n'(t) = f_n(t, y_1(t), \ldots, y_n(t)) \end{cases} \tag{7}$$

with the initial conditions

$$y_i(0) = y_{i0}, \qquad i = 1, 2, \ldots, n, \tag{8}$$

where each equation represents the first derivative of a function as a mapping depending on the independent variable x, and n unknown functions $f_1, \ldots, f_n$. The i-th equation of system (7) can be represented in common form

$$Ly_i(t) = f_i(t, y_1(t), \ldots, y_n(t)), \tag{9}$$

in which the operator L is the first order derivative compared to t. Considering $f_i = L_i + N_i$ with L_i and N_i as linear and nonlinear operators respectively and applying the inverse of the L operator, $L^{-1}(.) = \int_0^t (.)dt$, the equation (9) can be written as

$$y_i(t) = y_i(0) + \int_0^t (L_i(t, y_1(t), \ldots, y_n(t)) + N_i(t, y_1(t), \ldots, y_n(t)))dt, \tag{10}$$

which is called canonical form in Adomain schema. In order to apply the Adomain decomposition method, we let

$$y_i(t) = \sum_{j=0}^{\infty} y_{ij}(t), \tag{11}$$

$$L_i(t, y_1(t), \ldots, y_n(t)) = \sum_{k=1}^{n} \sum_{j=0}^{\infty} a_k y_{kj}(t), \tag{12}$$

$$N_i(t, y_1(t), \ldots, y_n(t)) = \sum_{j=0}^{\infty} A_{ij}(y_{i0}, \ldots, y_{ij}), \tag{13}$$

where $a_k,\ k = 0, 1, \ldots, n$ are scalers and $A_{ij},\ j = 0, 1, \ldots, n$ are the Adomian polynomials of $y_{i0}, \ldots, y_{in}$ given by

$$A_{ij} = \frac{1}{j!} \left[\frac{d^j}{d\lambda^j} N(\sum_{j=0}^{\infty} \lambda^i y_{ij}) \right]_{\lambda=0}. \tag{14}$$

Substituting (11-13) into (10) yields

$$\sum_{j=0}^{\infty} y_{ij}(t) = y_i(0) + \int_0^t \sum_{k=1}^{n} \sum_{j=0}^{\infty} a_k y_{kj}(t)\ dt + \int_0^t \sum_{j=0}^{\infty} A_{ij}(y_{i0}, \ldots, y_{ij})\ dt, \tag{15}$$

by which, we define

$$\begin{cases} y_{i0}(t) = y_i(0) & . \\ y_{i,j+1}(t) = \displaystyle\int_0^t \sum_{k=1}^{n} a_k y_{kj}(t)dt + \int_0^t A_{ij}(y_{i0}, \dots, y_{in})dt, & j = 0, 1, \dots \end{cases} \tag{16}$$

In practice, all terms of the series $y_i(t) = \sum_{m=0}^{\infty} y_{im}(t)$, can not be determined. So we use an approximation of the solution calculating following truncated series

$$\varphi_{ik}(t) = \sum_{j=0}^{k-1} y_{ij}(t), \qquad \text{with} \lim_{k\to\infty} \varphi_{ik}(t) = y_i(t). \tag{17}$$

Our procedure leads to a system of second kind Volterra integral equations, so by [22], the convergence of the method is proved.

3.3 Application

Consider the master equation (2). To solve this equation by the Adomian's schema, we first convert it to a system of first order differential equations. Choosing $y_1(t) = x(t)$ and $y_2(t) = \dot{x}(t)$ we obtain

$$\dot{y}_1(t) = y_2(t) \tag{18}$$
$$\dot{y}_2(t) = \mu y_2(t) - \mu y_1^2(t) y_2(t) - y_1(t) - \alpha y_1^3(t) \tag{19}$$

with the initial conditions $y_1(0) = 2$ and $y_2(0) = 0$. Applying ADM procedure and considering (17) we obtain

$$y_1(t) = x(t) \approx 2 - 1.072t^2 - 0.035t^3 + 0.088t^4 + 0.0035t^5 + \dots \tag{20}$$

The plot of (20) is presented in the figure 1.

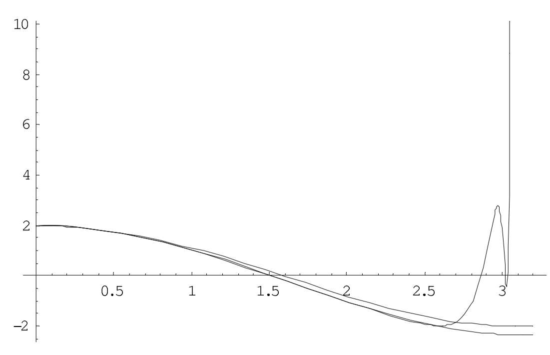

Figure 1. The plots of master system solution by three methods introduced in section 3

Figure 1 shows that the solution of ADM for VDPD equation diverges for $t > 1$.

3.4 Modified Adomian's decomposition method

As the solution in a long time interval is needed to synchronization method takes place, we try to modify the Adomian polynomials in such a way that find a long time stable solution. Our considerations in different nonlinear equations with the periodic solution showed that replacing the summation $\sum_{j=0}^{\infty} \lambda^i y_{ij}$ in the formula (14) with a periodic function by proper arguments and coefficients tends to the desired solution. Specially for the master system, we choose $cos(\lambda/10) + sin(\lambda/10)$ and follow the procedure of ADM introduced in section 3.2 to obtain

$$y_1(t) = x(t) \approx 2.5 + 0.5t - 1.297t^2 + 0.126t^3 + 0.105t^4 + 0.0084t^5 + \dots \quad (21)$$

The plot of $x(t)$ in figure 1 shows that the solution obtained by using the modified method has a wider range of convergence than standard ADM.

4 Master and slave solutions

As stated in section 2, the phase of DVP depends on initial conditions, so if two DVP oscillators run with different initial conditions, their trajectory will finally circulate on the same limit cycle with different phases. We use ADM to solve the master and slave systems and obtain related solutions as plotted in figure 2.

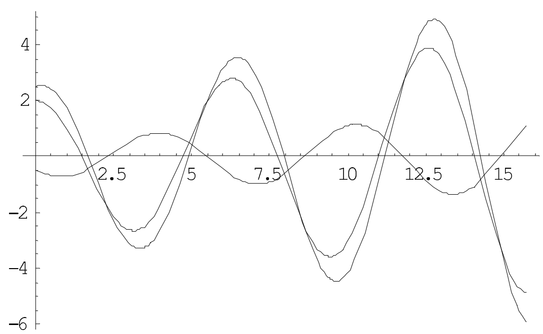

Figure 2. The asynchronous states of master and slave systems and their difference

In the figure 2, the master and slave responses and their difference are plotted as a function of time. This figure shows that the systems are asynchronous due to different initial conditions.

5 Synchronization

To synchronize the slave with the master system, we add a cosine excitation force to the right hand side of the slave system (3), so we obtain the excited slave system

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x + \alpha x^3 = 2.5cos(2t - \pi/2); \quad x(0) = 2.5, \ \dot{x}(0) = 0.5, \quad (22)$$

Converting to a system of first order differential equations and following the procedure introduced in section 2, we obtain the solutions of (22) as follow

$$\dot{y_1}(t) = x(t) \approx 2.5 + 0.035t - 1.328t^2 + 0.050t^3 + 0.109t^4 + 0.0046t^5 + \ldots \quad (23)$$

The solutions of the master and synchronized systems for $t \in [0, 3\pi]$ are plotted in Figure 3.

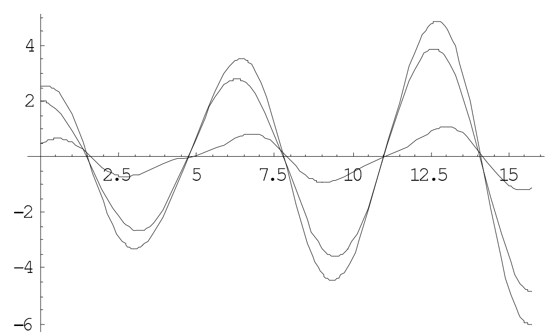

Figure 3. The synchronized states of slave and the master systems with their differences

In the figure 3, the phase synchronization of the master and slave systems can be seen.

To show the phase synchronization, analytically, we need to define the instantaneous phase of systems and their differences before and after synchronization. Toward this goal, consider a solution by the form $x(t) = A(t)cos(\omega t + \phi_0)$. Calculating $x'(t)$ and $x'(t)/x(t)$, the instantaneous phase, $(\omega t + \phi_0)$, obtains as follow

$$\phi(t) = \omega t + \phi_0 = arctan[\frac{1}{\omega}(\frac{A'(t)}{A(t)} - \frac{x'(t)}{x(t)})] \quad (24)$$

The study show that for our purpose a linear choose of $A(t)$ is valid. Choosing $A(t) = 2 + 0.15t$ for the master system and $A(t) = 2.5 + 0.18t$ for the slave system, and using the solutions of the systems, phase difference of the slave and synchronized systems with the master system, obtains as plotted in the figure 4.

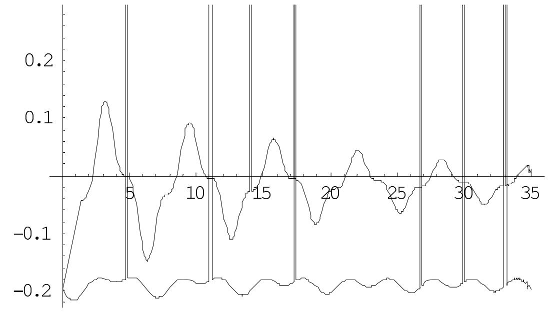

Figure 4. The phase differences of the slave and synchronized systems with the master system

Figure 4 show that the slave system has a negative periodic phase difference with the master system. After imposing the excitation term to the slave system, the phase difference tends to zero. As $x(t)$ is a $n-th$ order polynomial, from (23) one can conclude that the phase difference of the synchronized system tends to zero as $t \to \infty$, so we conclude that our synchronization method is stable.

6 Conclusions

Our study show that a cosine excitation force added to the DVP slave system, tends to phase synchronization between it and a similar system with different initial conditions as master system. We use Adomian's decomposition method to verify the method and show it's efficiency. This method gives the solutions as polynomials that make it easy to plot the curves and calculate the phases of the systems. Due to the simplicity of this method, we suggest this method as an suitable approach for nonlinear synchronization problems.

References

[1] G. Chen, X. Dong, From Chaos to Order: Methodologies, Perspectives and Applications, World Scientific, Singapore, 1998.

[2] J. H. Xiao, G. Hu, and Z. L. Qu, Phys. Rev. Lett. **77** (1996), 4162.

[3] Y. Liu and P. Davis, Phys. Rev. E **61** (2000), 2176.

[4] Z. G. Zheng and G. Hu, Phys. Rev. E **62** (2000), 7882.

[5] L. Pecora and U. Parlitz, Phys. Rev. Lett. **76** (1996), 1816.

[6] K. Pyragas, Phys. Rev. E **54** (1996) ,R4508.

[7] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76** (1996), 1804.

[8] J. Y. Chen, K. W. Wong, and J. W. Shuai, Chaos,**12** (2002), 100.

[9] D. Maza, A. Vallone, H. Mancini, and S. Boccaletti, Phys. Rev. Lett. **85** (2000), 5567.

[10] A. Pikovsky, M. Rosenblum, and J. Kurths, A Universal Concept in Nonlinear Sciences, Cambridge University Press, Cambridge, 2001

[11] G. Osipov, D. L. Valladares, and C. S. Zhou, Phys. Rep. **366** (2002), 1.

[12] A.N. Njah, U.E. Vincent, Chaos, Solitons and Fractals **37** (2008), 1356-1361.

[13] Dibakar Ghosh, A. Roy Chowdhury, Papri Saha. Communications in Nonlinear Science and Numerical Simulation **13** (2008), 790-803.

[14] Rene Yamapi, Giovanni Filatrella, Communications in Nonlinear Science and Numerical Simulation **13** (2008), 1121-1130.

[15] H.G. Enjieu Kadjia, R. Yamapi, Physica A **370** (2006), 16-328.

[16] G.Adomian, Nonlinear Stochastic Systems Theory and Applications to Physics, Kluwer, 1989.

[17] G.Adomian, Solving Frontier Problems of physics: The Decomposition Method, Kluwer, 1994.

[18] G.Adomian, J. Comput. Appl. Math. **11** (1984), 225-230.

[19] Y. Cherruault, G.Adomian, decomposition method: a new proof of convergence, Mathl. Comput. Modelling **18 (12)** (1993), 103-106.

[20] Y. Cherruault, convergence of Adomin,s method, Kybernetes **9 (2)** (1988), 31-38.

[21] Y. Cherruault, Y. Saccomandi and B. Some, Mathl. Comp. Modelling, **16 (2)** (1992), 83-93.

[22] E. Babolian, J. Biazar, Solution of a system of linear Volterra equations by Adomian decomposition method, Far East J. Math. Sci. **7 (1)** (2001), 1725.

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