

The Chern-Simons State and Topological Quantum Field Theory

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Abstract

We study a relation between the Chern-Simons state and topological quantum field theory. It is shown that the Chern-Simons state describes a topological state with unbroken diffeomorphism invariance in Yang-Mills theory and Einstein's general relativity in four dimensions. We give a clear explanation of "why" such a topological state exists.

1 Introduction

It has been known for a long time that Yang-Mills theory in four dimensions has an exact zero energy state of the Schrodinger equation [1], which is, what we call, the Chern-Simons state, and is expressed by the exponential of the Chern-Simons form

$$\Psi = \exp(\pm c S_{CS}), \quad (1)$$

with c being a suitable constant and S_{CS} being explicitly given by $S_{CS} = \int Tr(\frac{1}{2}A \wedge dA + \frac{1}{3}A \wedge A \wedge A)$. On the other hand, in the community of loop quantum gravity, this state is called the Kodama state since Kodama has first pointed out that the exponential of the Chern-Simons form solves the quantum Ashtekar constraints [2] by starting from the solution for Bianchi IX model and generalizing it [3]. This state has been extensively investigated by Smolin [4] since it shows that, at least for de Sitter space-time, loop quantum gravity *does* have a good low energy limit, thereby reproducing familiar general relativity and quantum field theory at the low energy as desired.

On the other hand, Witten has emphasized that the Chern-Simons state of Yang-Mills theory could not be the ground state of the theory since this state is highly unnormalizable and is not invariant under CPT, and in addition negative helicity states have not only negative energy but also negative

norm in expanding around this state [5]. In response to Witten's paper from the loop gravity side, Freidel and Smolin have argued that the Chern-Simons state is delta-functional normalizable in the Euclidean theory while it is not normalizable in the Lorentzian theory with the kinematical inner product [6]. They have also discussed that there is still the logical possibility such that the Chern-Simons state in the Lorentzian gravity theory might become normalizable if we take into account the contribution to the physical inner product from the measure to all order. Moreover, a relation of self-duality and the Chern-Simons state is studied in the abelian gauge theory [7].

The purpose of this article is to point out an interesting relationship between topological quantum field theory and the Chern-Simons state. To be precise, we will show the statement that the Chern-Simons state exists whenever a theory can be rewritten to the second Chern class $S = \int Tr F \wedge F$ under appropriate ansatz. In this context, we can clearly understand "why" such a topological state exists in Yang-Mills theory and general relativity in four dimensions which was one of motivations in the Witten's paper [5].

This paper is organized as follows. In Sec. 2 we review Horowitz's work [8] where it was mentioned that the Chern-Simons state also exists in topological quantum field theory in four dimensions. In Sec. 3, we show that the Chern-Simons state exists if we can cast a theory into the form of the second Chern class $S = \int Tr F \wedge F$. This fact implies that the Chern-Simons state is a topological state which describes a unbroken phase of diffeomorphism invariance. We conclude in Sec. 4 with a discussion of the results obtained in this article.

2 Review of Horowitz's topological quantum field theory

We begin with one of the simplest examples where there is a unique quantum state (up to an overall constant). This model has been already constructed by Horowitz [8] and clearly presents the reason why a unique quantum state exists in topological quantum field theory of cohomological type [9].

Let us consider an arbitrary scalar field theory described by a (Euclidean) action in D dimensional flat space and try to construct a $D + 1$ dimensional theory such that the theory is essentially controlled by the D dimensional boundary theory. The action is simply given by

$$S = \int dt \dot{I}(\phi) = \int dt \int d^D x F(\phi) \dot{\phi}, \quad (2)$$

where we have defined $F(\phi) = \frac{\delta I}{\delta \phi}$ and the dot denotes the differentiation over t . Now let us perform the canonical quantization of this theory. The momentum p conjugate to ϕ is given by $F(\phi)$, thereby giving rise to the primary

constraint $p - F(\phi) \approx 0$. Since the Hamiltonian is trivially vanishing owing to the reparametrization invariance, we introduce the extended Hamiltonian density which is proportional to the constraint

$$H_T = \lambda (p - F(\phi)), \quad (3)$$

where λ is a Lagrange multiplier field. Then, the first-order canonical action takes the form

$$S = \int dt d^D x [p\dot{\phi} - H_T] = \int dt d^D x [p\dot{\phi} - \lambda(p - F(\phi))]. \quad (4)$$

With the Poisson bracket $\{p, \phi\}_P = 1$, we can see that the constraint $p - F(\phi) \approx 0$ generates topological symmetry

$$\delta\phi = \varepsilon, \quad \delta p = \varepsilon \frac{\delta F}{\delta\phi}, \quad (5)$$

where ε is an arbitrary function. Since the constraint $p - F(\phi) \approx 0$ is the first-class one, using the Dirac quantization procedure, quantum states must satisfy

$$0 = (p - F(\phi)) \Psi = \left(-i \frac{\delta}{\delta\phi} - F(\phi) \right) \Psi, \quad (6)$$

where we have used the commutation relation $[p, \phi] = -i$. The unique solution to this equation is given by

$$\Psi(\phi) = e^{iI(\phi)}, \quad (7)$$

up to an overall normalization constant. An important lesson gained in this simple theory is that topological quantum field theory of cohomological type possesses the unique quantum state described by boundary action because of a large gauge symmetry, that is, topological symmetry.

Next, for later arguments, we shall consider a little more intricate topological quantum field theory in four dimensions which was also discussed by Horowitz [8]. The action that we consider is of form

$$S = \int_{M_4} Tr \left(B \wedge F - \frac{1}{2} B \wedge B \right), \quad (8)$$

where we have used the notation of differential forms and defined $F = dA + A \wedge A$. Here A_μ and $B_{\mu\nu}$ are a G-valued 1 form and 2 form, respectively. (For simplicity, we omit to write indices of gauge group G.) At first sight, it might appear that this action is BF theory [10], but is in fact of cohomological type,

since after performing the path integral over B field, we obtain the second Chern class

$$S = \frac{1}{2} \int_{M_4} \text{Tr} F \wedge F, \quad (9)$$

which is the gauge invariant action proposed by Baulieu and Singer [11] for Witten's topological quantum field theory [12].

We can express the action (8) in terms of the components

$$S = \frac{1}{4} \int_{M_4} d^4x \varepsilon^{\mu\nu\rho\sigma} \text{Tr} \left(B_{\mu\nu} F_{\rho\sigma} - \frac{1}{2} B_{\mu\nu} B_{\rho\sigma} \right), \quad (10)$$

where the Levi-Civita tensor density $\varepsilon^{\mu\nu\rho\sigma}$ is defined as $\varepsilon^{0123} = +1$ and we take the metric signature convention $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$. Since this action is linear in time derivative it is straightforward to cast it into canonical form whose result is given by

$$S = \frac{1}{2} \int d^4x \varepsilon^{ijkl} \text{Tr} \left[\dot{A}_i B_{jk} + A_0 D_i B_{jk} + B_{0i} (F_{jk} - B_{jk}) \right], \quad (11)$$

where i, j, k, \dots run over spatial indices 1, 2, 3 and $\varepsilon^{ijk} = +1$. Then the Hamiltonian takes the form

$$H = -\frac{1}{2} \int d^3x \varepsilon^{ijk} \text{Tr} \left[A_0 D_i B_{jk} + B_{0i} (F_{jk} - B_{jk}) \right], \quad (12)$$

which is purely a linear combination of the constraints

$$\varepsilon^{ijk} D_i B_{jk} \approx 0, \quad F_{ij} - B_{ij} \approx 0. \quad (13)$$

It is worthwhile to note that the former constraint can be derived from the latter ones because of the Bianchi identity $\varepsilon^{ijk} D_i F_{jk} = 0$, whose fact simply means that the usual gauge symmetry generated by the former constraint is included in topological symmetry generated by the latter constraints. Now we are not interested in constructing the "Donaldson invariants" so we do not take account of the "equivariant cohomology" [11] and neglect the former constraint. (Of course, even if we keep this constraint, the result obtained below is unchanged. Incidentally, we can show that spatial diffeomorphisms are included in the usual gauge transformation when the constraints are satisfied.)

As in the previous simplest theory (2), we can quantize this theory by using the Dirac procedure

$$0 = (F_{ij} - B_{ij}) \Psi = \left(F_{ij} - i\varepsilon_{ijk} \frac{\delta}{\delta A_k} \right) \Psi, \quad (14)$$

where we used the commutation relation $[B_{ij}(x_0, \vec{x}), A_k(x_0, \vec{y})] = i\varepsilon_{ijk}\delta(\vec{x} - \vec{y})$. Then, the unique solution is given by the exponential of the three dimensional Chern-Simons action

$$\Psi = e^{iS_{CS}(A)}. \quad (15)$$

Note that in comparison with the previous model (2), this result is easily expected through the famous identity $\frac{1}{2}\text{Tr}F \wedge F = dS_{CS}(A)$. Namely, the theory under consideration is topological quantum field theory of cohomological type in four dimensions whose boundary theory is the three dimensional Chern-Simons theory, so the unique quantum state is given by the exponential of the Chern-Simons action multiplied by i . Hence, we have shown that up to an overall constant the Chern-Simons state is the exact and unique quantum state of the topological quantum field theory (8), or equivalently, (9). This observation will be fully utilized in the next section in order to understand why the Chern-Simons state exists in Yang-Mills theory and general relativity in four dimensions.

3 The Chern-Simons state and topological quantum field theory

We now consider Yang-Mills theory in four dimensional flat space-time. Let us start with the Yang-Mills action

$$S = -\frac{1}{4} \int d^4x \text{Tr} F^{\mu\nu} F_{\mu\nu}. \quad (16)$$

This action can be rewritten by the electric fields $E_i = F_{0i}$ and the magnetic field $B_i = \frac{1}{2}\varepsilon_{ijk}F^{jk}$

$$S = \frac{1}{2} \int d^4x \text{Tr} (E_i^2 - B_i^2). \quad (17)$$

And up to the Gauss' law constraint the Hamiltonian takes the form

$$H = \frac{1}{2} \int d^3x \text{Tr} (E_i^2 + B_i^2). \quad (18)$$

Given the expression for the Hamiltonian, the Schrodinger equation is of form $H\Psi = E\Psi$ where E denotes the eigenvalue of energy. From the commutation relations $[E_i(x_0, \vec{x}), A_j(x_0, \vec{y})] = -i\delta_{ij}\delta(\vec{x} - \vec{y})$, the Schrodinger equation reduces to

$$\frac{1}{2} \int d^3x \text{Tr} \left(\frac{\delta}{\delta A_i} + B_i \right) \left(-\frac{\delta}{\delta A_i} + B_i \right) \Psi = E\Psi. \quad (19)$$

By using the identity $\frac{\delta S_{CS}(A)}{\delta A_i} = B^i$, the unique quantum state with zero energy, i.e., the ground state, is given by the Chern-Simons state [1, 5]

$$\Psi(A) = e^{\pm S_{CS}(A)}, \quad (20)$$

where $\left(\frac{\delta}{\delta A_i} \pm B_i\right) e^{\pm S_{CS}(A)} = 0$.

Now let us ask ourselves "why" this state exists in Yang-Mills theory in four dimensions, which was one of motivations in Ref. [5]. In this article, we will answer this question in a different way. To do that, we should first notice that in order to have the Hamiltonian with zero energy, we have to require the equations $E_i = \pm i B_i$ classically. It is then worth noting that the equations $E_i = \pm i B_i$ cause the deformation of the Yang-Mills action (16) into a topological term,

$$S = \pm i \int d^4x \text{Tr} E_i B_i. \quad (21)$$

Given that we define the second Chern class as $c_2(A) = \frac{1}{2} \int_{M_4} \text{Tr} F \wedge F = - \int d^4x \text{Tr} E_i B_i$, this reduced action is described by $S = \mp i c_2(A)$. (The presence of the factor $\mp i$ will be important in quantizing this theory shortly.) In other words, we could regard the equations $E_i = \pm i B_i$ as the requirements for picking up a topological phase among various ground states of Yang-Mills theory in four dimensions.

Under the ansatz $E_i = \pm i B_i$, let us quantize the Yang-Mills action. Since the action can be cast into the form similar to (8) by introducing the auxiliary fields $B_{\mu\nu}$

$$S = \mp i \int_{M_4} \text{Tr} \left(B \wedge F - \frac{1}{2} B \wedge B \right), \quad (22)$$

using again the Dirac quantization procedure, quantum states must satisfy

$$0 = (F_{ij} - B_{ij}) \Psi = \left(F_{ij} \pm \varepsilon_{ijk} \frac{\delta}{\delta A_k} \right) \Psi, \quad (23)$$

where we used the commutation relations $[B_{ij}(x_0, \vec{x}), A_k(x_0, \vec{y})] = \mp \varepsilon_{ijk} \delta(\vec{x} - \vec{y})$, which can be read off from the action (22). Here it is of importance to mention two remarks. One remark is that there is no appearance of i in the right-handed side of the commutation relations owing to the existence of $\mp i$ in front of the action (22). The other is that the commutation relations $[B_{ij}(x_0, \vec{x}), A_k(x_0, \vec{y})] = \mp \varepsilon_{ijk} \delta(\vec{x} - \vec{y})$ are consistent with the previous ones $[E_i(x_0, \vec{x}), A_j(x_0, \vec{y})] = -i \delta_{ij} \delta(\vec{x} - \vec{y})$ under the the ansatz $E_i = \pm i B_i$. Thus, as before, we have the Chern-Simons state (20) as the unique quantum state which satisfies Eq. (23). Of course, we can verify that, as a consistency condition, this state satisfies the equations

$$(E_i \mp i B_i) \Psi = \left(-i \frac{\delta}{\delta A_i} \mp i B_i \right) \Psi = 0. \quad (24)$$

In this way, we can understand the reason "why" the Chern-Simons state exists in Yang-Mills theory in four dimensions. Namely, the reason is that with the ansatz $E_i = \pm iB_i$ the Yang-Mills theory can be described as topological quantum field theory with the action $S = \mp \frac{i}{2} \int_{M_4} \text{Tr} F \wedge F$ so the Chern-Simons state corresponds to a topological state with unbroken diffeomorphism invariance among many ground states in Yang-Mills theory.

We next turn our attention to general relativity where it is known that there is a gravitational analog, commonly called the Kodama state in the community of loop quantum gravity [3], of the Chern-Simons state in Yang-Mills theory. We begin with the chiral action with the cosmological constant of general relativity [13]¹

$$S = \int_{M_4} \left[R_{AB} \wedge \Sigma^{AB} - \frac{\Lambda}{6} \Sigma_{AB} \wedge \Sigma^{AB} - \frac{1}{2} \psi_{ABCD} \Sigma^{AB} \wedge \Sigma^{CD} \right], \quad (25)$$

where we have used the $SL(2, C)$ spinor notation for the Lorentz group [15]. The indices A, B, \dots , therefore, run over 0 and 1. The equations of motion derived from this action read

$$\begin{aligned} R_{AB} - \frac{\Lambda}{3} \Sigma_{AB} - \psi_{ABCD} \Sigma^{CD} &= 0, \\ D \wedge \Sigma^{AB} &= 0, \\ \Sigma^{(AB} \wedge \Sigma^{CD)} &= 0, \end{aligned} \quad (26)$$

where the round bracket denotes the total symmetrization.

It is then easy to check that the ansatz [16]

$$R_{AB} = \frac{\Lambda}{3} \Sigma_{AB}, \quad (27)$$

not only lead to a class of solutions for the equations of motion but also give us a solution for the quantum Ashtekar constraints [2]. The consistency of the ansatz with the first equations of motion in Eq. (26) requires

$$\psi_{ABCD} = 0, \quad (28)$$

which implies that the anti-self-dual Weyl tensor should be vanishing. Note that in the Euclidean metric, Eq. (28) is the equations for the self-dual gravitational instantons.

It is then interesting to see that under the ansatz Eq. (27) and Eq. (28), the action (25) becomes a topological quantum field theory of cohomological type

$$S = \frac{3}{2\Lambda} \int_{M_4} R_{AB} \wedge R^{AB}. \quad (29)$$

¹See [14] for good review of 2-form gravity.

Therefore, following the same line of arguments as in Yang-Mills theory in four dimensions, it is straightforward to show that the unique quantum state is given by

$$\Psi(\omega) = e^{\frac{3}{\Lambda} S_{CS}(\omega)}. \quad (30)$$

where $S_{CS}(\omega)$ is the three dimensional Chern-Simons action for spin connections ω . This state is nothing but the Kodama state for quantum gravity. Why is the factor i missing in the Kodama state? This is traced back to the fact that the first-order Palatini action for general relativity is actually equivalent to $-i \int R_{AB} \wedge \Sigma^{AB}$ plus its anti-chiral part, so precisely speaking, we should put the factor $-i$ as an overall constant in the action (25) and then perform the Dirac quantization. Consequently, we can arrive at the quantum state (30).

4 Discussion

In this article, we have clarified "why" the Chern-Simons state exists in Yang-Mills theory and general relativity in four dimensions. This state is associated with a topological phase of unbroken diffeomorphism invariance and exists whenever one can rewrite the action into the form of the second Chern class. In a sense, we might say that the Chern-Simons state provides us a window of catching a glimpse of a relationship between general relativity (Yang-Mills theory) and topological quantum field theory.

In retrospect, it is now obvious to understand why in general relativity, the Chern-Simons (Kodama state) is not only the WKB wave functional for the classical solution but also an exact solution for the quantum constraints. In the WKB approach, we evaluate the action by substituting the ansatze Eq. (27) and Eq. (28), thereby reaching Eq. (29). The WKB wave function can be then obtained by exponentiating this classical action. The result of course coincides with (30), i.e., the Chern-Simons state. As seen in the identity $\frac{1}{2} Tr R \wedge R = dS_{CS}(\omega)$, the Chern-Simons state has its origin in topological quantum field theory so that this state has a large gauge symmetry which includes both the usual gauge symmetry and diffeomorphisms. Accordingly, the Chern-Simons state automatically satisfies the quantum Ashtekar constraints.

Even if we have understood a relation between the Chern-Simons state and topological quantum field theory to some degree, we have no idea whether such a topological state is relevant to real world or not. As an interesting remark, the recent observation by WMAP clearly shows that the cosmological constant is not exactly vanishing but has a small positive value. Relating to this remark, it is worthwhile to mention that the Chern-Simons state exists only when the cosmological constant is non-vanishing since there is a pole at the zero cosmological constant. This fact might give us some useful clue for understanding the relevance of the Chern-Simons state to real world.

Of course, one of big problems in future is to clarify whether the Lorentzian Chern-Simons state is normalizable under an appropriate inner product or not.

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