On Type II Superstrings: The Geometric Canonical Formalism for the (1,1) $\sigma$ Model

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Abstract

The geometric canonical exterior formalism on group manifold, for the heterotic supersymmetric (1,1) sigma model is constructed. This is done by starting from a classical 2D superconformal theory described by the Wess-Zumino-Witten model, where the world-sheet geometry is the (1,1) superspace. In this framework, the motion equations of the dynamical field and the constraints are found and analyzed from the geometric point of view. It can be seen how the use of the canonical exterior formalism is more adequate and simple because of its manifest covariance in all the steps. The relationship between the form brackets defined in the canonical exterior formalism and the Poisson-brackets is given. Later on, the Dirac-brackets are written by using the second class constraints provided by the canonical exterior formalism. As it can be seen the canonical exterior formalism allows to show how the canonical quantization of the heterotic supersymmetric sigma model is facilitated.

1 Introduction

From several years ago the interest of the people in studying two-dimensional models has been made evident. Two-dimensional gravity and supergravity models were constructed from different point of view. There is a vast literature on the subject matter and a complete list may not be feasible. Two years ago[1], the supersymmetric extension of the Jackiw-Teitelboim (1 + 1) linear gravity within the canonical exterior formalism (CEF) on group manifold was constructed. In this context the role of the several fields was well defined. The constraints and field equation were found and analyzed from a geometrical point of view. This supergravity model was also treated in the second order formalism. In that paper only an incomplete list of works were included.
The different type of 2D gravity or supergravity models can be briefly enumerated as follows:

1) A class of linear gravity theories is based on the Riemann scalar curvature $R$. The first model of two-dimensional gravity was constructed by Jackiw and Teitelboim (JT) by means of dimensional reduction of the usual Einstein-Hilbert action in $(2 + 1)$ dimensions\[2, 3, 4, 5, 6\].

2) Two-dimensional gravitational and vector gauge theories by reduction of $D = 3$ topologically massive models\[7\].

3) Later on, by starting from the gauge-theoretic formulation point of view, several works were realized\[8, 9, 10, 11, 12, 13, 14, 15, 16\]. The geometrical structure of the different models obtained in this framework are generally the de Sitter or anti-de Sitter groups (or the corresponding two-dimensional graded ones). All these models have the remarkable property of possessing a topological and gauge invariant formulation. In particular, in Refs.\[13,14\] by using non-geometrical fields other type of two-dimensional gravity models were constructed. These "string-inspired" models are based on the extended Poincaré group. It is possible to prove that "black-hole" solution appears in this kind of models and so its study becomes interesting from the quantum point of view.

In the last years the aforementioned research engendered much further works\[17, 18, 19, 20, 21\].

For instance, in Refs.\[19\] and \[20\], the two-dimensional reduction of the invariant action of the gravitational Chern-Simons model was studied. This was done by means of the Kaluza-Klein like ansatz, decomposing the three-dimensional metric into a two-dimensional metric, a $U(1)$ gauge field $A$ and a scalar field $\phi$. The dimensional reduction procedure yields a two-dimensional topological theory.

In Ref.\[19\] the main problem was to study local classical solutions, while in Ref.\[20\] the solutions are extended at global level in order to construct the Carter-Penrose diagrams. It is shown that two types of local classical solutions exist: symmetry breaking and kink solutions. It is interesting to note that the kink make possible an space whose geometry is asymptotically anti-de Sitter. At small distances the scalar curvature is positive and it vanishes at an intermediate point. So, the effect of the kink is analogous to a geometric gravitational force and it can be proved that the resulting two-dimensional action is formally similar to the action of the dilaton model. In Ref.\[20\], the action is written by using target space coordinates. As it can be seen, the use of such coordinates brings some advantages from classical as well as quantum point of view\[17, 18\]. Also, the Bogomolnyi-Prasad-Sommerfield black holes were studied in the framework of the two-dimensional dilaton supergravity\[21\].

On the other hand, as it is well known the 2D conformal supergravity is the proper framework for the description of superstring theories (see for instance
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Refs. [22-25] and bibliography quoted therein). This intuitive idea is originated by observing that two is the dimension of the world-sheet (WS) spanned by a one-dimensional object while propagating in an external space-time, named target manifold \( M_{\text{target}} \). The two-dimensional manifolds play an important role because they are responsible for the fundamental geometric structure of superstring theory. Moreover, in order to make local the graded algebra, the two-dimensional vielbein and the two-dimensional gravitino are needed. Of course, in a two-dimensional world, the action reduces to a pure divergence in both cases gravity or supergravity and so, the gravitational field is a non-dynamical one. The gravitational field must be interpreted as a Lagrangian multiplier for the corresponding constraints giving the vanishing condition of the matter fields stress-energy tensor. Consequently, the whole gravitational formalism reduces to a theory of boundary conditions in two-dimension and so, only its topology is the matter of interest. In fact, in the path-integral quantization framework, the two-dimensional different metrics become, after division by the diffeomorphism group, in a discrete sum over the topologies, labelled by a positive integer number \( g \), i.e. the genus of the surface. At fixed topology a multiple integral over a finite dimensional space of complex parameters defines the moduli space \( M_g \), whose coordinates label the conformal classes of the WS. Next, by means of a Wick rotation of the time variable, the superstring WS becomes a Riemann surface which can be treated by using all the results provided by the algebraic geometry.

Historically, the important works given in Refs. [22-24] are devoted to the study of the heterotic sigma model and conformal supergravity in two dimensions. These papers are developed in the context of the Lagrangian formalism in components via Noether theorem. By taking into account this last role of 2D conformal supergravity, and following the idea of Ref. [1] the motivation of the present paper is essentially to study - from a mathematical physics point of view - the supersymmetric \((1,1)\) sigma model of type II superstring in the framework of the CEF on group manifold. The first advantage is that this formalism is covariant in all its steps. Moreover, because of the direct relation between the form brackets provided by the CEF and the Dirac brackets, the canonical quantization of the heterotic sigma model is facilitated.

The paper is organized as follows: In Section 2, the main geometric definitions in 2D superconformal space used in the construction of superconformal field theory are given. The fundamental geometrical quantities of the group manifold \( G = M_{\text{target}} \) are written in terms of the left-invariant or right-invariant one-forms containing the Wess-Zumino-Witten field. The supersymmetric action of the \((1,1)\) \( \sigma \) model is written. In Section 3, starting from the geometric Lagrangian density which describes the \((1,1)\) heterotic \( \sigma \) model, the CEF on group manifold is constructed. In Section 4, the equations of motion are found
and their geometrical structure is analyzed. In Section 5, the relation between the CEF and the proper Hamiltonian formalism in components is given. Finally, The Dirac-brackets are defined in order to show that starting from the CEF the canonical quantization of the model is facilitated.

2 Preliminaries: Nomenclature and Definitions

It is well known that every consistent 2D conformal field theory corresponds to a possible string vacuum and it is a suitable starting point for the string perturbation theory. The Green functions of the 2D conformal field theory are then used to construct the string amplitudes. In the geometrical picture, closed string as a one-dimensional loop moving in a smooth target manifold $M_{\text{target}}$ was systematically studied (see for instance Ref.[24]). Hence, it is possible to regard as possible string vacua only those consistent conformal theories which are generated by embedding scalar functions $X^\mu(\xi^\alpha)$ from the world-sheet (WS) to that target space ($X^\mu \in M_{\text{target}}, \xi^\alpha \in WS$). In the two-dimensional framework the embedding scalar functions $X^\mu(\xi)$ must be viewed as scalar fields coupled to the 2D gravitational field with metric $g_{\alpha \beta}(\xi)$. The coupling is realized in such a way that the classical action must be invariant under both, diffeomorphisms and Weyl transformations, relating two different 2D conformal metrics. Moreover, in the case of superstrings, the two-dimensional action contains a convenient set of left-handed and right-handed 2D-fermions.

It is clear that a consistent conformal theory implies that the classical conformal theory mantains the classical Virasoro algebra also at the quantum level. This is done by choosing the field content in such a way that after quantization, all the central charges $c_i$ and the coboundaries $b_i$ corresponding to the different fields in the theory, sum up to zero. So, these quantum conformal theories, given by well defined choices of the target space, are suitable string vacua.

In this section the main geometric definitions used in the supergroup manifold approach for supergravity are recalled. This is done in order to construct geometrically the $\sigma$ model for superstrings of type II moving in an arbitrary target manifold $M_{\text{target}}$, in the framework of the CEF.

2.1 World-Sheet Geometry: $(1,1)$ Superspace

It is well known that the geometric structure underlying in the superspace named $(1,1)$ is that of $2D$ $N = 2$ superconformal algebra. This superalgebra contains: translations $V^a$, conformal boosts $K^a$, Q-supersymetry $\psi$, S-supersymmetry $\phi$, Lorentz rotations $\omega^{ab}$ and dilatations $W$.

In the dual language of the Maurer - Cartan equations the $N = 2$ superconformal algebra reads
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\[ \mathcal{D}V^a + W \wedge V^a - \frac{i}{2} \bar{\psi} \wedge \gamma^a \psi = 0 , \]  
(1)

\[ \mathcal{D}K^a - W \wedge V^a - \frac{i}{2} \bar{\phi} \wedge \gamma^a \phi = 0 , \]  
(2)

\[ \mathcal{D}\psi + \frac{1}{2} W \wedge \psi - i \gamma^a \phi \wedge V_a = 0 , \]  
(3)

\[ \mathcal{D}\phi - \frac{1}{2} W \wedge \phi - i \gamma^a \psi \wedge K = 0 , \]  
(4)

\[ d\omega^{ab} + \bar{\psi} \wedge \gamma^{ab} \phi - 2 V^{[a} \wedge K^{b]} = 0 , \]  
(5)

\[ dW - \bar{\psi} \wedge \phi + 2 V^a \wedge K^a = 0 . \]  
(6)

The two gravitini \( \psi \) and \( \phi \) are respectively Majorana-Weyl and Majorana-anti-Weyl spinors, i.e.

\[ \bar{\psi} \equiv \psi^\dagger \gamma^0 = \psi^T C ; \quad \gamma_3 \psi = \psi , \]  
(7)

\[ \bar{\phi} \equiv \phi^\dagger \gamma^0 = \phi^T C ; \quad \gamma_3 \phi = -\phi . \]  
(8)

These last equations are solved by setting\([25, 26]\)

\[ \psi = e^{-i \frac{\pi}{4}} \begin{pmatrix} \zeta \\ 0 \end{pmatrix} , \]  
(9)

\[ \phi = e^{-i \frac{\pi}{4}} \begin{pmatrix} 0 \\ \chi \end{pmatrix} . \]  
(10)

where \( \zeta^* = \zeta \) and \( \chi^* = \chi \).

A convenient basis is given by

\[ e^\pm = \frac{1}{2} (V^0 \pm V^1) , \quad k^\pm = \frac{1}{2} (K^0 \pm K^1) , \]  
(11)

\[ \omega^{ab} = \varepsilon^{ab} \omega , \quad \omega^\pm = W \pm \omega . \]  
(12)

On this basis the curvatures of the \( N = 2 \) superconformal algebra reads

\[ T^+ = de^+ + \omega^+ e^+ - \frac{i}{2} \zeta \zeta , \]  
(13)
\[ T^- = de^- + \omega^- e^- , \]  
(14)  
\[ \Sigma^+ = dk^+ - \omega^- k^+ , \]  
(15)  
\[ \Sigma^- = dk^- - \omega^+ k^- + \frac{i}{2} \chi \chi , \]  
(16)  
\[ \rho = d\zeta + \frac{1}{2} \omega^+ \zeta - 2\chi e^+ , \]  
(17)  
\[ \sigma = d\chi - \frac{1}{2} \omega^+ \chi - 2\zeta k^- , \]  
(18)  
\[ R^+ = d\omega^+ + 2i\zeta \chi + 8e^+ k^- , \]  
(19)  
\[ R^- = d\omega^- + 8e^- k^+ . \]  
(20)

The (1,1) superspace is described by two bosonic coordinates and two fermionic ones. A complete basis of one-forms is the supervielbein \((e^+, e^-, \zeta, \chi)\). The basic fields in this superspace are the one-forms \((V^a, K^a, \psi, \phi, \omega^{ab}, W)\). So, the torsion and curvature of the (1,1) superspace are respectively defined by

\[ T^+ = de^+ + \omega \wedge e^+ , \]  
(21)  
\[ T^- = de^- - \omega \wedge e^- , \]  
(22)  
\[ T^\bullet = d\zeta + \frac{1}{2} \omega \wedge \zeta , \]  
(23)  
\[ T^\circ = d\chi - \frac{1}{2} \omega \wedge \chi , \]  
(24)  
\[ R = d\omega , \]  
(25)

where \(\omega\) is the superspace spin connection \((\omega = \omega^{10})\).

Since the remaining one-forms \(\omega^\pm y k^\pm\), in the algebra (13-20), live on (1,1) superspace also can be expanded on the \((e^+, e^-, \zeta, \chi)\) basis.

In order to determine the geometry of the (1,1) superspace, constraints on the torsion and curvature in Eqs.[21-25] must be taken into account. Such constraints are imposed by the underlying \(N = 2\) superconforme algebra.

In the framework of the group manifold approach, the constraints come from the so called rheonomic parametrization ( see for instance Ref.[24]).
The parametrization on torsion and curvature must be consistent with the Bianchi identities of the superspace; i.e., directly by exterior differentiation of the definitions given in Eqs.(13-20) and Eqs.(21-25). Therefore the rheonomic parametrization for the superconformal curvatures (13-20) writes

\[ T^+ = 0 , \]  
\[ T^- = -\frac{i}{2} \chi \wedge \chi , \]  
\[ \Sigma^+ = \frac{i}{2} \zeta \wedge \zeta , \]  
\[ \Sigma^- = 0 , \]  
\[ \rho = \tau^\bullet e^+ \wedge e^- + P \chi \wedge e^+ , \]  
\[ \sigma = \tau^\circ e^+ \wedge e^- + P \zeta \wedge e^- , \]  
\[ R^+ = (8 + R)e^+ \wedge e^- + i\tau^\circ \chi \wedge e^+ - i\tau^\bullet \zeta \wedge e^- - iP\zeta \wedge \chi , \]  
\[ R^- = -Re^+ \wedge e^- - i\tau^\circ \chi \wedge e^+ + i\tau^\bullet \zeta \wedge e^- + i(P + 2)\zeta \wedge \chi . \]

Analogously, the rheonomic parametrization for the curvatures of the (1,1)superspace reads

\[ T^+ = \frac{i}{2} \zeta \wedge \zeta , \]  
\[ T^- = -\frac{i}{2} \chi \wedge \chi , \]  
\[ T^\bullet = \tau^\bullet e^+ \wedge e^- + (P + 2)\chi \wedge e^+ , \]  
\[ T^\circ = \tau^\circ e^+ \wedge e^- + (P + 2)\zeta \wedge e^- , \]  
\[ R = -R^- , \]  
\[ \omega = \omega^+ = -\omega^- , \]  

where in the above equations \( R, \tau^\bullet \) and \( \tau^\circ \) are the unique independent fields and \( P \) remains determined by
\[ D_\bullet \tau^* - (P + 2)^2 = \frac{1}{2} R \]  

(40)

2.2 Some definitions in the Target Space, Supervielbein and Geometric action for the (1,1) sigma model

We introduce an embedding function \( X^\mu(z, \bar{z}, \theta, \bar{\theta}) \) mapping the (1,1) superspace into an arbitrary target manifold \( M_{\text{target}} \), and describing the superstring propagation on \( M_{\text{target}} \). By considering the Bianchi identities of the \( M_{\text{target}} \) curvatures, the field equations for the vielbein \( V^a(z, \bar{z}, \theta, \bar{\theta}) \) de \( M_{\text{target}} \) can be determined. These equations can be seen as a 2D superfield i.e, a function of the (1,1) superspace coordinates. The same happen for the target spin connection \( \omega^{ab}(z, \bar{z}, \theta, \bar{\theta}) \).

On the other hand, the torsion and curvature of \( M_{\text{target}} \) are defined as usual

\[ T^a \equiv dV^a + \omega^{ab} \wedge V^b = T^{a}_{bc} V^b \wedge V^c \]  

(41)

\[ R^{ab} \equiv d\omega^{ab} + \omega^{ac} \wedge \omega^{bd} = R^{ab}_{bc} V^c \wedge V^d , \]  

(42)

where the vielbein of \( M_{\text{target}} \) can be expanded in the \((e^+, e^-, \zeta, \chi)\) basis of the (1,1) superspace as follows

\[ V^a = V^a_+ e^+ + V^a_- e^- + \lambda^a \zeta + \mu^a \chi . \]  

(43)

Therefore, the supervielbein \( V^a \) contains two bosonic fields \( V^a_+ \), \( V^a_- \) and two bidimensional fermionic fields \( \lambda^a \) y \( \mu^a \).

The supersymmetric action of the (1,1) \( \sigma \) model writes (see for instance Ref. [25])

\[ S = \int_{\partial M} [(V^a - \lambda^a \zeta - \mu^a \chi) \wedge (\Pi^a_+ e^+ - \Pi^a_- e^-) + 2i \lambda^a \nabla \lambda^a \wedge e^+ + 2i \mu^a \nabla \mu^a \wedge e^- + \lambda^a V^a \wedge \zeta - \mu^a V^a \wedge \chi + \Pi^a_+ \Pi^a_- e^+ \wedge e^- - \lambda^a \mu^a \zeta \wedge \chi + \frac{4i}{3} T^{abc} \lambda^a \lambda^b \lambda^c \zeta \wedge e^+ - 4i T^{abc} \mu^a \mu^b V^c \wedge e^- + 4 R^{ab} c d \lambda^a \lambda^b \mu^c \mu^d e^+ \wedge e^-] + \int_{M} H , \]  

(44)

where \( M \) is a three-dimensional manifold bounded by \( \partial M \) y \( H \) is a closed three-form, i.e, \((dH = 0)\). Moreover variations in \( \Pi^a_\pm \) yield \( \Pi^a_\pm = V^a_\pm \). Now, it is possible to consider the (1,1) \( \sigma \) model on group manifold, that is to say when \( M_{\text{target}} \) = group manifold \( G \). For this purpose we call \( g(z, \bar{z}, \theta, \bar{\theta}) \) the superfield mapping the super world-sheet into a group \( G \) describing the injection

\[ g(z, \bar{z}, \theta, \bar{\theta}) : \text{SW} S \rightarrow G . \]  

(45)
This theory is called the Wess-Zumino-Witten model (WZW).

All the geometrical quantities of $\mathcal{M}_{\text{target}} = G$ can be constructed in terms of left-invariant or right-invariant one-forms

$$\Omega = g^{-1} dg, \quad (46)$$

$$\bar{\Omega} = dg g^{-1}. \quad (47)$$

So, the Lie algebra-valued one forms $\Omega$ and $\bar{\Omega}$ are decomposed along a basis $t_A$ of the Lie algebra associated to the group manifold $G$.

$$\Omega = \Omega^A t_A \quad (48)$$

$$\bar{\Omega} = \bar{\Omega}^A t_A. \quad (49)$$

From the above definition it is obvious that $\Omega^A$ and $\bar{\Omega}^A$ satisfy the Maurer-Cartan equations

$$d\Omega^A + \frac{1}{2} f^{A}_{BC} \Omega^B \wedge \Omega^C = 0 \quad (50)$$

$$d\bar{\Omega}^A - \frac{1}{2} f^{A}_{BC} \bar{\Omega}^B \wedge \bar{\Omega}^C = 0, \quad (51)$$

for the structure constant $f^{A}_{BC}$ of the Lie algebra associated to the group manifold $G$.

Since the one-forms $\Omega^A$ and $\bar{\Omega}^A$ depend on the superspace coordinates $(z, \bar{z}, \theta, \bar{\theta})$, they can be written along a complete superspace basis of one-forms

$$\Omega^A = \Omega^A_+ e^+ + \Omega^A_- e^- + \lambda^A \zeta + \mu^A \chi, \quad (52)$$

and similarly for $\bar{\Omega}^A$.

As already was mentioned the 2D field theory under review is a particular case of a locally supersymmetric non-linear $\sigma$ model. The target space metric $g^{AB}$ is in this case the Killing metric $g^{AB} = f^{AMN} f^{BMN}$. Hence it is necessary to introduce a target space spin connection $\omega^{AB} = -\omega^{BA}$ besides of the target space metric. By means of the structure constant $f^{ABC}$ and by looking at the one form $\Omega^A$ as the vielbein of the group manifold $G$ it is possible to introduce a one-parameter family of spin connection defined by

$$\omega^{AB}_{(\alpha)} = \alpha f^{ABC} \Omega^C. \quad (53)$$

The two-forms torsion and curvature associated to the family of connections written in Eqs. (53), are respectively defined as usual by the expressions.
\[ T^{A}_{(\alpha)} = d \Omega^{A} + \omega^{AB}_{(\alpha)} \wedge \Omega^{B} = T^{ABC}_{(\alpha)} \Omega^{B} \wedge \Omega^{C}, \] 

(54)

\[ R^{AB}_{(\alpha)} = d\omega^{AB}_{(\alpha)} + \omega^{AC}_{(\alpha)} \wedge \omega^{CB}_{(\alpha)} = R^{ABMN}_{(\alpha)} \Omega^{M} \wedge \Omega^{N}, \] 

(55)

where

\[ T^{ABC}_{(\alpha)} = -\left(\alpha + \frac{1}{2}\right) f^{ABC}, \] 

(56)

\[ R^{ABMN}_{(\alpha)} = -\frac{1}{2} \alpha (1 + \alpha) f^{ABC} f^{CMN}. \] 

(57)

In this case the action is obtained from the above one (see Eq. [44]) by making the substitution

\[ V^{a} \rightarrow \frac{k}{2} \frac{1}{\Omega^{A}}. \] 

(58)

In terms of the parameter \( \alpha \) the action reads

\[
S_{(\alpha)} = \frac{k}{8\pi} \left\{ \int_{\partial M} \left[ (\Omega^{A} - \lambda^{A} \zeta - \mu^{A} \chi) \wedge (\Omega^{A}_{+} e^{+} - \Omega^{A}_{-} e^{-}) + 2i\lambda^{A} \nabla_{(\alpha)} \lambda^{A} \wedge e^{+} \right.ight.
\[ + 2i\mu^{A} \nabla_{(\alpha)} \mu^{A} \wedge e^{-} + \lambda^{A} \Omega^{A} \wedge \zeta - \mu^{A} \Omega^{A} \wedge \chi \]
\[ - \frac{4i}{3} (\frac{1}{2} + \alpha) f^{ABC} \lambda^{A} \lambda^{B} \lambda^{C} \wedge e^{+} + \Omega^{A}_{+} \Omega^{A}_{-} e^{+} \wedge e^{-} \]
\[ + \lambda^{A} \mu^{A} \zeta \wedge \chi + \frac{4i}{3} \left(\frac{1}{2} + \alpha\right) f^{ABC} \mu^{A} \mu^{B} \mu^{C} \chi \wedge e^{-} \]
\[ - 2i \left(\frac{1}{2} + \alpha\right) f^{ABC} \mu^{B} \mu^{C} \Omega^{A} \wedge e^{-} \]
\[ + \frac{1}{6} (1 + 2\alpha) \int_{M} f^{ABC} \Omega^{A} \wedge \Omega^{B} \wedge \Omega^{C} \right\}. \]

(59)

The covariant differential of a two-dimensional spinor \( \lambda^{A} \) is given by

\[ \nabla_{(\alpha)} \lambda^{A} = \mathcal{D} \lambda^{A} + \omega^{AB}_{(\alpha)} \lambda^{B}, \] 

(60)

where is well defined

\[ \mathcal{D} \lambda^{A} \equiv d\lambda^{A} + \frac{1}{2} \omega \lambda^{A}. \] 

(61)

Analogously to Eq.(52), the one-form \( \mathcal{D} \lambda^{A} \) is written along a complete superspace basis as follows

\[ \mathcal{D} \lambda^{A} = \mathcal{D}_{+} \lambda^{A} V^{+} + \mathcal{D}_{-} \lambda^{A} V^{-} + \Gamma^{A} \zeta + \Lambda^{A} \chi, \] 

(62)
where the outer component $\Gamma^A$ is given by

$$\Gamma^A = -\frac{i}{2} \Omega^A_\mp + \frac{1}{2} f^{ABC} \lambda^B \lambda^C .$$

(63)

and analogously for $\Lambda^A$.

In the geometric action Eq.(59) the dynamical variables are given by

a) The supergravity background one-form fields $e^+, e^-, \zeta, \chi$.

b) The WZW field $g$ contained in the one-form field $\Omega^A$ and the two fermion fields $\lambda^A$ and $\mu$.

c) The auxiliary 0-forms fields $\Omega^A_+, \Omega^A_-$ which play a double role: i) enforces the rheonomic parametrization, and ii) the field equation yields $\Omega^A_\pm = \text{tr}(g^{-1} \partial_\pm g t^A)$.

The above action for the WZW model is an example of a locally supersymmetric two-dimensional heterotic $\sigma$ model.

3 Canonical Exterior Formalism on Group Manifold

The CEF was constructed and applied to different models of gravity and supergravity in diverse dimensions, as well as their coupling to matter supermultiplets and to the Yang-Mills field[28, 29, 30, 32, 33]. The general response is that this formalism permits to find and study constraints, equation of motion and all the dynamical properties of such systems in a more simple way that following the usual Lagrangian method. As it was already commented, the CEF is covariant in all its steps because it is constructed by using only operation of the exterior algebra.

In the present paper the idea is to work by first time with the CEF applied to the description of the heterotic supersymmetric sigma model in which the supergravity field is a non-dynamical one

In the Eq. [59] the critical values for the parameter $\alpha$ are: $\alpha = 0$, $\alpha = -1$ and $\alpha = -\frac{1}{2}$. In order to simplify algebraic manipulations we will take as critical value for such parameter $\alpha = -\frac{1}{2}$ which correspond to choose a metric connection for which the torsion Eq.[54] vanish. In this case the Lagrangian density reads

$$L = \frac{k}{8\pi} \left[ (\Omega^A - \lambda^A \zeta - \mu^A \chi) \wedge (\Omega^A_+ e^+ - \Omega^A_- e^-) + 2i \lambda^A \nabla \lambda^A \wedge e^+ \\
+ 2i \mu^A \nabla \mu^A \wedge e^- + \lambda^A \Omega^A \wedge \zeta - \mu^A \Omega^A \wedge \chi + \Omega^A_+ \Omega^A_- e^+ \wedge e^- \\
+ \lambda^A \mu^A \zeta \wedge \chi \right].$$

(64)

In order to obtain the equation of motion, instead of the WZW field $g$ contained in the one-form field $\Omega^A$, we can use as dynamical variable the tangent variation i.e
\[
\delta y = g^{-1} \delta g = \delta y^A t_A ,
\]  
which is related to the variation of the one-form \( \Omega^A \) by

\[
\delta \Omega^A = d \delta y^A + f^{ABC} \Omega^B \delta y^C .
\]

The use of the 0-forms variables \( (y^A, \lambda^A, \mu^A) \) allows to obtain, for the three-variables, equation of motion having the same structure.

So, having in mind (61) and (62), and making in Eq.(63) the change of variables, apart from a total exterior derivative the Lagrangian density writes

\[
L = - (\lambda^A \zeta + \mu^A \chi) \wedge \Pi^A - y^A \wedge d\Pi^A + f^{ABC} \Omega^B \delta y^C \wedge \Pi^A
- 2i d\lambda^A \lambda^A \wedge e^+ - 2i d\mu^A \lambda^A \wedge e^-
+ 2i f^{ABC} d\lambda^A \lambda^B y^C \wedge e^+ + 2i f^{ABC} d\mu^A \mu^B y^C \wedge e^-
+ i f^{ABC} \lambda^A \lambda^B y^C \wedge e^+ + i f^{ABC} \mu^A \mu^B y^C \wedge e^-
- if^{ABC} f^{CDE} \lambda^A \lambda^B \Omega^D y^E \wedge e^+ - if^{ABC} f^{CDE} \mu^A \mu^B \Omega^D y^E \wedge e^-
- d\lambda^A y^A \wedge \zeta + d\zeta \lambda^A y^A + d\mu^A y^A \wedge \chi - d\chi \mu^A y^A
+ f^{ABC} \lambda^A \Omega^B y^C \wedge \zeta - f^{ABC} \mu^A \Omega^B y^C \wedge \chi
+ \lambda^A \mu^A \zeta \wedge \chi + \Omega^A_+ \Omega^A_- \wedge e^+ + e^-,
\]  
where was defined

\[
\Pi^A = \Omega^A_+ e^+ - \Omega^A_- e^- .
\]

Therefore, Eq.(67) will be our starting point in order to construct the first-order CEF.

By following Ref.[31], the first step is to define the canonical conjugate momenta to each one of the dynamical fields variables of the model under consideration. The dynamical fields variables are

\[
\mu^\Sigma = (y^A, \lambda^A, \mu^A, e^+, e^-, \zeta, \chi, \Omega^A_+, \Omega^A_-)
\]

for the compound index \( \Sigma \).

By means of the functional variation of the Lagrangian with respect to the "velocities" \( d\mu^\Sigma \), i.e: \( \pi^\Sigma = \delta L / \delta d\mu^\Sigma \), the canonical conjugate momenta remain defined as follows:

i) The momenta associated with the 0-forms \( y^A, \lambda^A \) respectively read

\[
P^A = \frac{\delta L}{\delta (dy^A)} = 0
\]

bosonic one-form,
\[ Q^A = \frac{\delta L}{\delta (d\lambda^A)} = -2i \lambda^A e^+ + 2i f^{ABC} \lambda^B y^C e^+ - y^A \zeta \]  
fermionic one form, and 
\[ R^A = \frac{\delta L}{\delta (d\mu^A)} = -2i \mu^A e^- + 2i f^{ABC} \mu^B y^C e^- + y^A \chi \]  
fermionic one form.

ii) The momenta associated with the supergravity background one-form fields \( e^+ \), \( e^- \), \( \zeta \) and \( \chi \) are

\[ \pi_+ = \frac{\delta L}{\delta (d e^+)} = i f^{ABC} \lambda^A \lambda^B y^C - y^A \Omega^A_+ \]  
bosonic 0-form,

\[ \pi_- = \frac{\delta L}{\delta (d e^-)} = i f^{ABC} \mu^A \mu^B y^C + y^A \Omega^-_+ \]  
bosonic 0-form,

\[ \pi_\zeta = \frac{\delta L}{\delta (d \zeta)} = \lambda^A y^A \]  
fermionic 0-form, and

\[ \pi_\chi = \frac{\delta L}{\delta (d \chi)} = -\mu^A y^A . \]  

iii) The momenta associated with the 0-forms \( \Omega^A_+ \), \( \Omega^-_+ \) are respectively the following bosonic one-form

\[ P^A_+ = \frac{\delta L}{\delta (d \Omega^A_+)} = -y^A e^+ , \]  
and

\[ P^-_A = \frac{\delta L}{\delta (d \Omega^-_+)} = y^A e^- \]  
bosonic one-form.

In the CEF it is necessary to define a suitable operation involving forms, capable of replacing the role of the classical Poisson brackets. Therefore, the graded form-brackets operation between pairs of canonical variables is defined and it is given by

\[ (\mu^\Sigma, \pi_\Lambda) = (-1)^{|a| + |A|} \delta^\Sigma_\Lambda , \]  
(78)
where $a$ and $|A|$ are respectively the degree and the Fermi grading of the form $\mu^\Sigma$. The remaining form-brackets properties for generic superforms were written in Eqs.(2.2) of Ref.[31].

In the present case the form-brackets between pairs of canonical variables writes

\begin{align}
(y^A, P_B) &= (P_B, y^A) = -\delta^A_B, \quad (79) \\
(\lambda^A, Q_B) &= -(Q_B, \lambda^A) = \delta^A_B, \quad (80) \\
(\mu^A, R_B) &= -(R_B, \mu^A) = \delta^A_B, \quad (81) \\
(e^+, \pi_+) &= (\pi_+, e^+) = 1, \quad (82) \\
(e^-, \pi_-) &= (\pi_-, e^-) = 1, \quad (83) \\
(\zeta, \pi_\zeta) &= -(\pi_\zeta, \zeta) = -1, \quad (84) \\
(\chi, \pi_\chi) &= -(\pi_\chi, \chi) = -1, \quad (85) \\
(\Omega^A_+, P^B_+) &= (P^B_+, \Omega^A_+) = -\delta^{AB}, \quad (86) \\
(\Omega^A_-, P^B_-) &= (P^B_-, \Omega^A_-) = -\delta^{AB}. \quad (87)
\end{align}

The set of momenta (69)-(77) define the following primary constraints

\begin{align}
\Phi^A &= P^A \approx 0, \quad (88) \\
\Psi^A &= Q^A + 2i \lambda^A e^+ - 2i f^{ABC} \lambda^B y^C e^+ + y^A \zeta \approx 0, \quad (89) \\
\Upsilon^A &= R^A + 2i \mu^A e^- - 2i f^{ABC} \mu^B y^C e^- - y^A \chi \approx 0, \quad (90) \\
\varphi_+ &= \pi_+ - i f^{ABC} \lambda^A \lambda^B y^C + y^A \Omega^A_+ \approx 0, \quad (91) \\
\varphi_- &= \pi_- - i f^{ABC} \mu^A \mu^B y^C - y^A \Omega^A_- \approx 0, \quad (92) \\
\varphi_\zeta &= \pi_\zeta - \lambda^A y^A \approx 0, \quad (93)
\end{align}
\[ \varphi_\chi = \pi_\chi + \mu_\chi y_\chi \approx 0, \quad (94) \]
\[ \Theta_\pm = \mathcal{P}_\pm + y_\pm e^\pm \approx 0, \quad (95) \]
\[ \Theta_\mp = \mathcal{P}_\mp - y_\mp e^- \approx 0. \quad (96) \]

By considering the definition and properties of the graded form-brackets written in Eqs. (2.2) of Ref. [31], it is possible to compute the form-brackets \( (\Phi_\Sigma, \Phi^\Lambda) \) for pairs of constraints. It is straightforward to prove that all the primary constraints (88)-(96) are second-class ones, that is
\[ (\Phi_\Sigma, \Phi^\Lambda) \neq 0. \quad (97) \]

In the CEF, the conserved first-class dynamical quantity describing the dynamics of the system is the extended Hamiltonian \( H_T \), and it is the bosonic two-form defined by (see Ref. [31])
\[ H_T = H_{can} + \Lambda_\Sigma \wedge \Phi_\Sigma, \quad (98) \]
where the Lagrange multipliers \( \Lambda_\Sigma \) can be unambiguously determined. When the fundamental equation of motion in the CEF is taken into account, it is possible to write the Hamiltonian equations for pairs of canonical variables,
\[ d\mu_\Sigma = \left( \mu_\Sigma, H_T \right), \quad (99) \]
\[ d\pi_\Sigma = \left( \pi_\Sigma, H_T \right). \quad (100) \]

From Eq. (99) and by using Eq. (98) the following general result is obtained
\[ \Lambda_\Sigma = d\mu_\Sigma. \quad (101) \]

In Eq. (98) the canonical Hamiltonian \( H_{can} = d\mu_\Sigma \wedge \pi_\Sigma - \mathcal{L} \) is given by
\[ \begin{align*}
H_{can} &= dy_\Sigma \wedge P_\Sigma + d\lambda_\Sigma \wedge Q_\Sigma + d\mu_\Sigma \wedge R_\Sigma + de_+ \wedge \pi_+ + de^- \wedge \pi_- \\
&+ d\zeta \wedge \pi_\zeta + d\chi \wedge \pi_\chi + d\Omega_+ \wedge \mathcal{P}_+ + d\Omega_- \wedge \mathcal{P}_- - \mathcal{L}, \quad (102)
\end{align*} \]
which after using Eq. (67) for the Lagrangian it results
\[ \begin{align*}
H_{can} &= (\lambda_\Sigma \chi + \mu_\Sigma \chi) \wedge \Pi_\Sigma - f^{ABC} \Omega_B y^C \wedge \Pi_\Sigma \\
&+ if^{ABC} f^{CDE} \lambda_\Sigma \lambda^B \Omega^D y^E \wedge e^+ \\
&+ if^{ABC} f^{CDE} \mu_\Sigma \mu^B \Omega^D y^E \wedge e^- \\
&- f^{ABC} \lambda_\Sigma \Omega_B y^C \wedge \zeta + f^{ABC} \mu_\Sigma \Omega^B y^C \wedge \chi \\
&- \lambda_\Sigma \mu_\Sigma \chi \wedge \chi + \Omega_+ \Omega_- \wedge e^+ + \Omega_- e^- . \quad (103)
\end{align*} \]
4 Motion Equations in the Canonical Exterior Formalism

In the CEF the field equations of motion are given by the consistency conditions on the primary constraints, i.e,

\[ d\Phi^\Sigma = (\Phi^\Sigma, H_T) \approx 0. \]  

(104)

As it was commented above the vielbein and the gravitino are not dynamical fields in 2D, therefore the motion equation for the supergravity background fields \( e^+, e^-, \zeta \) and \( \chi \) will be not considered. The supergravity background fields play the role of Lagrange multipliers associated to the primary constraints of the theory, that is the superstress-energy tensor and the supercurrent. In fact, the superstress-energy tensor and the supercurrent one-forms are respectively defined by making the variation of the action (59) with respect to the supervielbein \( (e^+, e^-, \zeta, \chi) \). As it is known in the classical theory these quantities are weakly zero ones. From the quantum point of view they are used to construct the BRST - charge.

About the variables \( \Omega_+^A \) and \( \Omega_-^A \) we remember that they are introduced to enforce the rheonomic parametrization.

Therefore, the main equations are those for the fields \( y^A, \lambda^A \) and \( \mu^A \), which respectively read

\[ d\Phi^M = (\Phi^M, H_T) \]
\[ = (P^M, H_{can}) + \Lambda^B \wedge (\Phi^M, \Phi_B) + \Sigma^B \wedge (\Phi^M, \Psi_B) + \Gamma^B \wedge (\Phi^M, \Upsilon_B) + \Lambda_+ \wedge (\Phi^M, \varphi_+) + \Lambda_- \wedge (\Phi^M, \varphi_-) + \Sigma^\zeta \wedge (\Phi^M, \varphi_\zeta) + \Sigma^\chi \wedge (\Phi^M, \varphi_\chi) + \Delta_+^A \wedge (\Phi^M, \Theta_+^A) + \Delta_-^A \wedge (\Phi^M, \Theta_-^A) \]
\[ + \text{weakly zero terms} = 0, \]  

(105)

\[ d\Psi^M = (\Psi^M, H_T) \]
\[ = (Q^M, H_{can}) + dy^B \wedge (\Psi^M, \Phi_B) - d\lambda^B \wedge (\Psi^M, \Psi_B) - d\mu^B \wedge (\Psi^M, \Upsilon_B) + de^+ \wedge (\Psi^M, \varphi_+) + de^- \wedge (\Psi^M, \varphi_-) + d\zeta \wedge (\Psi^M, \varphi_\zeta) + d\chi \wedge (\Psi^M, \varphi_\chi) + d\Omega_+^A \wedge (\Psi^M, \Theta_+^A) + d\Omega_-^A \wedge (\Psi^M, \Theta_-^A) \]
\[ + \text{weakly zero terms} = 0, \]  

(106)

\[ d\Upsilon^M = (\Upsilon^M, H_T) \]
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\[(R^M, H_{can}) + dy^B \wedge (\gamma^M, \Phi_B) - d\lambda^B \wedge (\gamma^M, \Psi_B)
- d\mu^B \wedge (\gamma^M, \Upsilon_B) + de^+ \wedge (\gamma^M, \varphi_+) + de^- \wedge (\gamma^M, \varphi_-)
+ d\zeta \wedge (\gamma^M, \varphi_\zeta) + d\chi \wedge (\gamma^M, \varphi_\chi)
+ d\Omega_+ \wedge (\gamma^M, \Theta_+^A) + d\Omega_- \wedge (\gamma^M, \Theta_-^A)
+ \text{weakly zero terms} = 0 \].

(107)

Looking at the above equation we see that the following explicit expressions for the form-brackets between constraints, must be considered:

\[(\Phi^A, \Phi^B) = 0, \quad (108)\]

\[(\Phi^A, \Psi^B) = 2ie^{ABC} \lambda^C e^+ - \zeta \delta^{AB}, \quad (109)\]

\[(\Phi^A, \varphi_+) = if^{ABC} \lambda^B \lambda^C - \Omega_+^A, \quad (110)\]

\[(\Phi^A, \varphi_-) = if^{ABC} \mu^B \mu^C + \Omega_-^A, \quad (111)\]

\[(\Phi^A, \varphi_\zeta) = \lambda^A, \quad (112)\]

\[(\Phi^A, \Upsilon^B) = 2ie^{ABC} \mu^C e^- + \chi \delta^{AB}, \quad (113)\]

\[(\gamma^A, \gamma^B) = -4ie^- \delta^{AB}, \quad (114)\]

\[(\Phi^A, \varphi_\chi) = -\mu^A, \quad (115)\]

\[(\Psi^A, \Psi^B) = -4ie^+ \delta^{AB}, \quad (116)\]

\[(\Psi^A, \gamma^B) = 0, \quad (117)\]

\[(\Psi^A, \varphi_+) = 2i\lambda^A, \quad (118)\]

\[(\Psi^A, \varphi_-) = 0, \quad (119)\]

\[(\Psi^A, \varphi_\zeta) = 0, \quad (120)\]

\[(\Psi^A, \varphi_\chi) = 0, \quad (121)\]
\[(\Upsilon^A, \varphi_+) = 0 , \quad (122)\]
\[(\Upsilon^A, \varphi_-) = 2i\mu^A , \quad (123)\]
\[(\Upsilon^A, \varphi_\zeta) = 0 , \quad (124)\]
\[(\Upsilon^A, \varphi_\chi) = 0 , \quad (125)\]
\[(\Phi^A, \Theta^B_+) = -\delta^{AB} e^+ , \quad (126)\]
\[(\Phi^A, \Theta^B_-) = \delta^{AB} e^- , \quad (127)\]
\[(\Psi^A, \Theta^B_+) = (\Psi^A, \Theta^B_-) = 0 , \quad (128)\]
\[(\Upsilon^A, \Theta^B_+) = (\Upsilon^A, \Theta^B_-) = 0 , \quad (129)\]
\[(\varphi_+, \Theta^A_+) = (\varphi_+, \Theta^A_-) = 0 , \quad (130)\]

where the form brackets between constraints involved in Eqs.(105),(106) and (107) were only written.

By replacing in the motion Eqs.(105),(106) and (107) the above expressions for the form-brackets between constraints, they respectively read

\[d\Phi^M = -d\Pi^M + f^{MBC} \Omega^B \wedge \Pi^C - \mathcal{D} \lambda^M \wedge \zeta - \lambda^M T^* \]
\[- f^{MBC} \Omega^B \lambda^C \wedge \zeta + 2i f^{MBC} \lambda^B \mathcal{D} \lambda^C \wedge e^+ - \frac{1}{2} f^{MAB} \lambda^A \lambda^B \zeta \wedge \zeta \]
\[+ i f^{MBC} f^{CDE} \Omega^B \lambda^D \lambda^E \wedge e^+ + \mathcal{D} \mu^M \wedge \chi + \mu^M T^0 \]
\[+ f^{MBC} \Omega^B \mu^C \wedge \chi + 2i f^{MBC} \mu^B \mathcal{D} \mu^C \wedge e^- - \frac{1}{2} f^{MAB} \mu^A \mu^B \chi \wedge \chi \]
\[+ i f^{MBC} f^{CDE} \Omega^B \mu^D \mu^E \wedge e^- \]
\[+ \text{ weakly zero terms} = 0 , \quad (131)\]

\[d\Psi^M = 4i \nabla \lambda^M \wedge e^+ - \zeta \wedge \Pi^M - \zeta \wedge \Omega^M \]
\[- \lambda^M \zeta \wedge \zeta + \mu^M \zeta \wedge \chi \]
\[+ \text{ weakly zero terms} = 0 , \quad (132)\]
\[
\begin{align*}
\text{d} \Upsilon^M &= 4i \nabla \mu^M \wedge e^- \wedge \Pi^M + \chi \wedge \Omega^M \\
&- \mu^M \chi \wedge \chi - \lambda^M \zeta \wedge \chi \\
&+ \text{weakly zero terms} = 0 .
\end{align*}
\]

The equations (131), (132) and (133) defined over the heterotic superspace are two-forms. Therefore, having the same structure they can be decomposed into the independent sectors corresponding to the inner-inner direction \(e^+ \wedge e^-\), the inter-outer directions \(e^+ \wedge \zeta\), \(e^- \wedge \zeta\), \(e^+ \wedge \chi\) and \(e^- \wedge \chi\), and the outer-outer direction \(\zeta \wedge \zeta\), \(\chi \wedge \chi\) and \(\zeta \wedge \chi\).

The first step is to consider the Maurer-Cartan two-form equation (50) and the one-forms defined in (52),(62) and (63) decomposed along the supergravity background one-form fields \((e^+, e^-, \zeta, \chi)\).

By straightforward calculation it can be shown:

i) Considering the Eq. (131) it can be seen that the coefficients of the inner-outer and outer-outer directions cancel automatically, when the rheonomic parametrization (eqs. (52), (62) and (63)) is introduced. On the other hand, the cancelation of the component \(e^+ \wedge e^-\) gives rise to the following condition

\[
\begin{align*}
\mathcal{D}_- \Omega^A_+ + \mathcal{D}_+ \Omega^A_- - \tau^* \lambda^A - 2if^{ABC} \lambda^B \mathcal{D}_- \lambda^C - if^{ABC} f^{CDE} \Omega^B_- \lambda^D \lambda^E \\
+ \tau^o \mu^A - 2if^{ABC} \mu^B \mathcal{D}_+ \mu^C - if^{ABC} f^{CDE} \Omega^B_+ \mu^D \mu^E = 0 .
\end{align*}
\]

ii) Analogously, by considering the Eq. (132) it can be seen that the coefficients of the inner-outer directions cancel automatically, while the cancelation of the component \(e^+ \wedge e^-\) gives rise to the following condition

\[
\begin{align*}
\mathcal{D}_- \lambda^A - \frac{1}{2} f^{ABC} \lambda^B \Omega^C_- = 0 ,
\end{align*}
\]

and the condition on the inner-inner component of Eq.(133) writes

\[
\begin{align*}
\mathcal{D}_+ \mu^A - \frac{1}{2} f^{ABC} \mu^B \Omega^C_+ = 0 .
\end{align*}
\]

Now, considering the different projections for the Maurer-Cartan equation (50), the following conditions are found:

iii) The coefficient cancelation of the components \(e^+ \wedge \zeta\) and \(e^- \wedge \zeta\) implicates respectively the following conditions

\[
\begin{align*}
\mathcal{D}_o \Omega^A_+ - \mathcal{D}_+ \lambda^A - f^{ABC} \Omega^B_+ \lambda^C = 0 ,
\end{align*}
\]

\[
\begin{align*}
\mathcal{D}_o \Omega^A_- - \mathcal{D}_- \lambda^A - f^{ABC} \Omega^B_- \lambda^C = 0 ,
\end{align*}
\]
and similar geometric conditions for the inner-outer components $e^+ \wedge \chi$ and $e^- \wedge \chi$.

Finally, the cancelation of the coefficient of, $e^+ \wedge e^-$ gives rise to the Bianchi identity, i.e

$$D_+ \Omega^A_+ - D_- \Omega^A_- - \tau^* \lambda^A - \tau^\circ \mu^A + f^{ABC} \Omega^B_+ \Omega^C_- = 0. \quad (139)$$

The coefficient of the outer-outer directions cancel automatically.

Therefore, the conclusion is that the motion field equations (131), (132) and (133) for the fields $y^A$, $\lambda^A$ and $\mu^A$ are reduced to the differential equations (135), (136), (137) and (138), and the remaining conditions are all geometrical ones.

5 Canonical exterior formalism versus proper Hamiltonian formalism

From the above construction it can be seen that the CEF is covariant in all the steps. But, as it was commented in detail in Ref.[31] the CEF is not a proper Hamiltonian formalism because the extended Hamiltonian $H_T$ defined in Eq.(98) is not a true generator of time evolutions. The form-brackets do not contain the same information as the Poisson brackets. Really, the Poisson-brackets contain more information than the form-brackets defined in the CEF. In fact, the CEF can be related with the Hamiltonian formalism in components, and so the form-brackets are related to the Poisson brackets but not in a trivial way[28, 31]. The integral relationship which relates the form-brackets $(A, B)$ to the Poisson brackets between forms $[A(x), B(y)]$ is given by

$$(-1)^{a+1} \int_\Sigma \alpha \wedge (A, B) \wedge \beta = \int \int_{\Sigma \times \Sigma} \alpha(x) \wedge [A(x), B(y)] \wedge \beta(y), \quad (140)$$

where $a$ is the degree of the form $A$ and $\alpha, \beta$ are text forms.

On the other hand, it is well known that the second order formalism is necessary when the model is considered from the quantum point of view. In fact, it is in the second order formalism where the dynamical degrees of freedom are separated from those of gauge degrees of freedom.

In Ref.[1] Sect. 4, the second order formalism by solving the torsion field equation in two-dimensional supergravity models was studied in detail. In this paper we only reproduce a few useful concepts.

When the spacetime decomposition is considered and a privileged time direction is chosen in the manifold $M^2$, the manifest covariance is lost. Usually, the time variable is chosen so that the one-form $dx^0$ can be detached. More
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precisely, we consider fields and forms defined on a spacelike \( x^0 = t = t^0 \) one-dimensional "surface" \( \Sigma \), by defining the injection map \( \chi : \Sigma \to M^2 \). Thus, the associated pullback \( \chi^* \) acts on any form by setting \( t = t^0 \) and \( dt^0 = 0 \).

Once the space-time decomposition is done and the surface \( \Sigma \) remains defined, the ordinary Poisson brackets are obtained by expanding the forms \( A(x) \) and \( B(y) \), given in Eq.(140), in the holonomic bases \( dx^i, dy^j \). Then, the ordinary Poisson brackets between fields and momenta components can be used.

All the quantities provided by the CEF, i.e the total Hamiltonian, the constraints and the field equations must be projected on the "surface" \( \Sigma \). Once the canonical conjugate momenta \( \pi_A \) are written in terms of the spatial components \( dx^i \) of the holonomic basis, the Poisson brackets between pairs of canonical variables remain defined as usual.

The final form of the Hamiltonian as the generator of time evolutions in the canonical component formalism is obtained by taking into account the metricity condition in one and two dimensions (see Eqs. (50)-(54) Sect. 4 of Ref.[1]).

Another question to take into account is that the CEF plays, with respect to the first order canonical component formalism, an analogous role to that played by the first order canonical component formalism with respect to the second order formalism. Therefore, we will consider that all the primary constraints in the CEF remain at least weakly zero in the canonical component formalism (see, for instance Refs.[28,31]). Consequently, we assume that the restrictions to \( \Sigma \) of the constraints (88),(89), (90), (93)-(96) are strongly equal to zero. For the remaining constraints (91),(92) \( \varphi_{\pm} \) the restriction to \( \Sigma \) is maintained as a weakly zero quantity, i.e:

\[
\chi^* \varphi_{\pm} \approx 0 .
\] (141)

The bosonic 2-form (98) provides by the CEF can be written as follows

\[
\int H_T = \int dx^0 \tilde{H} ,
\] (142)

where the time variable is chosen so that the 1-form \( dx^0 \) can be detached. The remaining bosonic one-form integrated in one dimension is the proper Hamiltonian generator of time evolutions and it turns out to be of the form

\[
\tilde{H} = \int dx \left( \frac{1}{2} \omega_0 \mathcal{H} + V_{a0} \mathcal{H}^a + \bar{\xi}_{0a} \mathcal{H}^a \right) ,
\] (143)

Finally, it can be proven that the constraints \( \mathcal{H}, \mathcal{H}^a \) and \( \mathcal{H}_c \) are the first-class constraints closing the following constraint superalgebra

\[
[\mathcal{H}_A(x), \mathcal{H}_B(y)] = \Lambda^C_{AB} \mathcal{H}_C(x) \delta(x - y) ,
\] (144)
where $\Lambda_{AB}^{C} = R_{AB}^{C} - C_{AB}^{C}$ are the structure functions for curvatures $R_{AB}^{C}$ and structure constant $C_{AB}^{C}$ of the graded Lie algebra. In particular it is easy to see that the antisymmetric weakly zero quantity $\mathcal{H}$ that appears in Eq.[143] is the generator of local Lorentz rotations, that in context of the CEF naturally appears when the space-time decomposition is carried out. Contrarily, starting from the component Hamiltonian formalism, the generator of local Lorentz rotations must be introduced 	extit{ad hoc} by demanding the closure of the constraint algebra.

By following the same steps as those given in Sect. 4 of Ref.[1], it is straightforward to explicitly write the first class constraints that verify the Eqs.[144].

Before concluding this section a further consideration about the exterior canonical formalism must be done: as it was said, all the primary constraints provided by the CEF are second-class ones, and so they are not related with the gauge symmetry of the model. Moreover, the possibility of using different Lagrangian densities means that there is not a unique set of canonical conjugate momenta and consequently there is not a unique set of primary constraints in the CEF. On the other hand, in the second-order formalism the second-class constraints must be eliminated. This is done by defining the Dirac brackets from the Poisson brackets. As it is well known the Dirac brackets $[F, G]^{D}$ for generic functional $F$ and $G$ are obtained from the set of second-class constraints $\Psi_A$ by means of the definition

$$[F, G]^{D} = [F, G] - [F, \Psi_A]C^{ABC}[\Psi_B, G],$$

(145)

where $C^{ABC}[\Psi_B, \Psi_C] = \delta^A_C$ for the compound indices $A, B, C$. To compute the Dirac brackets (145) we must consider the restriction to $\Sigma$ of all the second-class constraints (88)-(96).

As it is known, the main properties of the Dirac brackets are:

i) If one of the function $F$ or $G$ is first class, then

$$[F, G]^{D} \approx [F, G].$$

(146)

In particular, for the Hamiltonian $\mathcal{H}$ holds

$$[F, \mathcal{H}]^{D} \approx [F, \mathcal{H}].$$

(147)

This means that the same equations of motion are obtained by using the Poisson or the Dirac brackets. Thus, the rate of change in time of any functional $F$ of the canonical variables is also given by

$$\dot{F} = [F, \mathcal{H}]^{D}. $$

(148)

ii) For any functional $F$ of the canonical variables it is
\[ [\Psi_A, \mathcal{H}]^D = 0. \] (149)

Therefore, we can set \( \Psi_A = 0 \) either before or after evaluating the Dirac brackets.

Once the Dirac brackets are evaluated from the equation (145), the transition to quantum theory is realized as usual in a canonical formalism by replacing classical fields by quantum field operators acting on some Hilbert space. Consequently, the canonical Dirac brackets are replaced by quantum commutators between field operators.

Finally, let us a few words about the well known canonical quantization procedure that appears in the usual literature.

We consider as an example the WZW field \( g(z, \bar{z}) \) defined in Eq. [45] for the bosonic case. It can be shown that \( \Omega^A_+ \) becomes a conserved chiral current i.e, \( \mathcal{D}_- \Omega^A_+ = 0 \). This last equation implies \( \mathcal{D}_+ \Omega^A_+ = 0 \). Therefore, out of an on-shell WZW field \( g(z, \bar{z}) \) two analytic currents can be constructed

\[
J^A(z) = -i \frac{k}{2} \Omega^A_+ = i \frac{k}{2} \Omega^A_z = -i \frac{k}{2} \text{tr}(g^{-1} \partial_z g t^A) \tag{150}
\]

\[
J^A(\bar{z}) = -i \frac{k}{2} \Omega^A_\bar{z} = i \frac{k}{2} \Omega^A_{\bar{z}} = -i \frac{k}{2} \text{tr}(\partial_{\bar{z}} g g^{-1} t^A) \tag{151}
\]

Now, as it is known from the current literature, the Dirac brackets of the dynamical variables \( J^A(z) \) and \( \bar{J}^A(\bar{z}) \) write as follows

\[
[J^A(z), J^B(w)]^D = i \frac{f^{ABC} J^C(w)}{(z - w)} + \frac{k}{2} \frac{\delta^{AB}}{(z - w)^2}, \tag{152}
\]

\[
[J^A(z), \bar{J}^B(\bar{w})]^D = 0, \tag{153}
\]

\[
[J^A(\bar{z}), J^B(\bar{w})]^D = i \frac{f^{ABC} J^C(\bar{w})}{(\bar{z} - \bar{w})} + \frac{k}{2} \frac{\delta^{AB}}{(\bar{z} - \bar{w})^2}. \tag{154}
\]

After performing a Laurent series expansion of the two fields \( J^A(z) \) and \( \bar{J}^A(\bar{z}) \) and replacing the canonical Dirac brackets by the quantum commutators between field operators the canonical quantization is realized.

### 6 Conclusions

We conclude that due to its intrinsic geometrical language, the CEF can be used as an interesting formal resource to understand the structure of the supergravity field theories in diverse dimensions, as well as the heterotic supersymmetric sigma model describing type II superstrings.
The first remark is that the CEF is not a proper canonical formalism because it does not propagate data defined on an initial surface as it is required by a standard mechanical system. However, as it can be seen from the above construction, that the CEF is a powerful method at classical level. Due to the covariance of the CEF in all its steps this formalism allows to find the equations of motion and the constraints in a very simple way without introducing complicate algebraic manipulations.

Since all the primary constraints coming from the CEF are second-class ones, the Dirac brackets are easily defined by projecting these constraints on the surface Σ.

The relation between the CEF and the usual first-order canonical formalism written in components, was also given. This relation was done by means of a non trivial integral relationship between the form-brackets and the usual Poisson brackets.

In order to go over the second-order formalism, the space-time decomposition in $M^2$ was performed, losing the explicit covariance of all the equations. Once this is done, the Hamiltonian system is treated as usual according to the Dirac prescriptions. From the total Hamiltonian coming from the CEF, it is evaluated the proper Hamiltonian (143) as the generator of time evolution. As it was shown, the primary constraint obtained in the CEF also plays an important role in the construction of the proper Hamiltonian (143). Precisely, it is given in terms of the first-class constraints which close the constraint algebra. Therefore, all the Hamiltonian gauge symmetries remain determined and the apparent gauge degrees of freedom can be unambiguously removed leaving only the physical ones. When the model is considered from the quantum point of view this last step is necessary.

Finally, the CEF can be used for describing the heterotic $\sigma$ model on a general target space $M_{target}$ of dimension $D = d_{\text{Mink}} + d_{\text{compact}}$.

References


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Received: April 1, 2008