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Supersymmetry Methods in the Analysis of Ground-State Energy of Many-Particle Systems[1]

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Abstract

Supersymmetry methods are used to derive *rigorously* a lower bound for the exact ground-state energy of many-particle systems for a general class of interactions, in *arbitrary* dimensions, with main emphasis on the state of matter in bulk with Coulomb interactions. In particular, we derive a lower bound for the ground-state energy E_N of so-called "bosonic matter" as a *cubic* power of N - the number of negatively charged particles - valid for *all dimensions* $\nu \geq 2$ providing an upper as well as a lower bound for E_N for such matter in all such dimensions.

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1 Introduction

The study of the nature of the ground-state energy of Hamiltonians of interacting many-particle systems is of central importance for the investigation of the stability of such complex systems. Over the years much work has been done in deriving rigorous bounds (cf. [1, 2, 5, 7–9, 11–13, 15, 17]) on the exact ground-state energy of such Hamiltonians and, in turn, establish stability or instability of the underlying systems with main emphasis on systems pertaining to matter in bulk. The instability of so-called "bosonic matter", i.e., for matter obtained by relaxing the Pauli exclusion constraint (cf. [2, 9, 11–13, 15, 17]) is a result of a power law behaviour N^γ of the ground-state energy, where N is the number of negatively charged particles, *with* the exponent γ such that

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$\gamma > 1$. Such a power law behaviour, with $\gamma > 1$, implies instability of the underlying system, since the formation of such matter consisting of $(2N+2N)$ particles will be favourable over two separate systems brought into contact, each consisting of $(N+N)$ particles, and the energy released upon collapse, in the formation of the former system, being proportional to $[(2N)^\gamma - 2N^\gamma]$, will be overwhelmingly large for realistic large N, e.g., $N \sim 10^{23}$. It is interesting to point out that if collapse occurs, then the radial extension of such a system does not decrease faster than $N^{-1/3}$ [14] upon collapse, as N increases for large N . On the other hand, for ordinary matter, i.e., for which the Pauli exclusion constraint is invoked, the ground-state energy has the single power law behaviour $\sim N$ [8,19] consistent with stability. In this respect, as the number N is made to increase such matter inflates and its radial extension increases not any slower than $N^{1/3}$ [16]. In recent years there has been also much interest in physics of arbitrary dimensions (cf. [3,6,12,13,17,18]). In this respect it is also quite important to investigate if the change of the dimensionality of space will change the properties of many-particle systems and if a given property, such as instability, is a characteristic of the three-dimensional property of space. [Some present field theories speculate that at early stages of the universe, the dimensionality of space was not necessarily three and, by a process which may be referred to as compactification, the present three-dimensional character of space arose upon the evolution and the cooling of the universe.] The purpose of this communication is to use supersymmetry methods to derive *rigorously* lower bounds to a class of Hamiltonians, to be defined in the next section, with particular emphasis on "bosonic matter" in *arbitrary dimensions of space.* The basic idea of supersymmetry methods (cf. [10] for a pedagogical treatment) is to introduce generators $\mathbf{Q}$ and write the Hamiltonian H under consideration, or more precisely a part H' of the Hamiltonian, as $\mathbf{Q}^\dagger.\mathbf{Q}$, where $\mathbf{Q}^\dagger$ is the adjoint of $\mathbf{Q}$, and then use *positivity* constraints to derive a lower bound for H. In the concluding section, further comments on our findings are made.

2 Supersymmetry Methods and the Ground-State Energy: Application to "Bosonic Matter"

For an N-particle system, we introduce N real vector fields $G_j(\mathbf{x}_1, ..., \mathbf{x}_N; \varrho)$, $j = 1, ..., N$, as functions of N dynamical variables $\mathbf{x}_1, ..., \mathbf{x}_N \in \mathbb{R}^\nu$, which may also depend on some parameters which we denote collectively by ϱ. The space dimension is denoted by ν. We consider a class of potential energies $V(\mathbf{x}_1, ..., \mathbf{x}_N; \varrho)$ defined by

$$V(\mathbf{x}_1, ..., \mathbf{x}_N; \varrho) = -\sum_{j=1}^{N} \boldsymbol{\nabla}_j.\mathbf{G}_j(\mathbf{x}_1, ..., \mathbf{x}_N; \varrho) \tag{1}$$

where $\boldsymbol{\nabla}_j = \partial/\partial\mathbf{x}_j$, and define the multi-particle Hamiltonian by

$$H = \sum_{j=1}^{N} \frac{\mathbf{p}_j^2}{2m_j} \ + \ V(\mathbf{x}_1, ...\mathbf{x}_N; \varrho) \tag{2}$$

with $\mathbf{p}_j = -\mathrm{i}\hbar\boldsymbol{\nabla}_j$, and the m_j denoting the masses of the underlying particles.

Introduce the N operators

$$Q_j = \frac{\hbar\boldsymbol{\nabla}_j}{\sqrt{2m_j}} \ + \ \frac{\sqrt{2m_j}}{\hbar}\mathbf{G}_j \tag{3}$$

and their adjoints

$$Q_j^\dagger = -\frac{\hbar\boldsymbol{\nabla}_j}{\sqrt{2m_j}} \ + \ \frac{\sqrt{2m_j}}{\hbar}\mathbf{G}_j \tag{4}$$

$j = 1, ..., N$, and use the property $\boldsymbol{\nabla}_j.\mathbf{G}_j = (\boldsymbol{\nabla}_j.\mathbf{G}_j) + \mathbf{G}_j.\boldsymbol{\nabla}_j$ to obtain for any normalized state $|\Psi\rangle$

$$\begin{aligned} 0 \leq \sum_{j=1}^{N} \parallel Q_j\Psi \parallel^2 &= \sum_{j=1}^{N} \left\langle \Psi \left| Q_j^\dagger Q_j \right| \Psi \right\rangle \\ &= \sum_{j=1}^{N} \left\langle \Psi \left| \left[\frac{-\hbar^2\boldsymbol{\nabla}_j^2}{2m_j} \ - \ \boldsymbol{\nabla}_j.\mathbf{G}_j + \frac{2m_j}{\hbar^2}\mathbf{G}_j^2 \right] \right| \Psi \right\rangle \end{aligned} \tag{5}$$

an idea often used in supersymmetry methods, from which we obtain the basic lower bound

$$\langle \Psi | H | \Psi \rangle \geq -\sum_{j=1}^{N} \frac{2m_j}{\hbar^2} \left\langle \Psi \left| \mathbf{G}_j^2 \right| \Psi \right\rangle \tag{6}$$

for *any* Hamiltonian defined by (2), (1), giving a lower bound for the expectation value of the Hamiltonian in the state $|\Psi\rangle$.

A classic application of the above is to the Hamiltonian of matter given by

$$H = \sum_{j=1}^{N} \frac{\mathbf{p}_j^2}{2m} \; + \; \sum_{i<j}^{N} \frac{\mathrm{e}^2}{\mid \mathbf{x}_i - \mathbf{x}_j \mid} \; - \; \sum_{i=1}^{N} \sum_{j=1}^{k} \frac{Z_j \mathrm{e}^2}{\mid \mathbf{x}_i - \mathbf{R}_j \mid} + \; \sum_{i<j}^{k} \frac{Z_i Z_j \mathrm{e}^2}{\mid \mathbf{R}_i - \mathbf{R}_j \mid} \tag{7}$$

where k denotes the number of nuclei situated at $\mathbf{R}_1, ...\mathbf{R}_k$ with total charges $Z_1 \mid \mathrm{e} \mid, ..., Z_k \mid \mathrm{e} \mid$ such that $\Sigma_{j=1}^{k} Z_j = N$ for neutral matter.

The potential energy in (7) may be generated *exactly* from the vector fields $\mathbf{G}_j(\mathbf{x}_1, ..., \mathbf{x}_N; \mathbf{R}_1, ..., \mathbf{R}_k)$ defined by

$$\begin{aligned} \mathbf{G}_j(\mathbf{x}_1, ..., \mathbf{x}_N; \mathbf{R}_1, ..., \mathbf{R}_k) = & - \frac{\mathrm{e}^2}{(\nu - 1)} \sum_{\ell=1}^{j-1} \mathbf{n}_{j\ell} \; + \; \frac{\mathrm{e}^2}{(\nu - 1)} \sum_{\ell=1}^{k} Z_\ell \; \mathbf{k}_{j\ell} \\ & - \frac{\mathbf{x}_j \; \mathrm{e}^2}{\nu N} \sum_{i<\ell}^{k} \frac{Z_i Z_\ell}{\mid \mathbf{R}_i - \mathbf{R}_\ell \mid} \end{aligned} \tag{8}$$

with $\nu \geq 2$ the dimensionality of the space considered, and $\mathbf{n}_{j\ell}$, $\mathbf{k}_{j\ell}$ are *unit* vector fields defined by

$$\mathbf{n}_{j\ell} = \frac{\mathbf{x}_j - \mathbf{x}_\ell}{|\mathbf{x}_j - \mathbf{x}_\ell|}, \qquad \mathbf{k}_{j\ell} = \frac{\mathbf{x}_j - \mathbf{R}_\ell}{|\mathbf{x}_j - \mathbf{R}_\ell|} \tag{9}$$

by using, in the process, the facts that

$$\sum_{j=1}^{N} \boldsymbol{\nabla}_j . \mathbf{x}_j \; = \; \nu N \tag{10}$$

$$\sum_{j=2}^{N} \sum_{\ell=1}^{j-1} \boldsymbol{\nabla}_j . \mathbf{n}_{j\ell} \; = \; (\nu - 1) \sum_{\ell<j}^{N} \frac{1}{\mid \mathbf{x}_j - \mathbf{x}_\ell \mid} \tag{11}$$

$$\sum_{j=1}^{N} \sum_{\ell=1}^{k} \boldsymbol{\nabla}_j . \mathbf{k}_{j\ell} \; = \; (\nu - 1) \sum_{j=1}^{N} \sum_{\ell=1}^{k} \frac{1}{\mid \mathbf{x}_j - \mathbf{R}_\ell \mid} \tag{12}$$

giving

$$-\sum_{j=1}^{N} \boldsymbol{\nabla}_j.\mathbf{G}_j(\mathbf{x}_1,...,\mathbf{x}_N;\mathbf{R}_1,...,\mathbf{R}_k) = \sum_{i<j}^{N} \frac{\mathrm{e}^2}{\mid \mathbf{x}_i - \mathbf{x}_j \mid} \; - \; \sum_{i=1}^{N}\sum_{j=1}^{k} \frac{Z_j \mathrm{e}^2}{\mid \mathbf{x}_i - \mathbf{R}_j \mid} + \sum_{i<j}^{k} \frac{Z_i Z_j \mathrm{e}^2}{\mid \mathbf{R}_i - \mathbf{R}_j \mid} \tag{13}$$

which is the potential energy for matter in (7).

Due to the presence of the $\mathbf{x}_j$ factor in the last term on the right-hand side of (8), the lower bound in (6) for the Hamiltonian H *in* (7) will involve unmanageable terms such as $-\parallel \mathbf{x}_j \Psi \parallel^2$ for which no further lower bounds may be directly obtained. Accordingly, the definition in (8) suggests to introduce instead the vector fields $\mathbf{G}'_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...\mathbf{R}_N)$ given by

$$\mathbf{G}'_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...,\mathbf{R}_N) = -\frac{\mathrm{e}^2}{(\nu-1)} \sum_{\ell=1}^{j-1} \mathbf{n}_{j\ell} \; + \; \frac{\mathrm{e}^2}{(\nu-1)} \sum_{\ell=1}^{k} Z_\ell \; \mathbf{k}_{j\ell} \tag{14}$$

with the unit vector fields $\mathbf{n}_{j\ell}$, $\mathbf{k}_{j\ell}$ defined as before, yielding

$$-\sum_{j=1}^{N} \boldsymbol{\nabla}_j.\mathbf{G}'_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...,\mathbf{R}_N) = \sum_{i<j}^{N} \frac{\mathrm{e}^2}{\mid \mathbf{x}_i - \mathbf{x}_j \mid} \; - \; \sum_{i=1}^{N}\sum_{j=1}^{k} \frac{Z_j \mathrm{e}^2}{\mid \mathbf{x}_i - \mathbf{R}_j \mid} \tag{15}$$

From (6), (15), we then obtain the following lower bound for the expectation value of the Hamiltonian in (7) in a state $|\Psi\rangle$

$$\begin{aligned}\langle \Psi | H | \Psi \rangle &\geq -\frac{2m}{\hbar^2} \sum_{j=1}^{N} \left\langle \Psi \left| \mathbf{G}'^2_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...,\mathbf{R}_N) \right| \Psi \right\rangle \; + \; \sum_{i<j}^{k} \frac{Z_i Z_j \mathrm{e}^2}{\mid \mathbf{R}_i - \mathbf{R}_j \mid} \\ &\geq -\frac{2m}{\hbar^2} \sum_{j=1}^{N} \left\langle \Psi \left| \mathbf{G}'^2_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...,\mathbf{R}_N) \right| \Psi \right\rangle \end{aligned} \tag{16}$$

with $\mathbf{G}'_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...,\mathbf{R}_N)$ defined in (14).

Upon using the facts that $\mathbf{n}_{j\ell}$, $\mathbf{k}_{j\ell}$, defined in (9), are unit vector fields, i.e., $\mathbf{n}_{j\ell}.\mathbf{n}_{j\ell'} \leq 1$, $\mathbf{k}_{j\ell}.\mathbf{k}_{j\ell'} \leq 1$, $-\mathbf{n}_{j\ell}.\mathbf{k}_{j\ell} \leq 1$, we obtain from (14)

$$\left\langle \Psi \left| \mathbf{G}'^2_j(\mathbf{x}_1,...,\mathbf{x}_N,\mathbf{R}_1,...\mathbf{R}_N) \right| \Psi \right\rangle \leq \frac{\mathrm{e}^4}{(\nu-1)^2}(j-1+N)^2 \tag{17}$$

where we have used, in the process, the property $\sum_{j=1}^{k} Z_j = N$ for neutral matter.

Summing over j from 1 to N, (17), (16) give the following lower bound for the ground-state energy E_N for the Hamiltonian in (7)

$$E_N \geq -\frac{2m}{\hbar^2}\frac{\mathrm{e}^4}{(\nu-1)^2}\sum_{j=1}^{N}(j-1+N)^2 \tag{18}$$

or

$$E_N > -\Big(\frac{m\mathrm{e}^4}{2\hbar^2}\Big)\frac{16}{3}\frac{N^3}{(\nu-1)^2} \tag{19}$$

Needless to say for $\nu \to 1$, we do not obtain any contradiction with $-$ infinity as the lower limit of the set of real numbers - which is, however, not interesting.

3 Conclusion

We may combine the above result with an earlier one [17] which derives instead an *upper* bound for E_N valid also for all space dimensions ν and for $N \geq (2)^\nu$. The combined results now state that for the Hamiltonian H in (7) for so-called "bosonic matter"

$$-\Big(\frac{m\mathrm{e}^4}{2\hbar^2}\Big)\frac{N^{(2+\nu)/\nu}}{16\pi^2\nu^3(2)^\nu} \;>\; E_N \;>\; -\Big(\frac{m\mathrm{e}^4}{2\hbar^2}\Big)\frac{16N^3}{3(\nu-1)^2} \tag{20}$$

valid for *all* ν and for $N \geq (2)^\nu$. It is easy to check the consistency relation $16N^3/3(\nu-1)^2 > N^{(2+\nu)/\nu}/16\pi^2\nu^3(2)^\nu$ in relation to the above double inequalities. It is well known that for $\nu = 3$, the power 3 of N in the inequality on the right-hand side of (20) may be reduced to 5/3. Also for $\nu = 3$, for Fermionic, i.e., standard matter with the negatively charged particles obeying the Pauli exclusion principle, the power 3 of N is reduced to one, as mentioned in the introductory section, consistent with the stability criterion of matter. Our result obtained for *arbitrary* dimensions is obviously far from trivial. In (7), the so-called positively charged particles (nuclei) are treated non-dynamically being much heavier than the negatively charged particles which is the common practice. Our lower bound for the ground-state energy E_N given in (19) *is still valid in all dimensions* for an overall neutral system of bosonic charged particles with the positively charged particles treated dynamically as well with

the simplification that all the charges are equal in absolute values, provided m on the right-hand side of (19) denotes the *largest* mass in the set of masses of all the positively as well as negatively charged particles and N, being now even, denotes the total number of particles. The inequalities in (20) are consistent with a famous remark made by Dyson [1] concerning bosonic matter and the release of an overwhelmingly large amount of energy, as also discussed in the introductory section, when two such systems are brought into contact: "*[Bosonic] matter in bulk would collapse into a condensed high density phase. The assembly of any two macroscopic objects would release energy comparable to that of an atomic bomb ...*". Such a property will be also shared in higher dimensional spaces than three, as well as in two dimensions. We will not speculate on the physical significance of higher dimensional spaces (cf. [3, 6, 18]) except to re-iterate that it is important to investigate if the change of the dimensionality of space will change the properties of many-particle systems and if a given property, such as instability, is a characteristic of the three-dimensional property of space. Needless to say, two dimensional space, however, seems to be physically relevant at least in condensed matter physics.

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