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Some Comments on Contractions of Lie Algebras

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Abstract

We prove that two of the counterexamples of contractions inequivalent to a generalized Inönü-Wigner contraction exhibited in [1] are incorrect. An alternative characterization of general contractions is proposed.

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1 Introduction

In a paper recently published [1], a detailed list of all contractions depending on a parameter occurring among complex and real four dimensional Lie algebras was obtained. In this list, the different types of contractions existing in the literature where analyzed, and all relevant information for the contraction problem of Lie algebras computed. As a result of their analysis, the authors proved the existence of contractions that are not equivalent to a generalized Inönü-Wigner (gen. IW) contraction, showing that theorem 3.1 of [2] is incorrect. This important conclusion reopens the controversy around the characterization of contractions and their validity in physical applications.

In this note we point out that two of the four counterexamples given are invalid, since they are actually realizable as generalized Inönü-Wigner contractions, choosing an adequate representative of the isomorphism class different from that used in [1]. More specifically, we show that there exist orbit representatives of $A_{4,10}$ and $2A_{2,1}$ such that the contractions

$$A_{4,10} \longrightarrow A_{4,1} \longleftarrow 2A_{2,1} \tag{1}$$

can be realized via a diagonal matrix. We also propose a reformulation of the theorem proved invalid, in order to establish a weaker characterization of contractions, which however is more suitable for its application in physical problems.

2 Gen. IW contractions and counterexamples

The notion of contraction of Lie algebras, although generally given in terms of a representative of the isomorphism class, follows naturally from the geometry of orbits [1]. Given a Lie algebra $\mathfrak{g}$ with structure tensor C_{ij}^{k} over a basis $\{X_i\}, i = 1,..,n$, a linear redefinition of the generators via a matrix $M \in GL(n,\mathbb{R})$ gives the transformed structure tensor

$$C'^{n}_{ij} = A_i^k A_j^l (A^{-1})_m^n C_{kl}^m. \tag{2}$$

Taking into account all possible changes of basis, we obtain the orbit $\mathcal{O}(\mathfrak{g})$ of $\mathfrak{g}$ by the action of the linear group $GL(n,\mathbb{R})$, consisting of all Lie algebras that are isomorphic to $\mathfrak{g}$. Thus, for describing the Lie algebra, any of the elements (representatives) of the orbit can be chosen. In this coordinate free interpretation, a Lie algebra $\mathfrak{g}'$ is called a contraction of $\mathfrak{g}$ if $\mathfrak{g}$ belongs to the closure $\overline{\mathcal{O}(\mathfrak{g})}$ of the orbit. This naturally suggests the traditional presentation in terms of limits [5]. Considering a family $\Phi_t \in Aut(\mathfrak{g})$ of non-singular linear maps of $\mathfrak{g}$, where $t \in [1,\infty)$ [2], for any $X, Y \in \mathfrak{g}$ we define

$$[X,Y]_{\Phi_t} := \Phi_t^{-1}\left[\Phi_t(X), \Phi_t(Y)\right], \tag{3}$$

which obviously reproduce the brackets of the Lie algebra over the transformed basis. Actually this is nothing but equation (2) for a special kind of transformations. Now suppose that the limit

$$[X,Y]_{\infty} := \lim_{t\to\infty} \Phi_t^{-1}\left[\Phi_t(X), \Phi_t(Y)\right] \tag{4}$$

exists for any $X, Y \in \mathfrak{g}$. Then equation (4) defines a Lie algebra $\mathfrak{g}'$ which is a contraction of $\mathfrak{g}$, since it lies in the closure of the orbit. Now the contraction is non-trivial if $\mathfrak{g}$ and $\mathfrak{g}'$ are non-isomorphic, i.e., if $\mathfrak{g}'$ is a point of the frontier of $\mathcal{O}(\mathfrak{g})$, and trivial otherwise. In this sense contractions should not be understood as limits between Lie algebras given over fixed bases, but as limiting points of paths connecting two representatives of the orbits of the corresponding Lie algebras. Therefore two contractions $\mathfrak{g}_1 \overset{f_t}{\longrightarrow} \mathfrak{g}_2$ and $\mathfrak{g}'_1 \overset{g_t}{\longrightarrow} \mathfrak{g}'_2$ are equivalent if $\mathfrak{g}'_1 \in \mathcal{O}(\mathfrak{g}_1)$ and $\mathfrak{g}'_2 \in \mathcal{O}(\mathfrak{g}_2)$.

[1] See e.g. [3] and [4].

[2] Other authors use the parameter range $(0,1]$, which is equivalent to this by simply changing the parameter to $t' = 1/t$.

Table 1: Counterexamples

Lie algebra	Structure constants	Contracts onto[3]
$2A_{2,1}$	$[X_1, X_2] = X_1,\ [X_3, X_4] = X_3$	$A_{3,2} \oplus A_1, A_{4,1}$
$A_{3,2} \oplus A_1$	$[X_1, X_3] = X_1,\ [X_2, X_3] = X_1 + X_2$	
$A_{4,1}$	$[X_2, X_4] = X_1,\ [X_3, X_4] = X_2$	
$A_{4,10}$	$[X_1, X_3] = X_1,\ [X_2, X_3] = X_2,$	
	$[X_1, X_4] = -X_2,\ [X_2, X_4] = X_1$	$A_{3,2} \oplus A_1, A_{4,1}$

A contraction for which there exists some basis $\{Y_1, .., Y_n\}$ (i.e., some orbit representative of $\mathcal{O}(\mathfrak{g})$) such that the contraction matrix A_Φ is diagonal, that is, adopts the form

$$(A_\Phi)_{ij} = \delta_{ij} t^{n_j}, \quad n_j \in \mathbb{Z}, t > 0,$$

is called a generalized Inönü-Wigner contraction (see [2]). As commented above, in [1] the contractions

$$A_{4,10} \longrightarrow A_{4,1} \longleftarrow 2A_{2,1} \tag{5}$$

$$A_{4,10} \longrightarrow A_{3,2} \oplus A_1 \longleftarrow 2A_{2,1} \tag{6}$$

are given as non-equivalent to a generalized Inönü-Wigner contraction. The structure constants used there are given in Table 1. We now prove that the contractions (5) are diagonalizable.

2.1 The contraction $A_{4,10} \to A_{4,1}$

The contraction of $A_{4,10}$ onto $A_{4,1}$ is given in [1] by the matrix

$$U_3 = \begin{pmatrix} \epsilon^2 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & -1 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}.$$

This matrix is obviously non-diagonalizable. However, changing the representative of the isomorphism class of $A_{4,10}$, it is easy to construct a generalized Inönü-Wigner contraction onto $A_{4,10}$. To this extent, consider the change of basis given by $X_4' = -X_2 + X_4$ and $X_i' = X_i$ for $i \neq 4$. Then brackets over the new basis $\{X_1', .., X_4'\}$ are

$$[X_1', X_3'] = X_1', \quad [X_1', X_4'] = -X_2', \quad [X_2', X_3'] = X_2', \quad [X_2', X_4'] = X_1', \quad [X_3', X_4'] = X_2'.$$

[3]In this column, only the nontrivial contractions among the listed algebras are considered.

If we now take the family f_t of linear maps given by the diagonal matrix

$$f_t := \begin{pmatrix} t^{-3} & 0 & 0 & 0 \\ 0 & t^{-2} & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix}$$

and putting $X_i'' = f_t(X_i')$, the transformed brackets are easily seen to be:

$$[X_1'', X_3''] = t^{-1}X_1'', \quad [X_1'', X_4''] = -t^{-2}X_2'', \quad [X_2'', X_3''] = t^{-1}X_2'', \quad [X_2'', X_4''] = X_1'',$$
$$[X_3'', X_4''] = X_2''.$$

In the limit $\lim_{t\to\infty} f_t^{-1}\left[f_t X_i', f_t X_j'\right]$, the brackets are immediately seen to coincide with those of $A_{4,1}$, it is therefore isomorphic to it, and showing moreover that the contraction given by matrix U_3 is equivalent to one of generalized Inönü-Wigner type.

2.2 The contraction $2A_{2,1} \to A_{4,1}$

The contraction $2A_{2,1} \to A_{4,1}$ is realized via by the non-diagonalizable matrix

$$U_4 = \begin{pmatrix} -\epsilon^2 & -\epsilon & -1 & -1 \\ 0 & 0 & \epsilon & 0 \\ 0 & -\epsilon^2 & -\epsilon & 0 \\ 0 & 0 & \epsilon & \epsilon \end{pmatrix}.$$

Now consider the change of basis $\mathbf{X}' = A\mathbf{X}$ in $2A_{2,1}$, where A is the matrix given by:

$$A := \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Over this basis $\{X_i'\}$, the brackets of $2A_{2,1}$ are expressed by

$$[X_1', X_3'] = X_1', \quad [X_1', X_4'] = X_2', \quad [X_2', X_3'] = X_2', \quad [X_2', X_4'] = X_1', \quad [X_3', X_4'] = X_2'$$

Now, considering the family g_t of linear maps of $2A_{2,1}$ given by the diagonal matrix

$$g_t := \begin{pmatrix} t^{-4} & 0 & 0 & 0 \\ 0 & t^{-3} & 0 & 0 \\ 0 & 0 & t^{-2} & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix},$$

and putting $X_i'' = g_t(X_i')$, it is straightforward to verify that the transformed brackets are:

$$[X_1'', X_3''] = t^{-2}X_1'', \quad [X_1'', X_4''] = t^{-2}X_2'', \quad [X_2'', X_3''] = t^{-2}X_2'', \quad [X_2'', X_4''] = X_1'',$$
$$[X_3'', X_4''] = X_2'',$$

thus for $t \to \infty$ we get that the only non-vanishing brackets of $\lim_{t\to\infty} g_t^{-1}\left[g_t X_i', g_t X_j'\right]$ are

$$[X_2'', X_4''] = X_1'',\ [X_3'', X_4''] = X_2'',$$

and therefore the corresponding contraction is isomorphic to $A_{4,1}$. This contraction is equivalent to that given by matrix U_4, showing that $2A_{2,1} \to A_{4,1}$ is actually a generalized Inönü-Wigner contraction.

3 Concluding remarks

The remaining contractions

$$A_{4,10} \longrightarrow A_{3,2} \oplus A_1 \longleftarrow 2A_{2,1} \tag{7}$$

are genuine non-diagonalizable contractions, as follows from the analysis in [1] or the detailed analysis of the cohomological properties of the Lie algebras $A_{4,10}, A_{4,1}$ and $A_{3,2} \oplus A_1$ and their orbits.[4] It is worthwhile to observe that the latter contractions can be rewritten as the composition of two gen. IW. contractions, that is, there is an intermediary algebra in each case. Since the corresponding representatives for these diagonal contractions are different, the composition is not simultaneously diagonalizable. These two examples, which invalidate the conclusion of [2], reopen the question of characterizing contractions of Lie algebras. The analysis of contractions and analogous procedures in higher dimensions, where additional counterexamples to those of [1] can be constructed [6, 7], suggests the following restatement of the characterization of [2]:

Conjecture. Any contraction of Lie algebras $\mathfrak{g} \xrightarrow{f_t} \mathfrak{g}'$ is the composition of generalized Inönü-Wigner contractions.

From the physical point of view, the correctness of the previous conjecture allows to recover the physical sense of the limiting process. Generalized IW contractions correspond to re-scaling certain generators of the Lie algebra and taking a limit, and therefore does not alter their physical content, while arbitrary linear combinations mix operators with rather different physical meaning.

[4] For this approach, see e.g. [4].

The existence of a non-diagonalizable contraction which cannot be decomposed as the composition of diagonal ones would therefore seem rather unphysical, and a serious obstruction to the application of contractions in Physics. On the contrary, the repeated composition of contractions would correspond to limiting scaling transformations performed over different bases (from which the non-diagonalizable character of the contractions follows), but still preserving the physical content in each step. Whether the previous restatement of the characterization holds is still an open question, requiring more fundamental invariants of contractions than those known at the present.

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