Non-Global Potentials in Electrodynamics

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Abstract

In this paper, in connection with the ideas of Wu and Yang about magnetic monopoles, we make some comments on the consequences of the independence between the electric charges and the magnetic monopoles, and also, we point out some features on the use of non-global potentials in the Electrodynamics without the presence of magnetic monopoles.

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1 Introduction

In their celebrated paper of 1975 [1], Wu and Yang used a peculiar approach to describe the field strength of a point-like magnetic monopole $g$. They used two different patches to describe the $\mathbb{R}^3$ in such a way that, in each patch, the field strength of the magnetic monopole could be given by the vector potential of a so-called Dirac’s string [1, 2, 3]. The whole space $\mathbb{R}^3$ (except the point the monopole is lying) is described by the union of the two patches. In the overlap of the patches, Wu and Yang considered that both the potentials (one for each patch) must be linked by a kind of gauge transformation. With this procedure, Wu and Yang obtained the so called Dirac’s quantization [2].

In fact, to describe the magnetic monopole, Wu and Yang used a pair of non-global potentials in the sense that each of them were not defined in the whole $\mathbb{R}^3$, but only on a chart.
There are two parallel approaches for the Wu and Yang description of magnetic monopoles, the first is the one which we refered to above and is based on the use of potentials. The second approach is based on fibre bundles. Both of them are completely equivalent approaches.

In this paper we make, first, a discussion on the approach with potentials, and after, a discussion with fibre bundles.

Following the approach with potentials, the most simple version of the Wu and Yang procedure is to consider a first patch $P_I$ as the $\mathbb{R}^3$ with the semi-axis $z$-negative excluded. In this case, the field strength of a point-like magnetic monopole $g$ placed at the origin is given by the vector potential of a Dirac’s string lying along the negative $z$ axis

$$A^I = g \frac{1 - \cos \theta}{r \sin \theta} \hat{\phi}.$$  

where we used spherical coordinates with $\theta$ representing the polar angle. Notice that the potential (1) is well defined in the whole patch $P_I$, but not in the whole $\mathbb{R}^3$.

The second patch $P_{II}$ is taken to be the $\mathbb{R}^3$ with the semi-axis $z$-positive excluded, and the field strength of the monopole is given by the vector potential of a Dirac’s string lying along the positive $z$ axis

$$A^{II} = -g \frac{1 + \cos \theta}{r \sin \theta} \hat{\phi}.$$  

The potential (2) is well defined in the whole patch $P_{II}$.

Both the potentials (1) and (2) are non-global ones in the sence that each of them are not defined on the whole $\mathbb{R}^3$ but only on the charts $P_I$ and $P_{II}$, respectively.

The union of the two patches, $P_\cup = P_I \cup P_{II}$, is the whole $\mathbb{R}^3$ space, except the origin.

The overlap between the patches, $P_\cap = P_I \cap P_{II}$, is the manifold composed by the $\mathbb{R}^3$ space, except the whole $z$-axis. In this manifold, both the potentials (1) and (2) must be linked by the gauge transformation

$$A^T = A^I - A^{II} = 2g \frac{1}{r \sin \theta} \hat{\phi} = \nabla (2g\phi),$$

which has the structure of a Dirac-Aharonov-Bohm potential, and is the field produced by an infinit Dirac’s string lying along the whole $z$-axis.

In the classical context, there are no problems with the Wu and Yang approach used to describe the monopole, which avoids the use of a pair of four potentials [4], but in the context of Quantum Mechanics, this approach still remains valid only if the so-called Dirac’s quantization [1, 2, 3, 4] is valid, that is

$$qg = \frac{1}{2} m_{q,g},$$
where \( q \) stands for the possible electric charges a wave function may have, and \( m_{q,g} \) is a function which has integer values and whose variables are \( q \) and the possible values of the monopole intensity \( g \).

If the condition (4) is not satisfied, the two strings related to (1) and (2) produce different quantum effects, and the potentials (1) and (2) cannot describe the same physical system, namely, the point-like magnetic monopole at the origin.

In this paper we argue that a simple additional statement can lead to a more specific quantization condition to the electric charges and magnetic monopoles. We also argue that the deduction of the electric charge quantization by the Wu and Yang approach is much more a feature of the use of non-global potentials then properly a consequence of the existence of magnetic monopoles. Following the same lines used by Wu and Yang with non-global potentials we also obtain a quantization condition for the electric charges without the presence of monopoles.

2 An Additional Statement

In addition to the approach by Wu and Yang [1], let us consider an additional statement in order to obtain a more specific quantization condition for the electric charge; let us suppose the independence between the possible values of the monopole intensity \( g \) and the possible values of the electric charge \( q \) a wave function can have. It means that we can submit a wave function with any possible value of electric charge \( q \) to the field of any possible monopole intensity \( g \).

If we have this independence, and the quantization (4) is valid, the possible values of \( q \) and \( g \) must have, both of them, a injective correspondence to the set of integer numbers (see appendix), on the contrary, the relation (4) would not be valid. As a consequence, (4) can be written in the form

\[
qg = \frac{1}{2}m_q m_g
\]

where \( m_q \) is a function of integer values which depends only on the charge \( q \), and \( m_g \) is another function of integer values which depends only on \( g \). Note that, from (5), we have that \( q \) must be proportional to \( m_q \) and \( g \) must be proportional to \( m_g \), in special, we have that \( m_q \neq 0 \) if and only if \( q \neq 0 \), and similar situation for \( g \) and \( m_g \).

Applying equation (5) for the electron charge \( e \), a given value of the electric charge we know to exist in the nature, we have

\[
g = \frac{1}{2e} m_e m_g \Rightarrow g = \frac{M}{2e} m_g ,
\]
where we defined $M = m_e$, that is an integer non null constant.

Substituting the result (6) in (5) we are taken to the condition

$$q = \frac{e}{M} m_q,$$

that is, any value of the electron charge is an integer multiple of a given fraction of the electron’s charge $e$, what is in agreement with the quark model.

From the result (6) we have that any value of the monopole intensity $g$ can be written as an integer multiple of a standard monopole $M/2e$ ($M$ is an integer constant).

Due to the proportionality between $q$ and $m_q$, and taking into account that $m_e = M$, instead of the electron’s one, we could have used any value of electric charge as the standard one, and the results presented in (6) and (7) would be the same.

### 3 Remarks on the Wu and Yang Procedure

We want to extend the Wu and Yang idea on the use of non-global potentials to more general situations than the ones related to magnetic monopoles but, for future convenience, let us first emphasize, in this section, some points in the Wu and Yang approach [1] in what concerns the use of non-global potentials:

- **a-i) The patch $P_I$ (and also $P_{II}$) does note have the same topology of the whole $\mathbb{R}^3$ space, for instance, it is not a convex space, contrary to the $\mathbb{R}^3$ that is convex [5].**

- **a-ii) The overlap manifold $P_{\cap} = P_I \cap P_{II}$ does not have the same topology of each patch $P_I$ and $P_{II}$ separately, and not even the topology of the whole $\mathbb{R}^3$, for instance, the overlap $P_{\cap}$ is not a simply connected space [5].**

- **a-iii) Using concepts of Theory of Distributions, the rotational of the potential (1) in the whole $\mathbb{R}^3$ is given by [3, 6]

  $$\nabla \times A^I = \frac{g}{r^2} \hat{r} + 4\pi g \Theta(-z)\delta(x)\delta(y)\hat{z},$$

  where $\hat{r}$ is the unit vector in the $z$ direction, $\delta(x)$ is the Dirac delta function and $\Theta(\xi)$ is the step function which, as usually adopted, is non-defined for $\xi = 0$.

The result (8) encompasses the magnetic field of the monopole plus a singular one along the string. That is not a problem since, in the approach of Wu and Yang, it is taken only in the patch $P_I$, where the monopole field strength is correctly reproduced by this string potential (1).
Similarly, we have for the whole $\mathbb{R}^3$

$$\nabla \times A^{\text{II}} = \frac{g}{r^2}\hat{r} - 4\pi g \Theta(z) \delta(x) \delta(y) \hat{z},$$

(9)

and the correct monopole field strength in the patch $\mathcal{P}_{\text{II}}$ is reproduced. Also, in the whole $\mathbb{R}^3$ the rotational of (3) is

$$\nabla \times A^T = 4\pi g \delta(x) \delta(y) \hat{z},$$

(10)

that is null in the manifold where $A^T$ is defined, as should be for any gauge field.

• $a$-iv) If we wish to calculate the magnetic flux of the monopole through, for instance, a sphere $S$, centered at the origin, we could perform the surface integral of the magnetic monopole field, $(g/r^2)\hat{r}$, along the sphere, what gives the result

$$\Phi_B = \int_S \frac{g}{r^2} \hat{r} \cdot dS = 4\pi g. \quad (11)$$

Following an alternative procedure, we could use results on the Theory of Distributions and the Stokes Theorem, taking care about considering one of the potentials, (1) or (2), only in the patch where it is defined [3, 6]. In this case, we have to consider a hole in the sphere with boundary $\partial S$,

Thus, we have

$$\Phi_{B,\text{I}} = \int_{S_1} \nabla \times A^I \cdot dS$$

$$= \int_{\partial S_1} A^I \cdot d\ell_I = 4\pi g$$

$$\Phi_{B,\text{II}} = \int_{S_{\text{II}}} \nabla \times A^{\text{II}} \cdot dS$$

$$= \int_{\partial S_{\text{II}}} A^{\text{II}} \cdot d\ell_{\text{II}} = 4\pi g$$

If we had not considered the appropriated holes in the sphere, the magnetic flux through the sphere along each string, in both cases, would give a contribution $-4\pi g$ to the total flux, and we would have found $\Phi_{B,\text{I}} = \Phi_{B,\text{II}} = 0$, as might be for any closed surface and a rotational field.

As far as the magnetic flux produced by the Wu and Yang gauge field (3) is concerned, there is an analogous situation. The magnetic flux produced by (3) through any surface which crosses the $z$-axis can be calculated by
invoking the Theory of Distributions and the Stokes’ Theorem. In this case, care must be taken with the fact that $A^T$ is not defined on the $z$-axis, so it must be considered holes on the surface in such a way to avoid contributions to the flux along the $z$-axis. Considering this point and the fact that

$$\oint_{\Gamma} A^T \, d\ell = \oint_{\Gamma} \frac{\kappa}{r \sin \theta} \, \hat{\phi} \, d\ell = 2\pi \kappa$$

(13)

for any closed path $\Gamma$ which turns around once the $z$-axis, we have that $A^T$ does not produce any magnetic flux.

4 An Alternative Approach in Terms of Non-global Potentials

The main goal of this section is to extend the Wu and Yang ideas of the use of non-global potentials in electrodynamics to more general situations than the ones related to magnetic monopoles. For this task, inspired by the Wu and Yang proposal of using non-global potentials, let us take two alternative patches, $\mathcal{P}_{III}$ and $\mathcal{P}_{IV}$, to describe the $\mathbb{R}^3$ space. Let us take these patches in the following way:

- Both patches, $\mathcal{P}_{III}$ and $\mathcal{P}_{IV}$, are not disjoint.
- $\mathcal{P}_{III}$ is a given opened sub-set of $\mathbb{R}^3$ which contains the whole $z$-axis.
- $\mathcal{P}_{IV}$ is a given opened sub-set of $\mathbb{R}^3$ which does not contain any point of the $z$-axis, but contains an opened set around the whole $z$-axis.
- $\mathcal{P}_{III}$ and $\mathcal{P}_{IV}$ are taken in such a way that their union, $\mathcal{P}_{III} \cup \mathcal{P}_{IV}$, is the whole $\mathbb{R}^3$.

Indeed, $\mathcal{P}_{III}$ can be the whole $\mathbb{R}^3$ itself, and $\mathcal{P}_{IV}$ can be the $\mathbb{R}^3$ with a neighborhood around the $z$-axis excluded.

Some points concerning the need of an atlas for the $\mathbb{R}^3$ are discussed latter in the paper.

A given electromagnetic system can be described by different potentials in the patches $\mathcal{P}_{III}$ and $\mathcal{P}_{IV}$. Like in the case of Wu and Yang, the $z$ axis is excluded from the overlap manifold between $\mathcal{P}_{III}$ and $\mathcal{P}_{IV}$, $O = \mathcal{P}_{III} \cap \mathcal{P}_{IV}$, and a gauge transformation of the form (3) is well defined in this overlap. So, inspired by (3), let us take the gauge transformation

$$A_\kappa = A^{III} - A^{IV} = \nabla (\kappa \phi) = \frac{\kappa}{r \sin \theta} \, \hat{\phi} \, , \ \ r \sin \theta \neq 0 \ ,$$

(14)
in the overlap $\mathcal{O}$, where $\kappa$ is a gauge parameter, $A_{III}$ is the vector potential which describes, only in the patch $\mathcal{P}_{III}$, a given physical system and $A_{IV}$ is the vector potential which describes the same physical system, but only in the patch $\mathcal{P}_{IV}$.

Some questions concerning the magnetic flux and the gauge (14) are discussed later in the paper.

It is worthy emphasizing that, as in (3), the potential $A_\kappa$ presented in (14) is the same as the one produced by an infinite Dirac’s string lying along the whole $z$-axis, but, in this case, $A_\kappa$ is not related to any magnetic monopole, on the contrary to what happened in the Wu and Yang’s case (3). Here, $A_\kappa$ is written as a pure gauge potential.

By the same reasons which lead to equation (4), for the potentials $A_{III}$ and $A_{IV}$ describe the same physical system in the overlap region $\mathcal{O}$, the following condition must be fulfilled:

$$q_\kappa = n_{q,\kappa},$$

where $q$ is the value of electric charge that a wave function can have and $n_{q,\kappa}$ is a function of integer values whose variables are the parameter $\kappa$ and the charge $q$. With condition (15), it is assured that the potential $A_\kappa$ in (14) does not produce any quantum physical effect.

In addition to the statement of describing the $\mathbb{R}^3$ by the union of two different patches, $\mathcal{P}_{III}$ and $\mathcal{P}_{IV}$, let us also impose the independence between the gauge parameter $\kappa$ and $q$. It means that we can perform any posible gauge transformation of the form (15) with a wave function which has any possible value of electric charge $q$, it is, the gauge transformation does not depend on the charge.

With this independence, and given the condition (15), by the same reasons which lead to equation (5) from (4), we must have

$$q_\kappa = n_q n_\kappa,$$

where $n_q$ and $n_\kappa$ are functions of integer values whose variables are, respectively, the charge $q$ and the parameter $\kappa$.

Using in equation (16) the charge of the electron $e$, for example, we are taken to

$$\kappa = \frac{n_e n_\kappa}{e} = \frac{N n_\kappa}{e} = \frac{N_\kappa}{e},$$

where we defined $N = n_e$ which is a non-zero integer constant, and $N_\kappa = N n_\kappa$. Replacing equation (17) into (16), we have

$$q = \frac{n_q}{N} e,$$

We need also a scalar potential to describe an electromagnetic system but, once we are interested in the gauge (14), it shall not be relevant for our discussions.
which agrees with the result (7) obtained from the Dirac’s result (4).

We would like to stress on some points related to the result (18):

- **b-i)** The patch $P_{IV}$ does not have the same topology of the whole $\mathbb{R}^3$ space neither the topology of the patches $P_I$ and $P_{II}$ taken by Wu and Yang. For instance, $P_{IV}$ is not simply connected, on the contrary to $\mathbb{R}^3$, $P_I$ and $P_{II}$, which are simply connected.

  The patch $P_{III}$ can have the same topology of $\mathbb{R}^3$.

- **b-ii)** The overlap patch between $P_{III}$ and $P_{IV}$, $O$, is not simply connected, similarly to what happens for the overlap between $P_I$ and $P_{II}$, $P_I \cap P_{II}$, in the Wu and Yang case.

- **b-iii)** Similarly to what happens in the Wu and Yang case (10), if we take the rotational of the $A_\kappa$ field in the whole $\mathbb{R}^3$ we have

\[ \nabla \times A_\kappa = 2\pi \kappa \delta(x)\delta(y)\hat{z} \tag{19} \]

which is equal to zero in the region it is defined (that excludes the $z$ axis), what is expected for a gauge field.

- **b-iv)** To calculate the flux of the magnetic field produced by $A_\kappa$ through a surface which intercepts the $z$-axis, we must take care of not considering the contribution to the magnetic flux along the $z$-axis, once the $z$-axis is not on the manifold where the $A_\kappa$ field is defined.

The same situation happens in the Wu and Yang case for the gauge (3).

In the Wu and Yang approach, one could use two Dirac’s strings lying along any two distinct and semi-infinite curves ending on the monopole position. We have chosen two straight lines lying along the $z$-semi-axis only to make the analysis clearer. In our extended approach, we could also use, as a gauge field, the vector potential produced by a Dirac’s string lying along a curve which can be of two types: *i*) with the ends at the infinity, like the one presented in the text, but with no need to be a straight line; and *ii*) a finite closed curve, like a circle, for example.

The results (7) and (18) are equivalent, and to obtain both of them it was a necessary condition to describe the whole $\mathbb{R}^3$ with distinct patches. For the result (7), it was needed to state the independence between $q$ and $g$; for (18), it was needed to state the independence between $q$ and the gauge parameter $k$.

The independence between $g$ and $q$ also leads to the result (6), and similarly, the independence between $q$ and $\kappa$ leads to (17).

In spite of the results (18) and (7) to be equivalent, to obtain (18) there was no need to state the existence of a magnetic monopole, for which there is no experimental evidence, but there is the need to state the gauge (14). The
result (7) is established from the potential (3) and also with the existence of monopoles.

The arguments presented in the text can be inverted and the structure of the electric charges that are attributed to the elementary particles nowadays, along with the fact that there is no evidence for the existence of a magnetic monopole, can be seen as an indication that the statements which lead to equations (18) and (17) are valid, and the existence of gauge fields of the type (14) are justified in the overlap between specific patches.

5 An Analysis in Terms of Fibre Bundles

Having in mind that a natural geometrical setting for the gauge theories is the formulation in terms of fibre bundles, where gauge fields emerge as connections on the principal bundle, we believe that, for the sake of completeness, it would be worthwhile to present our ideas in terms of the fibre bundle description and, at the end, we see how we can re-obtain the results (17) and (18). That is what we do in the present section.

The Dirac’s quantization can be obtained by using fibre bundles, as exposed in reference [1, 7]. For this task we have to consider, first, that the magnetic field of a monopole \( g \) placed at the origin is defined in the \( \mathbb{R}^3 \) with the origin excluded, which is a manifold of the same homotopy type of the sphere \( S^2 \), that can be covered by an atlas composed by the charts

\[
U^I = \{ (\theta, \phi) | 0 \leq \theta \leq (\pi/2) + \alpha \}, \\
U^{II} = \{ (\theta, \phi) | 0 - \alpha \leq \theta \leq (\pi/2) \},
\]

where \( 0 < \alpha < (\pi/2) \).

In the charts \( U^I \) and \( U^{II} \) the monopole is described, respectively, by the so-called Wu and Yang forms

\[
A^I = ig(1 - \cos \theta) d\phi , \quad A^{II} = -ig(1 + \cos \theta) d\phi .
\]

The overlap between the charts \( U^I \) and \( U^{II} \), \( U_{\cap} = U^I \cap U^{II} \), is a manifold which does not contain the poles of the sphere, so in \( U_{\cap} \) we can find a submanifold which is homeomorphic to \( S^1 \), which we can take as being the equator, for simplicity. The structure group of the electromagnetic interactions is the \( U(1) \). We can define a map from \( S^1 \) (a sub-set in the overlap \( U_{\cap} \)) to the structure group \( U(1) \) as follows

\[
t_{I \rightarrow II} (\phi) = \exp [iq\varphi(\phi)] ,
\]

where \( q \) is any coupling constant of the structure gauge group \( U(1) \) (any electric charge).
For the forms (21) we have
\[ d\varphi = -i(A^I - A^{II}) = 2\ g \ d\phi , \] (23)
so, in a complete turn along \( S^1 \) the function \( \varphi(\phi) \) takes the range
\[ \Delta\varphi|_{\phi\rightarrow\phi+2\pi} = \int_{\phi}^{\phi+2\pi} d\varphi = 2g \int_{\phi}^{\phi+2\pi} d\phi = 4\pi g . \] (24)

Correspondingly, for the the map (22), we have
\[ t_{I\rightarrow II}(\phi + 2\pi) = \exp [iq\varphi(\phi + 2\pi)] = \exp [iq(\varphi(\phi) + \Delta\varphi|_{\phi\rightarrow\phi+2\pi})] = t_{I\rightarrow II}(\phi) \exp (i4\pi q g) \] (25)

From equation (25) we can see that for the map (22) to be uniquely defined, we must have the Dirac’s quantization (4).

With the additional statement of the independence between \( q \) and \( g \), we are taken to the results (7) and (6).

In order to follow an alternative analysis, let us consider an electromagnetic system where the fields are not defined on a given point, what can be an electric charge placed at the origin, for instance. Like in the monopole case, the electromagnetic field for this system is defined on a manifold which is homotopic to the \( S^2 \). Instead of the charts (20), let us take an atlas for \( S^2 \) composed by two patches, \( U^{III} \) and \( U^{IV} \), in the following way.

- \( U^{III} \) is a chart which contains both the poles of the sphere and a neighborhood around the equator.
- \( U^{IV} \) does not contain any pole of the sphere but contains a neighborhood around the equator.
- \( U^{III} \cup U^{IV} \) is the whole \( \mathbb{R}^3 \).

In fact, the chart \( U^{III} \) can be the whole \( S^2 \), and \( U^{IV} \) can be the \( S^2 \) with a neighborhood around each pole removed.

The overlap between \( U^{III} \) and \( U^{IV} \), \( \mathcal{U} = U^{III} \cap U^{IV} \), is a manifold which contains a neighborhood around the equator and does not contain the poles of the sphere. In the overlap \( \mathcal{U} \) we can find a submanifold which is homeotopic to \( S^1 \). We can also define a map from \( S^1 \) (a sub-set in the overlap \( \mathcal{U} \)) to \( U(1) \), the structure group of electrodynamics, which can be written as
\[ t_{III\rightarrow IV}(\phi) = \exp [i\psi(\phi)] . \] (26)

In the charts \( U^{III} \) and \( U^{IV} \) the electromagnetic field is described, respectively, by the forms \( \mathcal{A}^{III} \) and \( \mathcal{A}^{IV} \). Inspired by the Wu and Yang gauge (23), let-us take
\[ \mathcal{A}^{III} - \mathcal{A}^{IV} = i\kappa \ d\phi , \] (27)
where $\kappa$ is a given constant.

By similar arguments used for the Wu and Yang gauge (23), equations (26) and (27) lead to the result (15).

With the additional statement of the independence between $q$ and $\kappa$ we are taken to the results (17) and (18).

6 Conclusions and Final Remarks

In this paper we showed that the additional statement of the independence between the electric charges and the magnetic monopoles can lead to a more restrictive quantization condition for the electric charges than the Dirac’s one. This statement is physically natural once any electric charge can interact with any magnetic monopole which can exist in nature. The more restrictive quantization condition is consistent with the quark model.

We also argued that the deduction of the Dirac’s quantization condition for the electric charges by the Wu and Yang procedure is much more a feature of the use of non-global potentials than properly a consequence of the existence of magnetic monopoles.

We extended the Wu and Yang idea of the use of non-global potentials to more general situations than the ones related to magnetic monopoles and showed that we can obtain a quantization condition for the electric charges without magnetic monopoles and consistent with the quark model. In other words, it was argued that the electric charge quantization can be a consequence of the extension of the gauge symmetry between patches covering a manifold where the field strength is defined.

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Appendix

Consider two non-zero numbers, $u$ and $v$, which can vary independently from the other one and satisfy the relation

$$uv = \ell_{u,v},$$

(28)
where $\ell_{u,v}$ is a function of the variables $u$ and $v$ which can assume only integer values, that is, its image is a sub set of the integer numbers $\mathbb{Z}$.

In order to analyze the expression (28), let us consider two distinct cases:

1) $\ell_{u,v} = 0$: for this trivial situation we can see from (28) that $\ell_{u,v} = 0$ if, and only if, one or both the variables, $u$ and $v$, are zero, and we can not establish any restriction for $u$ and $v$.

2) $\ell_{u,v} = 0$: in this case, both $u$ and $v$ are not equal to zero, and there arises a restriction for the independent variables $u$ and $v$. In fact, for each one, there must be an injective relation into the integer numbers, on the contrary, the condition (28) would not be satisfied due to the independence between $u$ and $v$. So we can write

$$u = C_1 n_u, \quad v = C_2 n_v.$$  \hspace{1cm} (29)

where $C_1$ and $C_2$ are constants, and $n_u$ and $n_v$ are functions of integer values of the variables $u$ and $v$ respectively.

The result (29) can be seen noticing that $v$ is non null, and considering equation (28) which allows us to write

$$u = \frac{\ell_{u,v}}{v}.$$  \hspace{1cm} (30)

Once the left-hand side of (30) is not dependent on $v$, the right-hand side must also be $v$-independent, therefore, equation (30) leads to the first equation (29). The second equation (29) can be stated following similar arguments.

Substituting equation (29) back to (28), we have

$$C n_u n_v = \ell_{u,v}.$$  \hspace{1cm} (31)

From the equation above, we see that $C = C_1 C_2$ must be the ratio between the integers $\tilde{N}$ and $\tilde{M}$:

$$C = \frac{\tilde{N}}{\tilde{M}},$$  \hspace{1cm} (32)

and the functions $n_u$ and $n_v$ can be written as

$$n_u = \tilde{M} m_u, \quad n_v = \tilde{M} m_v$$  \hspace{1cm} (33)

where $m_u$ and $m_v$ are new functions of integer values.

If we did not have the validity of (32) and (33), the left-hand side of (31) could assume non-integer values, what would be inconsistent with the right-hand side.

Substituting (33) and (32) back to (31) and using (28), we have

$$uv = \tilde{N} \tilde{M} m_u m_v = \ell_{u,v}.$$  \hspace{1cm} (34)
Redefining one or both of the functions, $m_u$ and $m_v$, leads to
\[ uv = \ell_u \ell_v \]  \hfill (35)
where $\ell_u$ and $\ell_v$ are functions of integer values of the variables $u$ and $v$ respectively.

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