Analysis of Foam Drainage Problem Using Variational Iteration Method

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Abstract

In this paper, we use variational iteration method (VIM) to handle the foam drainage equation. Foaming occurs in many distillation and absorption processes. The drainage of liquid foams involves the interplay of gravity, surface tension, and viscous forces. The results reveal that the variational iteration method (VIM) is more efficient than the ADM and it is very effective, convenient and quite accurate to systems of non-linear partial differential equations. It is predicted that VIM can be found widely applicable in engineering.

Keywords: Foam drainage; nonlinear partial differential equation; variational iteration method.

1 Introduction

Foams are of great importance in many technological processes and applications, and their properties are subject of intensive studies from both practical and scientific points of view [1]. Liquid foam is an example of soft matter (or complex fluid) with a very well-defined structure, first clearly described by Joseph plateau in the 19th century. Weaire [2] showed in his work that simple first-order answers to many such questions exist but no going experiments continue to challenge our understanding.
Foams and emulsions are well known to scientists and the general public alike because of their everyday occurrence [3, 4]. Foams are common in foods and personal care products such as creams and lotions, and foams often occur, even when not desired, during cleaning (clothes, dishes, scrubbing) and dispensing processes (c.f. [5]). They have important applications in the food and chemical industries, firefighting, mineral processing, and structural material science (c.f. [6]). Less obviously they appear in acoustic cladding, lightweight mechanical components, and impact absorbing parts on cars, heat exchangers and textured wallpapers (incorporated as foaming inks) and even have an analogy in cosmology.

The packing of bubbles or cells can form both random and symmetrical arrays, such as sea foam and bees’ honeycomb. History connects foams with a number of eminent scientists, and foams continue to excite imaginations [7], although there are now many applications of polymeric foams [8] and more recently metallic foams, which are foams made out of metals such as aluminum [9]. Some commonly mentioned applications include the use of foams for reducing the impact of explosions and for cleaning up oil spills. In addition, industrial applications of polymeric foams and porous metals include their use for structural purposes and as heat exchange media analogous to common “finned” structures [10].

Polymeric foams are used in cushions and packing and structural materials [11]. Glass, ceramic, and metal foams [12] can also be made, and find an increasing number of new applications. In addition, mineral processing utilizes foam to separate valuable products by flotation. Finally, foams enter geophysical studies of the mechanics of volcanic eruptions [5].

Recent research in foams and emulsions has centered on three topics which are often treated separately, but are in fact interdependent: drainage, coarsening, and rheology; see Fig.1. We focus here on a quantitative description of the coupling of drainage and coarsening.

Foam drainage is the flow of liquid through channels (plateau borders) and nodes (intersections of four channels) between the bubbles, driven by gravity and capillarity [13-15]. (see [6]). During foam production the material is in the liquid state and fluid can rearrange while the bubble structure stays relatively unchanged. The flow of liquid relative to the bubbles is called drainage. Generally drainage is driven by gravity and/or capillary (surface tension) forces and is resisted by viscous forces (c.f. [5]).

Because of their limited time stability and despite the numerous studies reported in the literature, many of their properties are still not well understood, in particular the drainage of the liquid in between the bubbles under the influence of gravity [16,17]. Drainage plays an important role in foam stability: indeed, when foam dries, its structure becomes more fragile; the liquid films between adjacent bubbles being thinner, then can break, leading to foam collapse. In the case of aqueous foams, surfactant is added in to water and it adsorbs at the surface of the films, protecting them against rupture (c.f. [18]).
Most of the basic rules that explain the stability of liquid gas foams were introduced over 100 years ago by the Belgian, Joseph Plateau, who was blind before he completed his important book on the subject. This modern-day book by Weaire and Hutzler provides valuable summaries of plateaus work on the laws of equilibrium of soap films, especially useful since the original 1873 French text does not appear to be in a fully translated English version.

Weaire and Hutzler note that Sir W. Thompson (Lord Kelvin) was simulated by plateau's book to examine the optimum packing of free space by regular geometrical cells. His solution to the problem remained the best until quite recently. Why does this area of theoretical research, still active today, have connections with the apparently frivolous theme of bubbles? It is because the packing of free space involves the minimization of the surface energy of the structure (i.e. least amount of boundary material). Thus one might ask why such an often-observed medium as a foam has not provided the optimum solution to this problem much earlier; perhaps this shows that observation is often biased towards what one expects to see, rather than to the unexpected. In nature also, there are packing problems, such as the bees' honeycomb. Its shaped ends provide a nice example of plateau's rules in a natural environment (c.f. [7]).

Recent theoretical studies by Verbist and Weaire describe the main features of both free drainage [19, 20], where liquid drains out of a foam due to gravity, and forced drainage [21], where liquid is introduced to the top of a column of foam. In the latter case a solitary wave of constant velocity is generated when liquid is added at a constant rate [22]. Forced foam drainage may well be the best prototype for certain general phenomena described by non-linear differential equations, particularly the type of solitary wave which is most familiar in tidal bores.

The model developed by Verbist and Weaire idealizes the network of Plateau borders, through which the majority of liquid is assumed to drain, as a set of N identical pipes of cross section A, which is a function of position and time [23].

In this paper, we will apply variational iteration method [24-29] to study the following non-linear foam drainage equation [30].
\[ \frac{\partial A}{\partial t} + \frac{\partial}{\partial x} \left( A^2 - \frac{\sqrt{A}}{2} \frac{\partial A}{\partial x} \right) = 0, \]  

(1)

Where \( x \) and \( t \) are scaled position and time coordinates respectively. In the case of forced drainage, the solution can be expressed as [30]:

\[ A(x,t) = c \tanh^2(\sqrt{c}(x - ct)), \quad x \leq ct, \]

\[ = 0, \quad x > ct, \]  

Where \( c \) is the velocity of the wavefront [21]. The study will be carried out by using variational iteration method.

2 Variational iteration method

To clarify the basic ideas of VIM, we consider the following differential equation:

\[ Lu + Nu = g(t), \]  

(3)

Where \( L \) is a linear operator, \( N \) is a nonlinear operator and \( g(t) \) is a homogeneous term. According to VIM, we can write down a correction functional as follows:

\[ u_{n+1}(t) = u_n(t) + \int_0^t \lambda (Lu_n(\tau) + Nu_n(\tau) - g(\tau))d\tau \]  

(4)

Where \( \lambda \) is a general lagrangian multiplier which can be identified optimally via the variational theory. The subscript \( n \) indicates the \( n \)th approximation and \( u_n \) is considered as a restricted variation, i.e., \( \delta u_n = 0 \).

3 The foam drainage equation

We will apply the variational iteration method to handle the foam drainage equation. Foam drainage equation (1) can be written as [30]:

\[ u_t + 2uu_x - \frac{\sqrt{u}}{2} u_{xx} - \frac{1}{4\sqrt{u}} (u_x^2) = 0 \]  

(5)

With exact solution given in (2).
3.1 Application of variational iteration method

\[ u_{n+1}(x,t) = u_n(x,t) + \int_0^\tau \left[ \frac{\partial u_n(x,\tau)}{\partial \tau} + 2u_n(x,\tau)\left\{ \frac{\partial}{\partial x}u_n(x,\tau) \right\} - \right. \]
\[ \left. \int_0^\tau \lambda \left( \frac{\partial}{\partial x}u_n(x,\tau) \right)^2 - \frac{1}{2} \left( \frac{\partial^2}{\partial x^2}u_n(x,\tau) \right) \sqrt{u_n(x,\tau)} \right] d\tau \]

We obtain the lagrangian multiplier:

\[ \lambda = -1 \]

As a result, we obtain the following iteration formula:

\[ u_{n+1}(x,t) = u_n(x,t) + \int_0^\tau \left[ \frac{\partial u_n(x,\tau)}{\partial \tau} + u_n(x,\tau)\left\{ \frac{\partial}{\partial x}u_n(x,\tau) \right\} - \right. \]
\[ \left. \int_0^\tau (-1) \left( \frac{\partial}{\partial x}u_n(x,\tau) \right)^2 - \frac{1}{2} \left( \frac{\partial^2}{\partial x^2}u_n(x,\tau) \right) \sqrt{u_n(x,\tau)} \right] d\tau \]

Now we start with an arbitrary initial approximation that satisfies the initial condition:

\[ u_0(x,t) = 3 \tanh^2(\sqrt{3}x) \]

Using the above variational formula (8), we have

\[ u_1(x,t) = u_0(x,t) + \int_0^\tau \left[ \frac{\partial u_0(x,\tau)}{\partial \tau} + u_0(x,\tau)\left\{ \frac{\partial}{\partial x}u_0(x,\tau) \right\} - \right. \]
\[ \left. \int_0^\tau (-1) \left( \frac{\partial}{\partial x}u_0(x,\tau) \right)^2 - \frac{1}{2} \left( \frac{\partial^2}{\partial x^2}u_0(x,\tau) \right) \sqrt{u_0(x,\tau)} \right] d\tau \]

Substituting Eq. (9) in to Eq. (10) and after simplifications, we have:
\[ u_i(x,t) = 3 \tanh(\sqrt{3}x)(\tanh(\sqrt{3}x) - 12 \tanh^2(\sqrt{3}x) + \sqrt{3} + 12 \tanh^4(\sqrt{3}x) + 6 \sqrt{3} c \sgn(\tanh(\sqrt{3}x)) - 18 \sqrt{3} c \sgn(\tanh(\sqrt{3}x)) + \tanh^2(\sqrt{3}x) + 12 \sqrt{3} c \sgn(\tanh(\sqrt{3}x)) \) \tag{11} \]

And so on. In the same way the rest of the components of the iteration formula can be obtained.

### 4. Numerical results and some illustrative figures

Tables 1, 2, and 3 investigated comparison between errors of ADM [30] and variational iteration method (VIM). Figs. 2, 3 and 4 show comparison between results of VIM and exact solution. Results show that variational iteration method is more efficient than the ADM

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Table 3: Comparison between errors of Adomian method and variational iteration method for $t = 0.1$

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Fig. 2. Comparison between results of the VIM with exact solution at $t = 0.001$

Fig. 3. Comparison between results of the VIM with exact solution at $t = 0.01$

Fig. 4. Comparison between results of the VIM with exact solution at $t = 0.1$
5 Conclusion
In this work, we proposed variational iteration method to handle the foam drainage equation, and to emphasize the strength of this method in handling nonlinear differential equations. The results obtained here were compared with results of exact solution. The results revealed that the variational iteration method is more efficient than the ADM and it is a powerful mathematical tool for solutions of non-linear differential equations in terms of accuracy and efficiency.

References


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