A Stationary Solution of Einstein’s Equations

Representing a Universe in Differential Rotation

William Davidson\textsuperscript{1,2}

\textsuperscript{1}Mathematics and Statistics Department, University of Otago, Dunedin, New Zealand

\textsuperscript{2}Permanent Address: 21 Rowbank Way, Loughborough, Leicestershire, UK, LE11 4AJ
wdav295@btinternet.com

Abstract. A solution of Einstein’s equations is given describing a perfect fluid of infinite extent in steady differential rotation. The equation of state is \( \lambda = (\gamma - 1) \mu \), \( \lambda \) being the fluid pressure, \( \mu \) the density. The metric is algebraically general of Petrov type I. A sub-class for which \( \lambda = \mu/2 \) is selected to illustrate examples of properties of the rotating fluid and geometry of the metric. There is a special case of the original metric for which the fluid is stiff (\( \lambda = \mu \)); this is of Petrov type D, rigidly rotating, of vanishing magnetic Weyl tensor and homothetic.

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1. INTRODUCTION

As remarked elsewhere [1,p 338], there are few GR solutions in the literature for a perfect fluid in steady differential rotation. A review of results was given in [1]. The solutions by Senovilla [2], and Garcia [3] and more recently by Davidson [4] have an equation of state \( \lambda = \mu + \text{const} \). In this paper we add a solution whose equation of state is \( \lambda = (\gamma - 1) \mu \) (\( \gamma = \text{const} \)).

2. THE METRIC

We start from the metric form
\[
\text{ds}^2 = a_0 e^{ar+ibc} \left( dr^2 + dz^2 \right) + e^{ar+ibc} \left( c(r) d\phi^2 - d(r) dr^2 - 2k(r) dr d\phi \right),
\]

(2.1)

where \( a_0 (> 0) \), \( a, b, m \) and \( n \) are constants and, as indicated, \( c, d \) and \( k \) are functions of \( r \). The coordinates \( r \) and \( z \) are radial and axial, respectively, while \( \phi \) is the azimuthal coordinate. Killing vectors are \( \partial_r = \xi \) and \( \partial_\phi = \eta \), reflecting stationarity and axial symmetry, respectively, and these evidently commute.

For a perfect fluid the Einstein tensor and energy tensor are related by

\[
G^{\mu}_{\nu} = T^{\mu}_{\nu} = (\lambda + \mu) u^i u_j \delta^\mu_{\nu} + \lambda \delta^\mu_{\nu},
\]

(2.2)

where \( u^i \) is the fluid 4-velocity in the coordinates \( x^i = (t, r, z, \phi) \) for \( i = 0, 1, 2 \) and 3, respectively. In the assumed stationary, system we shall have

\[
u^i = \left( u^0, 0, 0, u^3 \right),
\]

(2.3)

\( u^0 \) and \( u^3 \) being functions of \( r \) and \( z \).

We find that Einstein’s equations can be satisfied by the form (2.1) when, specifically:

\[
a = \frac{bn}{m} \left( \frac{2m^2 - (m^2 - n^2)}{m(3m^2 + n^2) - 2f} \right),
\]

(2.4)

\[
b = \frac{n}{3m^2 + n^2 - 2f},
\]

\[
f = \left( 3m^4 + 3m^2n^2 + n^4 \right)^{1/2},
\]

with

\[
c(r) = c_0 \left[ 1 - \frac{\left( m^2 + n^2 \right)^2}{2f - 3m^2 - n^2} \left( b_0 + r \right)^2 \right],
\]

(2.5)

\[
d(r) = \frac{k_0^2 \left( 2f - 3m^2 - n^2 \right)}{c_0 \left( m^2 + n^2 \right)^2},
\]

\[
k(r) = k_0 \left( b_0 + r \right),
\]

\( b_0, c_0 \) and \( k_0 \) being positive constants. We then have

\[
\lambda = \frac{m^2 + n^2}{2a_0} e^{(ar+ibc)},
\]

(2.6)

\[
\frac{\lambda}{\mu} = \frac{2f - 3m^2 - n^2}{m^2 + n^2}.
\]

Thus there is a \( \gamma \) law equation of state for the fluid.

In this solution all variables depend on the parameters \( m \) and \( n \) and the constants \( a_0, b_0, c_0 \) and \( k_0 \). In particular the ratio \( \lambda/\mu \) can be written in the form
\[ \frac{\lambda}{\mu} = \frac{2(1+3x^2+3x^4)^{1/2} - 3x^2 - 1}{1+x^2}, \quad x = \frac{m}{n}. \quad (2.7) \]

We have therefore a stiff fluid \((\lambda = \mu)\) at \(x = 0\) and as \(x\) increases \(\lambda/\mu\) steadily decreases and is asymptotic to the value \(2\sqrt{3} - 3 \approx 0.46\) as \(x \to \infty\).

3. SOME PROPERTIES OF THE METRIC

The parameters \(m\) and \(n\) are independent, but we shall consider a mathematically more tractable subset as an example of the solutions provided by the metric (2.1), which has led to (2.4) and (2.5). We choose

\[ m = -\sqrt{7} j, \quad n = j, \quad b_0 = \sqrt{7} f(4j), \quad j > 0. \quad (3.1) \]

For this set we find

\[ f = 13 j^2, \quad a = \sqrt{7} j/2, \quad b = 3j/2 \]
\[ \lambda = \frac{4j^2}{a_0} e^{-j(\sqrt{7}/2)} \]
\[ \mu = \frac{1}{2}. \]

Evidently the fluid pressure and density tend to zero when \(r \to \infty\) and when \(z \to +\infty\), but become infinite when \(z \to -\infty\), so that we would expect the metric to have a singularity at \(z = -\infty\). This is confirmed if we calculate, for example, the Ricci scalar curvature for the metric specified by (3.1):

\[ R = -\frac{4j^2}{a_0} e^{-j(\sqrt{7}/2)} \]

which indicates that \(z = -\infty\) is the only singularity of the metric. Also, since from (3.2) both \(a\) and \(b\) are positive, we see that the extent of the fluid is infinite in both the radial and axial directions.

Calculation of the fluid 4-velocity (2.3) for the case (3.1) yields

\[ u^0 = 8\frac{\sqrt{2}}{3} e^{1/2} (j^2 / k_0) r e^{i(\sqrt{7}/2)}, \]
\[ u^3 = -\frac{1}{\sqrt{6}} e^{-1/2} j e^{i(\sqrt{7}/2)}. \quad (3.4) \]

It follows that the rate of angular rotation of the fluid is

\[ \Omega = \frac{u^3}{u^0} = -\frac{k_0}{16c_0 j^2 r}. \quad (3.5) \]

The rotation is therefore differential, infinite at \(r = 0\), and tending steadily to zero as \(r \to \infty\). But, as noted above, \(r = 0\) is not a singularity of the metric. However,
$r = 0$ is not a regular axis of symmetry. In fact the infinite ocean of fluid has nowhere a regular axis of symmetry since in (2.1) the $t, \phi$ determinant of the metric, namely $c(r)d(r)+k(r)^2$ does not vanish for any value of $r$.

Choice of an appropriate complex null tetrad for the spacetime shows that the usual discriminant $I^2 - 27J^2$ (cf. [1]) is complex so that the metric is of Petrov type I. Both the electric and magnetic Weyl tensors are therefore non-zero [5,6]. In fact, for the case (3.1) we find the invariants:

$$E_{ab}E_{ab}^{\text{in}} = \frac{160}{9} j^4 a_0^2 e^{-j(\sqrt{\gamma}+3\gamma)/2},$$

$$H_{ab}H_{ab}^{\text{in}} = -\frac{256}{9} j^4 a_0^2 e^{-j(\sqrt{\gamma}+3\gamma)/2}.$$  \hfill (3.6)

These results imply that the principal null directions of the Weyl tensor are linearly independent [6]. The vorticity $\omega$ (acting in the $z$ direction) is given by

$$\omega^2 = \frac{49}{9} j^2 a_0^{-1} e^{-j(\sqrt{\gamma}+3\gamma)/2},$$

while the shear invariant is

$$\sigma^2 = \frac{j^2}{9 a_0} e^{-j(\sqrt{\gamma}+3\gamma)/2}. \hfill (3.8)$$

Thus both $\omega$ and $\sigma$ tend steadily to zero as $r \rightarrow \infty$ and as $z \rightarrow +\infty$ but become infinite at the singularity $z = -\infty$.

For the fluid acceleration we obtain

$$\dot{u}_i = \left(0, \sqrt{\frac{7}{6}} j, \frac{1}{2} j, 0\right). \hfill (3.9)$$

In conclusion, as a GR solution we have an infinite universe of perfect fluid obeying a $\gamma$ law, undergoing a steady motion with acceleration, shear and differential rotation.

4. THE STIFF FLUID CASE

The choice $x = 0$ at (2.7) led to a stiff fluid ($\lambda = \mu$) and this requires special attention. We have to consider the following metric (letting $m \rightarrow 0$ in (2.4) and (2.5)):

$$ds^2 = e^{\alpha_1} \left[ a_0 \left(dr^2 + dz^2 \right) + c_0 \left(1 - n^2 \left(b_0 + r \right)^2 \right)d\phi^2 - \frac{k^2}{c_0^2}d\theta^2 - 2k_0 \left(b_0 + r \right)dr d\phi \right].$$ \hfill (4.1)

We then find the following physical and kinematic results:
The fluid characteristics therefore depend only on $z$. Since $u^3 = 0$ the coordinates are now comoving with the fluid, and since $\sigma^2 = 0$ the rotation is rigid. The pressure, density and vorticity are finite everywhere except at the singularity $z = -\infty$.

Analysis of the Weyl tensor shows that the Petrov type is D; also, that the magnetic Weyl tensor vanishes. In fact we find the invariants

$$E_{ab}E^{ab} = \frac{1}{6} n^4 a_0^{-2} e^{-2nz},$$

$H_{ab}H^{ab} = 0$.\hspace{1cm} \hspace{1cm} \hspace{1cm} (4.3)

That is, for the case $m = 0$ the Weyl tensor relative to the fluid velocity has an electric component only.

Another property of the stiff fluid case is that the metric admits a homothety:

$$L_\zeta g_{ij} = 2sg_{ij},$$

where

$$\zeta = \partial_z, \quad s = n/2.$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} (4.5)

REFERENCES


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