

Heterotic $D = 2(1/3, 0)$ Susy Models

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Abstract

Following our previous work on fractional spin symmetries (FSS) [2, 4] and references therein, we build here a superspace representation of the heterotic $D = 2(1/3, 0)$ superalgebra and derive a field theoretical model invariant under this symmetry.

1 Introduction

We focus here to contribute, once again, to the study of fractional spin symmetries (FSS) [1, 2, 3, 4], a subject that emerges remarkably in coincidence with the growing interest in high energy and condensed matter physics through quantum field theory [5], conformal symmetries [6] and string theory [7].

In previous works, [2, 4], we worked out the conformal field representations of the TIM and TPM of [8, 9, 10, 11] to any arbitrary value of the spins $s = 1/k; k = 2, 3, \dots$. This is a particular realization showing that the $D = 2$ spins $1/2$ and $1/3$ theories are really the two leading examples of a more general two dimensional spin $1/k$ supersymmetric theories.

Presently, this feature motivates us to develop a superspace formulation for these kind of theories. Indeed, focusing our attention to the $D = 2(1/3, 1/3)$, we will construct first a superfield representation of the heterotic spin $1/3$ algebra. We will built also a Lagrangian invariant under this symmetry. The basic actors are the two dimensional fields of spin $0, 1/3$ and $2/3$. There, we will show that the obtained model admits in fact a spin $4/3$ superconformal symmetry generated by a spin $4/3$ conserved current $G_{4/3}(z)$ in addition to the usual spin two conformal current $T_2(z)$. The field realization of these currents as well as unitarity are discussed.

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2 The Heterotic $D = 2(1/3, 0)$ Model

We Start by defining the $D = 2(1/3, 0)$ supersymmetric algebra as the set of operators generated by $Q_{1/3}^-$, $Q_{1/3}^+$ and P satisfying:

$$Q^{-3} = P^- \quad (1)$$

$$Q^{+3} = P \quad (2)$$

where we have dropped out the values of spin for reasons of simplicity. This superalgebra is invariant under two kinds of discrete symmetries: First the Z_3 symmetry operating as:

$$Q^+ \longrightarrow qQ^+ \quad (3)$$

$$Q^- \longrightarrow \bar{q}Q^- \quad (4)$$

$$P = q^3 P = \bar{q}^3 P, \quad (5)$$

where $q^3 = q^{-3} = 1$. Second, the Z_2 symmetry, generated by the charge conjugation operator C , acting on Q^\pm and P as:

$$CQ^\pm C^{-1} = Q^\mp \quad (6)$$

$$CP = PC \quad (7)$$

Since eqs (1) and (2) are interchanged under the Z_2 -symmetry, we shall focus hereafter our attention on one equation only say eq(1). Hermiticity of P is then lost. We shall forget about this physical requirement for the moment. Later on we will show how this basic feature may be restored. To construct field theoretical models exhibiting the algebra eq. (1) as a symmetry, we shall proceed by analogy with the $D = 2(1/2, 0)$ supersymmetric theory by introducing the left spin 1/3 superspace (z, θ^+) where $\theta^+ = \theta_{-1/3}^+$ is a parafermionic variable carrying plus one Z_3 -charge and a spin $s = -1/3$ and obeying:

$$\theta^3 = 0, \quad (8)$$

$$\theta^+ z = z\theta^+ \quad (9)$$

The objects θ such that $\theta^k = 0, k \geq 2$ are some how mysterious since they are not well common among the c -numbers. For $k = 2$, θ is just a Grassmann variable often used in the superspace formulation of supersymmetric theories [12]. For $k > 2$, however, such objects are new in the sense that they were not used previously. To our knowledge, similar quantities obeying higher non linear constraints were postulated in [13]. There, they were used in the interpretation of the Sine Gordon integrable models as a $N = 2$ supersymmetric Landau-Ginsburg theory.

We shall not discuss here the space of solutions of these constraints although we shall suppose that such space exist and is non empty. Various motivations in favor of this assumption may be quoted. In addition to technical arguments, there are also physical indications supporting this assumption. One of which is the existence of models exhibiting fractional spins symmetries [14]. The latter's are generated by spin $s = \frac{1}{k}$ charge operators among which the two dimensional $s = \frac{1}{2}$ supersymmetric algebra is just the leading example. As for the Bose-fermi local theory, two dimensional consistent fractional spin supersymmetric theories are expected to exist and wait to be discovered. Nevertheless, as far as the constraint equation $\theta^k = 0$ is considered, solutions can be worked out. We give hereafter two natural ones. The first solution is given by $k \times k$ nilpotent matrices.

Taking for instance θ as $\varphi\Lambda_1$, where φ is a c -number and where Λ_1 and more generally Λ_n ; $-k < n < k$ are $k \times k$ matrices, expected in terms of the $e_{i,j}$, $1 \leq i, j \leq k$ matrix generators, having one at the site (i, j) and zero elsewhere, as $\Lambda_n = \sum_{1 \leq i \leq k} e_{i, i+n}$. These matrices obey among other properties the identity $(\Lambda_1)^p = \Lambda_p$ for $p < k$ and $(\Lambda_1)^p = 0$ for $p \geq k$ so that $\theta^k = \varphi^k \Lambda_k = 0$ [16]. The second solution of $\theta^3 = 0$ is obtained by using two Grassmann variables $\psi_{1/3}^+$ and $\eta_{2/3}^-$ allowing to rewrite the equation in the following linearized form:

$$\psi^+ \eta^- + \eta^- \psi^+ = 0 \tag{10}$$

$$\psi^+ \psi^+ + \psi^+ \psi^+ = 2\eta^- \tag{11}$$

where we have dropped out the index of the spin. In forthcoming works, we shall explore this kind of solution and its relation with $N = 2$ supersymmetry. Under our hypothesis, the superspace realization of eq.(1) generalizing the usual supersymmetric derivative may be worked out by using covariance and dimensional arguments. We find

$$D^- = \partial/\partial\theta^+ + \theta^{+2}\partial/\partial z \tag{12}$$

$$Q^- = (1 + q)^{1/3} D^- \tag{13}$$

$$P = \partial/\partial z \tag{14}$$

where $\partial/\partial\theta^+$ should be understood as a q -deformed, see [2] for more details. To check that these operators form indeed a differential representation of the algebra eq.(1), we first calculate the square of D^- . This is a spin $2/3$ object carrying a Z_3 charge $n = 1 \pmod{3}$. It reads as:

$$D^{-2} = \partial^2/\partial\theta^{+2} + (1 + q)\theta^+ \partial/\partial\theta^+ \partial/\partial z + (1 + q^2)\theta^{+2} \partial^2/\partial\theta^{+2} \partial/\partial z \tag{15}$$

In deriving this relation we have used the following basic property of the deformed q -derivation

$$\partial/\partial\theta^+(\theta^{+2}) = (1 + q)\theta^+ + q^2\theta^{+2}\partial/\partial\theta^+, \tag{16}$$

where $q = \exp(2i\pi s)$, $s = 1/k$, k positive integer. This is a general identity valid for any value of the spin of the parafermions parameter θ^+ . For a spin $s = 0$ parameter $\theta^+ = x$, the deformation parameter q is equal to one and then eq(16) reduces to $[\partial/\partial x, x^2] = 2x$ or also $[\partial/\partial x, x] = 1$. For a spin $s = 1/2$ parameter q i.e. a Grassmann variable verifying $q^2 = 0$, eq(16), reads as $[\partial/\partial\theta, \theta^2] = 0$ since $q = -1$. An equivalent identity using the anticommutator is $\{\partial/\partial\theta, \theta\} = 1$. More generally, using the definition of q -deformation of the derivative $\partial/\partial\theta^+$, we have on one hand

$$ad\partial/\partial\theta^+(\theta^+)^n = \partial\theta^{+n}/\partial\theta^+ - q^n(\theta^{+n})\partial/\partial\theta^+ \tag{17}$$

and on the other hand by using the above mentioned commutation rules:

$$ad\partial/\partial\theta^+(\theta^+)^n = (1 + q + q^2 + \dots + q^{n-1})\theta^{+n-1} - q^n\theta^{+n}\partial/\partial\theta^+ \tag{18}$$

Making appropriate choices of the value of the spin, taking into account the constraints, one recovers the known results. We turn now to complete the proof of the equation $D^{-3} = (1 + q)P$. Repeating the same analysis, we find:

$$D^{-3} = (1 + q)\partial/\partial z + \partial^3/\partial\theta^{+3} + (1 + q + q^2)[(1 + q)\theta^+\partial/\partial\theta^+ + \theta^{+2}\partial^2/\partial\theta^{+2}]\partial/\partial z \tag{19}$$

The first term of the r.h.s. of this equation is the energy momentum vector up to the coefficient $(1 + q)$. The remaining terms vanish individually either by using the identity $\theta^{+3} = 0$ or also by help of the property $(1 + q + q^2 = 0)$, the sum of all roots of $q^M = 1$ is identically zero.

Superfields $\phi_r^n(z, \theta^+)$ are superfunction defined on the superspace (z, θ^+) carrying fractional values of the spin. In our present case, the allowed spin values of r are multiples of $1/3$. Moreover, because of the nilpotency property of the variable q , the superfield ϕ_r may be expanded as follows

$$\phi_r^n = \varphi_r^n + \theta^+\psi_{r+1/3}^{n-1} + \theta^{+2}\chi_{r+2/3}^{n-2} \tag{20}$$

Under a spin $1/3$ supersymmetric infinitesimal transformation $\delta\theta^+ = \epsilon^+$, this superfields varies as $\delta\phi_r = \epsilon^+D^-\phi_r$. In terms of the component fields, we have:

$$\delta\phi_r^n = \epsilon^+\psi_{r+1/3}^{n-1} \tag{21}$$

$$\delta\psi_{r+1/2}^{n-1} = \epsilon^+\chi_{r+2/3}^{n-2} \tag{22}$$

$$\delta\chi_{r+1/3}^{n+1} = \epsilon^+(1 + q)\partial\varphi_r^n \tag{23}$$

where we have used the identity $\chi_{r+2/3}^{n-2} = \chi_{r+2/3}^{n+1}$. From these equations, one learns at least two things: 1. Any irreducible representation \mathfrak{R} of the

algebra eq.(1) is three dimensional. The spin values of its field components are $s = 0, 1/3$ and $2/3(mod\ 1/3)$. Their Z_3 charges n are $n = 0, \pm 1$ modulo three. Note that as in spin $D = 2(1/2, 0)$ supersymmetric representation theory, this ensures that the trace of spin $1/3$ supersymmetric number operator q^F on \mathfrak{R} vanishes identically since

$$Tr(q^F) \sim (1 + q + q^2) \tag{24}$$

The second thing we would like to note is that the highest component of the expansion of ϕ_r eq.(20) transforms as a total derivative under a spin $1/3$ supersymmetric transformation see eqs.(21-23). Supersymmetric invariant actions S are then defined as:

$$S \sim \int d^2z d^2\theta^+ L^- \tag{25}$$

where the superlagrangian L^- is required to have $1/3$ and 1 as left and right scale dimensions respectively. It should carry also a minus one Z_3 charge. As an example, we may take the non hermitean lagrangian L^- as:

$$L^- \sim (D^- \phi_1^n) \bar{\partial} \phi_2^{-n}, \quad n = 0, \pm 1, \tag{26}$$

where $\bar{\partial} = \partial/\partial\bar{z}$ and where ϕ_1^n and ϕ_2^{-n} are non hermitean superfields of charges n and $-n$ respectively. Note that superfields given by eq.(20) carry in general Z_3 charges. This is easily seen by remarking that the representation eq. (12, 13, 14) of the algebra eq.(1) admits a Z_3 symmetry acting as

$$\theta^+ \rightarrow q \theta^+ \tag{27}$$

$$z \rightarrow q^3 z \tag{28}$$

$$D^- \rightarrow \bar{q} D^- \tag{29}$$

$$P \rightarrow \bar{q}^3 P \tag{30}$$

The Z_3 -symmetry involved here is generated by the group element $q = exp(2i\pi/3)$. This is a discrete subgroup of the continuous $U(1)$ group of phases $U(a) = exp(ia)$; $\alpha \in [0, 2\pi[$. For $\alpha = \pi$, one obtains the Z_2 -symmetry of the spin $1/2$ supersymmetric algebra. For a arbitrary, one has the full $U(1)$ symmetry of the $N = 2U(1)$ theory [16]. We shall examine the analogy between the $\phi_{1,3}$ deformation of the TPM and $N = 2$ supersymmetric Z_3 invariant models later. Using eqs.(12, 13, 14, 20) and integrating with respect to θ^2 , the field components Lagrangian L reads as:

$$L \sim \partial\varphi^- \bar{\partial}\bar{\varphi}^+ + \bar{q}\psi_{1/3}^+ \bar{\partial}\bar{\chi}_{2/3}^- - q\chi_{2/3}^0 \bar{\partial}\bar{\psi}_{1/3}^0 \tag{31}$$

where we have used the commutation rules $\psi^+\theta^+ = q\theta^+\psi^+$ and $\chi^0\theta^+ = q^2\theta^+\chi^0$. In this equation, we have chosen $n = -1$ as a Z_3 charge of the superfield ϕ_r as suggested by eq.(16). The equations of motion of the free fields are solved by

$$\varphi = \varphi(z) + \varphi(\bar{z}), \tag{32}$$

$$\bar{\psi}^0 = \bar{\psi}^0(z), \tag{33}$$

$$\psi^+ = \psi^+(z), \tag{34}$$

$$\bar{\chi}^- = \bar{\chi}^-(z), \tag{35}$$

$$\chi^0 = \chi^0(z), \tag{36}$$

In addition to its manifest Z_3 -symmetry, the above non hermitean lagrangian admits a global spin 1/3 supersymmetric invariance generated by the conserved charge $Q_{1/3}^-$

$$Q_{1/3}^- = \int dz G_{4/3}^- + d\bar{z} G_{-2/3}^- \tag{37}$$

where $G_{4/3}^-$ and $G_{-2/3}^-$ are fractional spin currents satisfying the usual conservation law namely

$$\bar{\partial}G_{4/3}^- + G_{-2/3}^- = 0 \tag{38}$$

Using the transformation laws eqs.(21, 22, 23) and following the Nother method, one may calculate explicitly these currents. We find that $G_{-2/3}^- = 0$ and

$$G_{4/3}^- = \partial\varphi^- \bar{\psi}_{1/3}^0 - q\psi_{1/3}^+ \bar{\partial}\bar{\varphi}^+ + \bar{q}\chi_{2/3}^0 \bar{\chi}_{2/3}^- \tag{39}$$

This is an analytic current showing that eq(31) admits indeed a huge symmetry namely a spin 4/3 superconformal symmetry. Computing the variation of the lagrangian L under space time translations, we find the following field realization of the conserved spin two tensor

$$\begin{aligned} T_c = & \partial\varphi^- \bar{\partial}\bar{\varphi}^+ + \bar{q}\psi_{1/3}^+ \bar{\partial}\bar{\chi}_{2/3}^- - q\chi_{2/3}^0 \bar{\partial}\bar{\psi}_{1/3}^0 \\ & - \partial\left(\frac{2}{3}\bar{q}\psi_{1/3}^+ \bar{\chi}_{2/3}^- - \frac{q}{3}\chi_{2/3}^0 \bar{\psi}_{1/3}^0\right) \end{aligned} \tag{40}$$

The other components of $T_{\mu\nu}$ namely \bar{T} and Θ vanish identically as required by conformal invariance. The analyticity of G^- and T follows directly from eqs.(32, 33, 34, 35, 36).

Note that T_c is a complex current, this feature was expected since the underlying constant of motion $P = \int(dzT_c + d\bar{z}\Theta)$ involved in the algebra eq.(1) is not hermitean too. To restore this basics property, we require, to the spin two current eq.(40), to be hemitean. This can be achieved in two ways leading to two different theories. The first way is to treat the degrees of freedom involved in ϕ_1^- and ϕ_2^+ as unconstrained fields so that the hermitean energy momentum current T is equal to $T_c + T_c^+$ where T_c^+ is the adjoint conjugate to T_c

$$\begin{aligned} T_c^+ &= \partial\varphi^+\partial\bar{\varphi}^- + q\psi_{1/3}^-\partial\bar{\chi}_{2/3}^+ - \bar{q}\bar{\chi}_{2/3}^0\partial\psi_{1/3}^0 \\ &\quad - \partial\left(\frac{2}{3}q\psi_{1/3}^-\bar{\chi}_{2/3}^+ - \frac{\bar{q}}{3}\bar{\chi}_{2/3}^0\psi_{1/3}^0\right), \end{aligned} \quad (41)$$

where $\varphi^+ = (\varphi^-)^+$ and so on. The $4/3$ supersymmetric conserved current G^+ is obtained in a similar way. We find

$$G^+ = \partial\varphi^+(\bar{\psi}_{1/3}^0)^c + \bar{q}\psi_{1/3}^-\partial\bar{\varphi}^- + q(\bar{\chi}_{2/3}^0)^c\bar{\chi}_{2/3}^+ \quad (42)$$

where $(\bar{\psi}_{1/3}^0)^c$ and $(\bar{\chi}_{2/3}^0)^c$ are the conjugates of $(\bar{\psi}_{1/3}^0)$ and $(\bar{\chi}_{2/3}^0)$ respectively.

In this approach, one needs at least four dynamical scalar fields φ^\pm and their conjugates $\bar{\varphi}^\pm$. The resulting free lagrangian reads then as

$$L_0 \sim (\partial\varphi^-\partial\bar{\varphi}^+ + \partial\varphi^+\partial\bar{\varphi}^-) + (\bar{q}\psi_{1/3}^+\partial\bar{\chi}_{2/3}^- + q\psi_{1/3}^-\partial\bar{\chi}_{2/3}^+) \quad (43)$$

$$-(q\chi_{2/3}^0\partial\bar{\psi}_{1/3}^0 + \bar{q}\bar{\chi}_{2/3}^0\partial\psi_{1/3}^0) \quad (44)$$

The second way is to require

$$\bar{\varphi}^+ = (\varphi^-)^+ \quad (45)$$

$$\chi_{2/3}^- = (\chi_{1/3}^+)^+ \quad (46)$$

$$\chi_{2/3}^0 = (\psi_{1/3}^0)^+ \quad (47)$$

relating the component fields of ϕ_1^- and ϕ_2^+ . This conjugation which affect both the charge and the value of the spin is not new as it was introduced, although differently, in the Zamolodchikov's Z_N parafermionic theory of central

charge $c = 2\frac{(N-1)}{(N+2)}$ [17]. Presently, the eqs.(45-47) may be wondered already at the level of the algebra (1). Indeed imposing the hermiticity of $P = (Q_{1/3}^-)^3$, one gets

$$(Q_{1/3}^-)^+ = (Q_{1/3}^-)^2 \quad (48)$$

and vice versa. In this case the $G_{4/3}^+$ current does not exist. The hermitean lagrangain L_0 under the conjugation eqs.(45-47) coincides with the original one eq(31).

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