Stationary Axisymmetric Solutions

of Einstein’s Equations for a Perfect Fluid in Rigid or Differential Rotation

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Abstract. Two stationary metrics are derived as solutions of Einstein’s equations, one representing rigid rotation of a perfect fluid and the other differential rotation. These are obtained as solutions of the two second order non-linear ordinary differential equations which describe a rotating family first presented by Senovilla [15]. The properties of the solutions are analysed and physical interpretations in terms of rotating fluid objects tentatively advanced.

Keywords: Stationary, Rotation, Rigid, Differential

1. INTRODUCTION

Rather few exact stationary solutions of Einstein’s equations of general relativity that involve a fluid in rotation are to be found in the literature. Dust solutions and Einstein-Maxwell cases, which we shall not enumerate here (see e.g. [17]), have featured but very few for a perfect fluid, and especially few in differential rotation. In this paper we shall be concerned with a perfect fluid in steady rigid or differential rotation.

A well known case of rigid rotatory (and therefore shearfree) motion of a perfect fluid is the Petrov type D solution due to Wahlquist [18], who used generalised oblate-spheroidal coordinates. For certain values of the parameters there is a bounding pressure free surface inside which there is no singularity. For a
limiting case of the parameters the solution becomes spherically symmetric with the density $\mu$ and pressure $p$ satisfying $\mu + 3p = \text{const.}$

A specially developed coordinate system for rigid rotation by Bonanos and Sklavenites [1] reduced the problem to two partial differential equations for two unknown functions. Using this approach solutions were found by Sklavenites [16] and Kyriakopoulos [10], [11], the equipressure surfaces being planes. A feature of Sklavenites’ solutions was the inclusion of the condition that the magnetic Weyl tensor vanishes. Earlier, Collins [4] had shown that this condition, in a motion where a perfect fluid is rotating but shearfree and satisfying an equation of state $p = p(\mu)$ ($\mu + p \neq 0$), leads to the fluid’s expansion being zero. Collins inferred that in general this class of spacetime was stationary and axisymmetric (admitting a 2-dimensional Abelian isometry group). He also showed that in this class the vortex lines are geodesic and hypersurface orthogonal, and that all such solutions are of Petrov type D.

Among the various specialisations and assumptions used to obtain stationary solutions, Kramer [9] postulated the presence of a proper conformal Killing vector orthogonal to the orbits of the 2-dimensional group of motions. Collins [4] had shown that a stationary and shearfree motion which has a zero magnetic Weyl tensor will also admit a conformal vector, directed parallel to the vorticity. In this way Kramer was able to reproduce a solution obtained earlier by Senovilla [14]. Senovilla had derived that solution with the assumptions that the metric was of Petrov type D, rigidly rotating and having the fluid 4-velocity in the plane spanned by the two repeated null directions of the Weyl tensor. This metric, in comoving coordinates, had an equation of state $\mu = p + \text{const.}$ The solution contains a log of the $r$ coordinate which induces a singularity at $r = 0$.

Turning to solutions with differential rotation, as remarked elsewhere [17,p. 338), there are few. Especially there are few that may represent an isolated body. Fluids in differential rotation will have both shear and vorticity in general.

A Petrov type I solution with $p = \mu$ and zero vorticity was obtained by Chinea and Gonzalez-Romero [3]. Hermann [8], using non-comoving coordinates, derived a metric having a homothetic vector, and an equation of state $p = \gamma \mu$. This solution also has a log of $r$ term. Chinea [2] employed a tetrad approach and found a solution with both shear and vorticity, having $p = \mu / 2$.

A type D solution with zero magnetic Weyl tensor and a fluid 4-velocity in the plane of the two null Weyl eigenvectors has been derived by Senovilla [15], up to two ordinary second order non-linear differential equations. This includes his earlier rigidly rotating metric [14] as a special case. The equation of state for the class is $p = \mu + \text{const.}$ As Mars and Senovilla have remarked [12], it is quite difficult to find solutions to the two defining differential equations. One solution was given by Senovilla [15], but does not seem to be physically reasonable [12]. Another solution of the equations was found by Garcia [6], but this has $\mu < p$.

In this paper we shall concentrate on Senovilla’s interesting equations to find solutions. A multi-parameter rigidly rotating axisymmetric case will be presented and discussed. Mars and Senovilla [12] have studied general features
of the rigidly rotating case using comoving coordinates. Our solution here will feature non-comoving coordinates. It belongs to a general class already known but has features of special interest. A solution manifesting differential rotation will also be presented. Both cases will receive possible physical interpretations.

2. GENERAL PROPERTIES OF THE METRIC

The metric we are to examine is [15, 12]:
\[
ds^2 = a(r,z)dr^2 + b(r,z)dz^2 + c(r,z)d\phi^2 - d(r,z)dt^2 - 2k(r,z)dtd\phi,
\]
and specifically
\[
ds^2 = \left(n(z)\right)^2 \left\{dr^2 / \left(mh + s^2\right) + dz^2 + hdp^2 - mdr^2 - 2stdp\right\},
\]
where \(m, h\) and \(s\) are functions of \(r\) only. The Einstein field equations require
\[
n'' = \varepsilon a^2 n (a = \text{const}),
\]
\[
m'' + s'^2 = 0,
\]
\[
 mh'' + h'm'' + 2ss'' + m'h' + s'^2 + \varepsilon 4a^2 = 0.
\]
A prime indicates differentiation and \(\varepsilon = +1, -1\) or 0. In this paper we shall adopt \(\varepsilon = +1\) and express the solution to (2.3) in the form
\[
n(z) = fe^{az} + ge^{-az},
\]
f and g being constants.

For a perfect fluid the Einstein tensor and energy tensor are related by
\[
G_{ij} = T_{ij} \equiv (\mu + p)u_iu_j + \delta_i^j p,
\]
where \(u_i\) is the fluid 4-velocity. In the coordinate system \(x^i = (t, r, z, \phi)\) for \(i = 0, 1, 2\) and 3, respectively, we have for the assumed stationary system:
\[
u^i = (u^0, 0, 0, u^3),
\]
\(u^0\) and \(u^3\) being functions of \(r\) and \(z\). The stationariness will be reflected by the Killing vector \(\partial_t\) and the axial symmetry by the Killing vector \(\partial_\phi\), which evidently commute.

The field equations provide for \(\mu\) and \(p\) the relations
\[
p = \frac{1}{4} \left(mh' + s'^2\right) \left(fe^{az} + ge^{-az}\right)^2 + a^2 \left(f^2 e^{2az} - 10 fg + g^2 e^{-2az}\right),
\]
\[
\mu = p + 24a^2 fg.
\]
For the fluid 4-velocity we find
\[
u^0 = \frac{-s'' \left(fe^{az} + ge^{-az}\right)}{\left(-hm'^2 - 2sm's^2 + ms'^2\right)^{1/2}},
\]
\[
u^3 = \frac{m'' \left(fe^{az} + ge^{-az}\right)}{\left(-hm'^2 - 2sm's^2 + ms'^2\right)^{1/2}}.
\]
The field equations allow us to write the denominator of (2.11) and (2.12) as 
\[(m^*)^{1/2} \left( m'h' + s^2 + 4a^2 \right)^{1/2}, \]
so that acceptable solutions will have 
\[m'' > 0 \] (2.13)
and hence \(h'' < 0\). We can now confirm [15] that the rotational angular velocity of the fluid is (independent of \(z\)):
\[\Omega = \frac{u^3}{u^0} = -\frac{m^*}{s^*}. \] (2.14)

The rotation will be rigid only if \(\Omega = \text{const.}\)

For the acceleration vector we calculate
\[\ddot{u}_i = u_i j u' = (0, \dot{u}_1, \dot{u}_2, 0), \] (2.15)
\[\ddot{u}_1 = \frac{-\left( h'm'^2 + 2s'm's'' - m's'^2 \right)}{2 \left( h'm'^2 + 2s'm's'' - m's'^2 \right)}, \] (2.16)
\[\ddot{u}_2 = -\frac{a \left( f^{au} - ge^{-au} \right)}{fe^{au} + ge^{-au}}. \] (2.17)

The vorticity vector is \(\omega^i = \frac{1}{2} \eta^{ijk} u_j u_k = (0, 0, \omega^2, 0)\) where
\[\omega^2 = \frac{s'^2(m's' - sm') + hs'm'^2 + \left( s'm'' - m's'' \right) - h'm's' + \left( ms'^2 - sm'^2 \right) h'm^* \left( fe^{au} + ge^{-au} \right)^2}{2 \left( h'm'^2 + 2s'm's'' - m's'^2 \right)}. \] (2.18)

The shear invariant is
\[\sigma^2 = \frac{1}{2} \sigma_L^{ij} \sigma_L^{ij} = \frac{\left( s'm'' - m's'' \right)^2 \left( mh + s^2 \right)^2 \left( fe^{au} + ge^{-au} \right)^2}{4 \left( h'm'^2 + 2s'm's'' - m's'^2 \right)^2}. \] (2.19)

As Senovilla has pointed out [15], the family of metric (2.2) is of Petrov type D, the fluid 4-velocity lying in the plane of the two principal null directions. Indeed calculation shows that, in the notation of [17], the invariant I is > 0 while the invariant J is real and the invariant \(M = I^3 - 27J^2 (= 0)\) is non-negative. Hence the magnetic Weyl tensor must vanish [13]. It has been indicated by Sklavenitis [16] that, at least in the case of a rigid rotation, zero \(H_i\) is associated with an equation of state \(\mu = p + \text{const.}\), as indeed we find in (2.10).

Another feature of the family (2.2) is that it admits a proper conformal Killing vector [4,9,15]. Thus, for the metric \(g_{ij}\) in (2.2) we can show that the vector \(\zeta = \partial_z\) satisfies the conformal Lie differential relation
\[\mathcal{L}_\zeta g_{ij} = F(z) g_{ij}, \] (2.20)
\[F(z) = -2a \left( fe^{au} - ge^{-au} \right) / \left( fe^{au} + ge^{-au} \right). \] (2.21)

This would become a homothety if \(f \) or \(g = 0\). Evidently \(\zeta\) commutes with both \(\xi\) and \(\eta\).
3. A RIGIDLY ROTATING AXISYMMETRIC SOLUTION

We shall seek a solution to (2.4) and (2.5) when \( m, h \) and \( s \) have a quadratic dependence on \( r \):

\[
m = m_0 + m_1 r + m_2 r^2,
\]
\[
h = h_0 + h_1 r + h_2 r^2,
\]
\[
s = s_0 + s_1 r + s_2 r^2.
\]

Then (2.4) requires (noting from (2.13) that \( m_2 > 0 \))

\[
h_2 = -s_2^2 / m_2,
\]

while from (2.5) we have

\[
h_1 = s_2 \left( m_2 s_2 - 2 m_2 s_1 \right) / m_2^2,
\]
\[
m_0 = m_2 \left( 4a^2 + 2m_2 h_0 + 4s_0 s_2 + m_1 h_1 + s_1^2 \right) / \left( 2s_2^2 \right).
\]

We require the spacetime to be axisymmetric and it will be felicitous to have the \( z \) axis (\( r = 0 \)) as the axis of symmetry. From (2.1) the requisite behaviour is

\[
\frac{c(r, z)}{a(r, z)} \approx Dr^2 \text{ as } r \to 0,
\]

where \( D \) is a constant that can be made unity. The result (3.5) can be achieved if we set

\[
h_0 = s_0 = 0.
\]

We therefore have an axisymmetric solution to (2.4) and (2.5) as follows:

\[
m = \frac{4a^2 m_2^2 + (m_2 s_2 - m_2 s_1)^2}{2m_2 s_2^2} + m_1 r + m_2 r^2,
\]
\[
h = s_2 \left( m_2 s_2 - 2m_2 s_1 \right) r - \frac{s_2^2}{m_2} r^2,
\]
\[
s = s_1 r + s_2 r^2,
\]

and the constant \( D \) can be made unity by setting

\[
\left( 4a^2 m_2^2 + (m_2 s_2 - m_2 s_1)^2 \right) \left( m_2 s_2 - 2m_2 s_1 \right)^2 = 2m_2^5. \tag{3.8}
\]

There are therefore six independent parameters in the solution if we include \( f \) and \( g \). We can regard (3.8) as an equation for \( a \). For example, if we set

\[
m_1 = 1, \ m_2 = 105/2, \ s_1 = \sqrt{7}/2, \ s_2 = 315\sqrt{7} / \left( 2\sqrt{2} \right),
\]

then

\[
a = 2. \tag{3.10}
\]

We may calculate the physical and kinematic properties of the solution from Section 2. In particular, from (2.14) we find
\[ \Omega = -\frac{m_2}{s_a}, \]  
so that the rotation is \textit{rigid}. There is vorticity:

\[ \omega^i = -\frac{1}{2} \left( f e^{az} + g e^{-az} \right)^2 \frac{(m_i s_2 - m_2 s_i)}{m_2} \delta^i_a, \]  
but vanishing shear. It may be verified that \( \omega^i \) is geodesic as foreshadowed in Section 1 (cf. [4]). The acceleration in this rigid rotation is in the \( z \) direction:

\[ \delta_i = -a \left( f e^{az} - g e^{-az} \right)/\left( f e^{az} + g e^{-az} \right) \delta^i_a \]  
The vorticity and acceleration are therefore parallel.

As stated in Section 2, the magnetic Weyl tensor \( H_{ij} \) must vanish. This condition, plus the fact that the vorticity and acceleration are parallel vectors means that the metric is locally rotationally symmetric \([4, 5]\), and admits a \( G_4 \) isometry multitransitive on timelike hypersurfaces \( (t, r, \phi, s) \). For the electric Weyl tensor \( E_{ij} = C_{ij, k l} u^k u^l \) we find for our case the invariant

\[ E_{ij} E^{ij} = \left( f e^{az} + g e^{-az} \right)^2 \left( 4a^2 m_2^2 + (m_i s_2 - m_2 s_i)^2 \right) \]  
\[ \left( 24m_2^4 \right). \]  

\textbf{The limit surface} \( p = 0 \)

The surfaces of equipressure are the planes \( z = \text{const} \) on which (from (2.9)):

\[ p = \left( f e^{az} + g e^{-az} \right)^2 \left( 4a^2 m_2^2 + (m_i s_2 - m_2 s_i)^2 \right) / \left( 4m_2^2 \right) - 12a^2 fg. \]  

In this paper we shall assume \( f, g > 0 \). The pressure becomes infinite at \( z = \pm \infty \). To check on the possible singularities of the spacetime we calculate the scalar curvature

\[ R = -\left( f^2 e^{2az} + g^2 e^{-2az} \right)^2 \left( 4a^2 m_2^2 + (m_i s_2 - m_2 s_i)^2 \right) + 2 \left( (m_i s_2 - m_2 s_i)^2 - 4a^2 m_2^2 \right) fg / \left( 2m_2^2 \right) \]  
\[ \left( 2m_2^2 \right) \]  

which confirms a singularity at \( z = +\infty \) and one at \( z = -\infty \) and nowhere else.

The two planes on which \( p = 0 \) will be given by

\[ c_{u_2} = \frac{\left( m_i s_2 - m_2 s_i \right)^2 \pm 4 \sqrt{3} a m_2 \left( 8a^2 m_2^2 - (m_i s_2 - m_2 s_i)^2 \right)^{1/2}}{4a^2 m_2^2 + (m_i s_2 - m_2 s_i)^2} \]  
\[ g \]  
\( f \)

We may now fix our attention on the slab of rigidly rotating fluid between the plane \( z = -\infty \), which embraces a singularity, and what we may suppose is a free upper surface of coordinate \( z \) given by (3.17), on choosing the \(-\)ve sign there. We may illustrate by adopting the particular parameters in (3.9) and (3.10). The free upper surface will then be at \( z = \bar{z} \) given by

\[ e^\bar{z} = \left[ 11 - 4 \sqrt{6} \frac{g}{f} \right]^{1/4}. \]  

\[ \bar{z} \]

The fluid will be of infinite density at its lower bound, steadily decreasing to the value \( 96fg \) at its free surface. The slab will have proper thickness (referring to (2.2) and (2.6)): 

\[ \]
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\[ d = \frac{\tilde{r}}{\sqrt{f(\bar{e}^z \bar{e}^{-z})}} = \frac{1}{2\sqrt{fg}} \tan^{-1} \left( \sqrt{f/g} \ e^{2z} \right), \quad (3.19) \]

which is finite.

Changing the perspective from a slab to a sphere

It is interesting to make a transformation of coordinates so that the \( r, z \)
2-space becomes isotropic. This can be achieved if we set

\[ r = \frac{(A + B)}{2} \left( 1 - \frac{Z}{(R^2 + Z^2)^{1/2}} \right), \quad (3.20) \]

\[ z = \log \left( C - \frac{1}{2} \log (R^2 + Z^2) \right), \]

where \( C (> 0) \) is a constant, and

\[ A = 2a^2 \left( m_1 s_2 - 2m_2 s_1 \right), \]

\[ B = \left( m_1 s_2 - 2m_2 s_1 \right) \left( m_1 s_2 - m_2 s_1 \right)^2 / \left( 2s_2 a m_2^3 \right), \quad (3.21) \]

provided

\[ a^2 = \left( 2m_2^2 + (m_1 s_2 - m_2 s_1)^2 \right) / (4m_2^2). \quad (3.22) \]

The \( R, Z \) 2-space then has the metric

\[ d\sigma^2 = \frac{C^2a \left( R^2 + Z^2 \right)^{a-1}}{\left( fC^2a + g \left( R^2 + Z^2 \right)^a \right)^{1/2}} (dR^2 + dZ^2). \quad (3.23) \]

If (3.22) is now combined with (3.8), eliminating \( a^2 \), the relationship between \( m_1, m_2, s_1 \) and \( s_2 \) is

\[ m_2^2 \left( m_2^3 - m_1^2 s_2^2 + 4m_1 m_2 s_1 s_2 - 4m_2^3 s_1^2 \right) = \left( m_1^2 s_2^2 - 3m_1 m_2 s_1 s_2 + 2m_2^2 s_1^2 \right)^2, \quad (3.24) \]

so that the transformed metric has five independent parameters. (3.24) is of course satisfied by (3.9).

The fluid pressure can now be expressed as

\[ p = \left( fC^a \rho^{-a} + gC^{-a} \rho^a \right) \left( 4a^2 m_2^2 + (m_1 s_2 - m_2 s_1)^2 \right) / (4m_2^2) - 12a^2 f g, \quad (3.25) \]

\[ \rho = \left( R^2 + Z^2 \right)^{1/2}. \quad (3.26) \]

Thus the singularities which were originally at \( z = \pm \infty \) now form a singularity at \( \rho = 0 \). The surfaces of equipressure are now the coordinate spheres \( \rho = \text{const.} \)

In particular, the limit surface of interest previously has now the coordinate radius \( \bar{\rho} \) given by

\[ \bar{\rho}^2a = \frac{C^2a \left[ - \left( m_1 s_2 - m_2 s_1 \right)^2 + 20a^2 m_2^2 \right] - 4\sqrt{3}am_2 \left( 8a^2 m_2^2 - (m_1 s_2 - m_2 s_1)^2 \right)^{1/2}}{4a^2 m_2^2 + (m_1 s_2 - m_2 s_1)^2} \frac{f}{g}, \quad (3.27) \]
If we again apply the particular parameters (3.9) and (3.10), the pressure becomes
\[ p = 3 \left( \frac{5C^8 f^2 - 22C^4fg \rho^4 + 5g^2 \rho^8}{2C^8 \rho^4} \right), \] (3.28)
and we obtain the bounding sphere of coordinate radius
\[ \bar{\rho} = C \left( \frac{11 - 4\sqrt{6}}{5} f \right)^{1/4}. \] (3.29)

The proper distance from \( \rho = 0 \) to the boundary is, from (3.23):
\[ d = \int_0^\infty \frac{C^a \rho^{a-1} d\rho}{fC^{2a} + g \rho^{2a}} = \frac{1}{a \sqrt{fg}} \tan^{-1} \left( \frac{(\bar{\rho}/C)^a}{\sqrt{f/g}} \right), \] (3.30)
which is finite. The pressure decreases steadily from \(+\infty\) at the centre of the sphere to zero at its boundary \( \rho = \bar{\rho} \). For the special parameters, and for the arbitrary choice of \( f = 2, \ g = 1 \), Figure 1 shows how \( p \) varies as a function of \( x = \rho/C \) along a radius within the limit surface.

According to this perspective the metric would in principle apply to a rigidly rotating isolated finite sphere of perfect fluid, possessing a singularity at its centre, provided an appropriate vacuum metric could be matched at its free surface.

4. A SOLUTION WITH DIFFERENTIAL ROTATION

A solution of (2.4) and (2.5) is sought involving differential rotation with the tentative format
\[ m = m_0 + m_2 r^2 + m_3 r^{2+k} + m_4 r^{2-k}, \]
\[ h = h_0 + h_2 r^2 + h_3 r^{2+k} + h_4 r^{2-k}, \]
\[ s = s_0 + s_2 r^2 + s_3 r^{2+k} + s_4 r^{2-k}. \] (4.1)

It transpires that there is a solution with \( k = \sqrt{11/2} \) and the constant coefficients depend on \( a \) and constants \( b \) and \( c \) as follows:
\[ m = br^{2-\sqrt{11/2}}, \]
\[ h = \frac{2(a^2 - c^2 r^2)}{c} - \frac{16c^2}{27b} r^{2+\sqrt{11/2}} - br^{2-\sqrt{11/2}}, \] (4.2)
\[ s = \frac{(a^2 - c^2 r^2)}{c} - br^{2-\sqrt{11/2}}. \]

By (2.13) we shall require \( b > 0 \) and we adopt \( c > 0 \).

For the fluid velocity (2.8) we find
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\[ u^0 = \frac{\sqrt{3}}{2r^{3/2}} \left[ 2 \left( 5 - \sqrt{22} \right) (9a^2 + 11c^2r^2) b \right]^{1/2}, \]

\[ u^3 = \frac{3\sqrt{3}}{2r^{3/2}} \left( 5 - \sqrt{22} \right) b \left( fe^{xz} + ge^{-xz} \right). \]

(4.3)

Hence the angular velocity of rotation is

\[ \Omega = \frac{u^3}{u^0} = \frac{3 \left( 5 - \sqrt{22} \right) b}{3 \left( 5 - \sqrt{22} \right) b + 4cr^{3/2}}, \]

(4.4)

so that the rotation is *differential*, decreasing as \( r \) increases. The acceleration has components in the \( r \) and \( z \) directions:

\[ \dot{u}_1 = \frac{-11c^2 r}{9a^2 + 11c^2r^2}, \]

\[ \dot{u}_2 = \frac{-a \left( fe^{xz} - ge^{-xz} \right)}{fe^{xz} + ge^{-xz}}. \]

(4.5)

There is shear

\[ \sigma^2 = \frac{11 \left( 27a^4 - 54a^2c^2r^2 + 11c^4r^4 \right)^2 \left( fe^{xz} + ge^{-xz} \right)^2}{288 \left( 9a^2 + 11c^2r^2 \right)^2 c^2r^2}, \]

(4.6)

while the fluid vorticity is

\[ \omega^0 = \frac{-\sqrt{11} \left( 3a^2 - 7c^2r^2 \right) \left( fe^{xz} + ge^{-xz} \right)^2}{12\sqrt{2cr} \delta_r} \]

(4.7)

The limit surface \( p = 0 \)

For the fluid pressure we obtain

\[ p = \frac{11}{9} c^2 r^2 \left( fe^{xz} + ge^{-xz} \right)^2 + a^2 \left( f^2 e^{2xz} - 10fg + g^2 e^{-2xz} \right). \]

(4.8)

This would indicate physical singularities at \( z = \pm \infty \), (and at \( r = \infty \) ) and this is confirmed if we calculate the curvature invariant

\[ R = -\frac{2}{9} \left[ \left( 9a^2 + 11c^2r^2 \right) \left( f^2 e^{2xz} + g^2 e^{-2xz} \right) - 22 \left( 9a^2 - c^2r^2 \right) fg \right]. \]

(4.9)

However, there is a limit surface \( p = 0 \) which is closed and does not include these singularities. Thus the limit surface will be given by

\[ \frac{11c^2r^2}{9a^2} = -\frac{f^2 e^{4xz} - 10fg e^{2xz} + g^2}{f^2 e^{4xz} + 2fg e^{2xz} + g^2}. \]

(4.10)

Recalling that we are adopting \( f > 0, g > 0 \), we set \( fe^{2xz} / g = x \) so that (4.10) becomes
\[
\frac{11c^2r^2}{9a^2} = \frac{-x^2 + 10x - 1}{x^2 + 2x + 1},
\] (4.11)

Figure 2 shows how \( y = 11c^2r^2 / (9a^2) \) varies on the limit surface. The surface meets the \( z \) axis at \( x = 5 \pm 2\sqrt{6} \approx (9.899, 0.101). \) The maximum radial extension of the surface is \( y_{\text{max}} = 2, \) \( r_{\text{max}} = 3\sqrt{2} a / (\sqrt{11} c), \) and this occurs at \( x = 1. \)

Inside the limit surface the pressure \( p \) is everywhere < 0 while the density \( \mu \) is everywhere > 0. Thus for a given \( x \) \((5 - 2\sqrt{6} < x < 5 + 2\sqrt{6})\) the value of \( p \) is least on the \( z \) coordinate axis, rising to zero on the limit surface, and the least value on the axis is \( p_{\text{min}} = -8a^2 f g \) and this occurs at \( x = 1. \) Hence by (2.10) we have \( \mu_{\text{min}} = p_{\text{min}} + 24a^2 f g = 16a^2 f g \) and so \( \mu \) is positive everywhere within the limit surface. The region within the surface is also free of singularity. Outside the limit surface both \( p \) and \( \mu \) are positive but with the singularities previously identified.

In this differential rotation the coordinate \( z \) axis is not the axis of symmetry. The axis of symmetry will be where

\[
 mh + s^2 = 0 = 11c^4 r^4 - 54a^2 c^2 r^2 + 27a^4.
\] (4.12)

There is therefore an axis of symmetry at

\[
r = r_\parallel = \frac{\sqrt{3} \left(9 - 4\sqrt{3}\right)^{1/2}}{\sqrt{11}} \frac{a}{c} \approx 0.752 \frac{a}{c}.
\] (4.13)

Then it may be verified that for regularity at this axis we require the relation

\[
b = \frac{4\sqrt{3}}{9} c r_\parallel^{1/2}.
\] (4.14)

If we were to ascribe tentatively a physical interpretation to the regime, we might think of an isolated rotating fluid object whose boundary is the specified limit surface and whose interior comprises a fluid of positive density under negative pressure, or a tension not exceeding the fluid density. Again this assumes that an exterior vacuum metric can be matched at the surface.

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REFERENCES


Figure 1. The graph of $p$ as a function of $x = \rho / C$ along a radius of the fluid sphere for the parameters (3.9) and (3.10) and the arbitrary values $f = 2$, $g = 1$; $p = +\infty$ at $x = 0$ and vanishes on the free surface at $x = 0.83$. 
Figure 2. The graph shows $y = \frac{11 c^2 r^2}{9a^2}$ as a function of $x = \frac{fe^{2az}}{g}$ on the limit surface $p = 0$. As indicated, the limit surface meets the $z$ axis ($r = 0 \Rightarrow y = 0$) at $x = 0.10$ and at $9.90$, and $y$ has a maximum value of 2 on the limit surface at $x = 1$.

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