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On Products of Random Matrices from Groups of 2×2 Matrices and the Verlinde Algebra

Jafar Shaffaf

Institute for Studies in Theoretical Physics and Mathematics (IPM) and
Mathematics Department, Sharif University of Technology
Tehran, P.O. Box: 11365-9415, Iran
shaffaf@ipm.ir

Abstract

The determination of the density functions for products of random elements from specified classes of matrices is a basic problem in random matrix theory and is also of interest in theoretical physics. For connected simple Lie groups of 2×2 matrices and conjugacy and spherical classes a complete solution is given here. The problem/solution can be re-stated in terms of the structure of certain Hecke algebras attached to groups of 2×2 matrices and the S-matrix associated to the Kac-Moody Lie algebra $\widehat{\mathfrak{sl}_2}$ and the fusion rule of the Verlinde algebra.

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1 Introduction

Let (G, K) be a symmetric pair where G is a semi-simple Lie group with finite center. For G compact let $\mathcal{H}_c = \mathcal{H}_c(G)$ be the algebra under convolution generated by the invariant measures concentrated on conjugacy classes in G, and $\mathcal{H}_s = \mathcal{H}_s(G, K)$ be the convolution algebra generated by the invariant measures on spherical classes $\mathcal{O}_a = KaK \subset G$. It is elementary that $\mathcal{H}_c$ and $\mathcal{H}_s$ are commutative algebras with unit. These algebras arise naturally in random matrix theory, and theoretical physics. In fact, the product structures for generators of these algebras are given by the density functions of products of random matrices chosen according to invariant measures on conjugacy or spherical classes; and in string theory generic D-branes are localized along the product of (twisted) conjugacy classes of the Lie group. We will not discuss specific physical applications but the relationship with the structure of the Verlinde algebra and the S-matrix of the Kac-Moody Lie algebra $\widehat{\mathfrak{sl}_2}$ are given in §5. The reader is referred to [12], and references thereof, for a detailed discussion of the issues of interest in string theory.

This work may be regarded as the initial attempt at understanding the structure of the Hecke algebras $\mathcal{H}_c$ and $\mathcal{H}_s$ and considers only the special case of 2×2 matrices. The support of the density function for products of two generators of $\mathcal{H}_c(SU(n))$ is determined in [2] and is exhibited by a set of linear inequalities on the Lie algebra of a maximal torus. The result in [2] is essentially a reformulation of a theorem about singular connections on a holomorphic vector bundle on a Riemann surface with marked points [3], or equivalently a theorem of Mehta and Seshadri in algebraic geometry. Neither theorem is applicable directly for the computation of the density function for the product of two generators of the Hecke algebra. However the connection between algebraic geometry, the Verlinde algebra and the S-matrix matrix of the Kac-Moody Lie algebra $\widehat{\mathfrak{sl}_n}$ appears to be the correct framework for the determination of the distribution function (see §5).

A complete structure theorem for $\mathcal{H}_c(SU(2))$ is given in Theorem 2.1 below. For spherical classes attached to groups of 2×2 matrices we consider the symmetric pairs $(SU(2), S(U(1)\times U(1)))$, $(SL(2,\mathbb{R}), SO(2))$ and $(SL(2,\mathbb{C}), SU(2))$. Theorem 2.2 gives density functions for products of spherical classes. In the final section we present some numerical results in relation to products of conjugacy classes in $SU(2)$. Each conjugacy class, with the induced metric, is a copy of S^2 equipped with a metric of constant curvature. It is noticed (perhaps surprisingly) that if S^2 is discretized according to the prescription of Thomson's problem (minimizing Coulomb potential or intuitively "the best equally spaced distribution") then the convergence to the predicted measure is much slower than if the points were chosen randomly.

In [9] the question of whether a product of conjugacy classes $\mathcal{C}_{\alpha_1}, \dots, \mathcal{C}_{\alpha_n}$ contains the identity element and/or is the entire group $SU(2)$ is studied. By successive applications of Theorem 2.1 of this paper one can recover the results in [9] and in fact give a more precise version of it, but this subject will not be elaborated on here.

2 Statement of main results

A conjugacy class $C_\theta \subset SU(2)$ is uniquely determined by the eigenvalues $e^{\pm i\theta}$ of a matrix in C_θ. Each conjugacy class is a homogeneous space, and for C_θ this measure μ_θ is uniquely normalized to be $4\pi \sin^2\theta$ in accordance with Weyl's integration formula and the normalization vol.$(SU(2)) = 4\pi^2$.

Theorem 2.1 *Let μ_α and μ_β be the invariant measure on the conjugacy classes C_α and C_β respectively (regarded as singular distribution on G). Then $\mu_\alpha \star \mu_\beta$ is an absolutely continuous conjugation invariant measure on G relative to the*

Haar measure and its density is given by

$$\mu_\alpha \star \mu_\beta = \begin{cases} 4\pi^2 \sin\alpha \sin\beta \sin\theta, & \text{for} \quad \alpha - \beta \le \theta \le \alpha + \beta, \\ -4\pi^2 \sin\alpha \sin\beta \sin\theta, & \text{for} \quad -\alpha - \beta \le \theta \le -\alpha + \beta, \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

where $0 < \beta \le \alpha < \pi$.

Let $K = S(U(1) \times U(1)) \subset SU(2)$, then (G, K) is a symmetric pair of compact type [7]. Orbits of the action of $K \times K$ on G via

$$a \longrightarrow kak', \qquad (k, k') \in K \times K, \quad a \in G$$

are called spherical classes and are denoted by $\mathcal{O}_a$. $\mathcal{O}_a$ is a homogeneous space and it is a simple calculation that $|a_{11}|$ is constant on $\mathcal{O}_a$ and uniquely determines it. Here $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SU(2)$ is any matrix in $\mathcal{O}_a$. Each spherical class $\mathcal{O}_a$ carries an invariant measure λ_a. The total mass of the measure λ_a of a spherical class $\mathcal{O}_a$ is $2|a_{11}|$, and we set $r(a) = |a_{11}|$.

In the non-compact cases $(SL(2,\mathbb{R}), SO(2))$ and $(SL(2,\mathbb{C}), SU(2))$ the real Cartan subgroup A is

$$A = \{a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \mid t \in \mathbb{R}\}$$

$A_+ = \{a_t \in A \mid t > 0\}$ parametrizes spherical classes. The Haar measure for the Cartan (polar) decomposition $G \simeq KAK$ is

$$\int_G f(g)dg = c \int_K \int_K \int_0^\infty f(k_1 a k_2)\delta(t) dt dk_1 dk_2 \tag{2.2}$$

where c is a suitable constant (see [8]), $\delta(t) = \sinh^\epsilon t$, and $\epsilon = 1$ or 2 according as $G = SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$. The volume (area) of the spherical class $\mathcal{O}_{a_t}$ is

$$\text{vol.}(\mathcal{O}_{a_t}) = [4\pi^2 c \sinh t]^\epsilon. \tag{2.3}$$

Theorem 2.2 *The product formula for spherical classes in the three cases of symmetric pairs attached to groups of* 2×2 *matrices are:*
(A)- *Let* λ_a *and* λ_b *be two (singular) spherical measures concentrated on the spherical classes* $\mathcal{O}_a$ *and* $\mathcal{O}_b$ *respectively. Then* $\lambda_a \star \lambda_b$ *is absolutely continuous relative to the Haar measure on* $SU(2)$ *and given by*

$$\begin{cases} \frac{16\pi^2 |a_{11} b_{11}| u}{\sqrt{c_1^2 - (u^2 - c_0)^2}}, & \text{for} \quad \sqrt{c_0 - c_1} \le u \le \sqrt{c_0 + c_1}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.4}$$

where c_0 and c_1 are symmetric functions of a and b and given by

$$\begin{aligned} c_0 &= r^2(a)r^2(b) + (1-r^2(a))(1-r^2(b)), \\ c_1 &= 2r(a)r(b)\sqrt{(1-r^2(a))(1-r^2(b))}. \end{aligned}$$

(B) - *Let $\lambda_{a_{t_1}}$ and $\lambda_{a_{t_2}}$ be the (singular) invariant measures concentrated on the spherical classes $\mathcal{O}_{a_{t_1}}$ and $\mathcal{O}_{a_{t_2}}$ in $SL(2,\mathbb{R})$. Then $\lambda_{a_{t_1}} \star \lambda_{a_{t_2}}$ is spherical and absolutely continuous relative to the Haar measure, and for a continuous spherical function f on $SL(2,\mathbb{R})$ we have*

$$\lambda_a \star \lambda_b(f) = 4c^2\pi^2 \sinh t_1 \sinh t_2 \int_{I_{t_1,t_2}} f(r) \frac{\sinh r}{\sqrt{c_2^2 - (c_1 - \cosh r)^2}}\, dr\ ,$$

where $I_{t_1,t_2} = [t_2 - t_1, t_2 + t_1]$, $c_1 = \cosh t_1 \cosh t_2$, and $c_2 = \sinh t_1 \sinh t_2$.
(C) - *Let $\lambda_{a_{t_1}}$ and $\lambda_{a_{t_2}}$ be the (singular) invariant measures concentrated on the spherical classes $\mathcal{O}_{a_{t_1}}$ and $\mathcal{O}_{a_{t_2}}$ in $SL(2,\mathbb{C})$. Then $\lambda_{a_{t_1}} \star \lambda_{a_{t_2}}$ is spherical and absolutely continuous relative to the Haar measure, and for a continuous spherical function f on $SL(2,\mathbb{C})$ we have*

$$\lambda_{a_{t_1}} \star \lambda_{a_{t_2}}(f) = 32c^2\pi^6 \sinh t_1 \sinh t_2 \int_{I_{t_1,t_2}} f(r) \sinh r dr\ ,$$

where $I_{t_1,t_2} = [t_2 - t_1, t_2 + t_1]$.

It may be of interest to normalize the measures on the spherical classes to probability measures and determine the corresponding empirical measure of products. For such a normalization the density functions determined in Theorem 2.2 become

$$\text{A}: \ \frac{1}{2\pi}\frac{u}{\sqrt{c_1^2 - (u^2 - c_0)^2}}, \quad \text{B}: \ \frac{1}{\pi}\frac{\sinh r}{\sqrt{c_2^2 - (c_1 - \cosh r)^2}}, \quad \text{C}: \ \frac{\sinh r}{2 \sinh t_1 \sinh t_2}. \tag{2.5}$$

Remark 2.1 Let $\tilde{\mathcal{H}}_c$ and $\tilde{\mathcal{H}}_s$ denote the completions of $\mathcal{H}_c$ and $\mathcal{H}_s$ in the weak topology. Then in all cases considered $\tilde{\mathcal{H}}_c$ and $\tilde{\mathcal{H}}_s$ contain the corresponding L^1 space as a dense ideal. Furthermore, by the above analysis, for every pair of generators μ_1 and μ_2, we have $\mu_1 \star \mu_2 \in L^1$.

3 Proof of Theorem 2.1

First we show that $\mu_\alpha \star \mu_\beta$ is an L^p function for $p \leq 2$. The Fourier expansion of the singular measure μ_α is given by

$$\sum_{\rho\in\hat{G}} d_\rho \mathrm{Tr}\big[\rho(\mu_\alpha)\rho(g)\big], \tag{3.1}$$

where $\hat{G}$ is the set of irreducible representations of G and the Fourier transform of a measure μ is defined as

$$\rho(\mu) = \int \rho(g^{-1})d\mu.$$

This series does not converge in the ordinary sense of convergence of series of functions since the measure μ_α is singular, however, it converges in the weak sense. The convolution product $\mu_\alpha \star \mu_\beta$ can be calculated from

$$\sum_{\rho\in\hat{G}} d_\rho \mathrm{Tr}\big[\rho(\mu_\alpha)\rho(\mu_\beta)\rho(g)\big]. \tag{3.2}$$

Since the character $\chi_\rho(g)$ is independent of $g \in C_\alpha$, and is given by $\frac{\sin(n+1)\alpha}{\sin\alpha}$ for ρ is the symmetric n^{th} representation of $G = SU(2)$,

$$\rho(\mu_\alpha) = \frac{4\sin^2\alpha}{d_\rho}\mathrm{Tr}(\rho(g))I = \frac{4\chi_\rho(C_\alpha)\sin^2\alpha}{d_\rho}I.$$

Applying the Plancherel theorem to $\mu_\alpha \star \mu_\beta$, we obtain

$$\begin{aligned} ||\mu_\alpha \star \mu_\beta||^2 &= \sum \frac{16\sin^2\alpha\sin^2\beta\chi_\rho(C_\alpha)\chi_\rho(C_\beta)}{d_\rho^4}d_\rho^2 \\ &= \sum_{n\geq 1}\frac{16}{n^2}\sin^2(n+1)\alpha\sin^2(n+1)\beta. \end{aligned}$$

This series converges absolutely and therefore $\mu_\alpha \star \mu_\beta$ is a square integrable function. From the Cauchy-Schwartz inequality it follows that $\mu_\alpha \star \mu_\beta$ is absolutely integrable.

Therefore the measure $\mu_\alpha \star \mu_\beta$ is a conjugation invariant function and it can be interpreted as the (defective) density function for the space of solutions c to the equation

$$abc = e, \quad \text{where} \;\; a \in C_\alpha,\; b \in C_\beta.$$

Since this density is conjugation invariant we represent it as the function $\nu(\theta)$ on the maximal torus of diagonal matrices given by

$$\nu(\theta) = ((\mu_\alpha \star \mu_\beta) \star \mu_\theta)(e).$$

Let $\mathcal{R}$ denote the left regular representation of G, and $\Delta_\alpha = 4\pi \sin^2 \alpha$ be the factor appearing in Weyl integration formula for conjugacy invariant functions. The Fourier transform of a function ψ on G at a representation ρ is defined as

$$\rho(\psi) = \int \rho(x^{-1})\psi(x)dx.$$

Let d_ρ denote the dimension of the representation ρ, χ_ρ its character and $\hat{G}$ the space of (complex) irreducible representations of G. Taking Fourier transform and decomposing the regular representation $\mathcal{R}$ of G in the usual way we obtain

$$\begin{aligned}
\nu(\theta) &= ((\mu_\alpha \star \mu_\beta) \star \mu_\theta)(e) \\
&= \frac{1}{\mathrm{vol.}(G)} \mathrm{Tr}\mathcal{R}((\mu_\alpha \star \mu_\beta) \star \mu_\theta) \\
&= \frac{1}{\mathrm{vol.}(G)} \sum_{\rho \in \hat{G}} d_\rho \mathrm{Tr}\rho((\mu_\alpha \star \mu_\beta) \star \mu_\theta) \\
&= \frac{1}{\mathrm{vol.}(G)} \sum_{\rho \in \hat{G}} d_\rho \frac{\Delta_\alpha \Delta_\beta \Delta_\theta}{d_\rho^3} \chi_\rho(\alpha)\chi_\rho(\beta)\chi_\rho(\theta)\mathrm{Tr}(I) \\
&= \frac{\Delta_\alpha \Delta_\beta \Delta_\theta}{\mathrm{vol.}(G)} \sum_{\rho \in \hat{G}} \frac{\chi_\rho(\alpha)\chi_\rho(\beta)\chi_\rho(\theta)}{d_\rho}.
\end{aligned}$$

Therefore

$$\nu(\theta) = 16\pi \sin^2 \alpha \sin^2 \beta \sin^2 \theta \sum_{\rho \in \hat{G}} \frac{\chi_\rho(\alpha)\chi_\rho(\beta)\chi_\rho(\theta)}{d_\rho} \tag{3.3}$$

Irreducible representations of $G = SU(2)$ are determined by dimension $k \geq 2$. The character of the k-dimensional representation ρ_k is:

$$\chi_k(g) = \sum_{j=0}^{k} e^{(j-2)i\theta} = \frac{\sin(k+1)\theta}{\sin\theta}$$

where $e^{\pm i\theta}$ is the eigenvalue of the conjugacy class of g. Substituting in (3.3) we obtain

$$\nu(\theta) = 16\pi \sin\alpha \sin\beta \sin\theta \sum_{k=0}^{\infty} \frac{\sin(k+1)\alpha \ \sin(k+1)\beta \ \sin(k+1)\theta}{k+1}. \tag{3.4}$$

Now let $f(\theta)$ denote the function defined by (2.1). Since f is an even function its Fourier expansion is of the form $f(\theta) = \frac{a_0}{2} + \sum a_n \cos nx$, and

$$a_n = \frac{2}{\pi}\Big[\frac{1}{n+1}\sin(n+1)\alpha \sin(n+1)\beta - \frac{1}{n-1}\sin(n-1)\alpha \sin(n-1)\beta\Big]$$

Substituting in the Fourier expansion we obtain

$$f(\theta) = 8\pi \sin\alpha \sin\beta \sum_{m=0} \left(\frac{\sin(m+1)\alpha \sin(m+1)\beta}{m+1} - \frac{\sin(m-1)\alpha \sin(m-1)\beta}{m-1}\right) \cos m\theta.$$

Using the elementary identity $\cos m\theta - \cos(m+2)\theta = 2\sin(m+1)\theta \sin\theta$ the series becomes telescopic and simplifies to

$$f(\theta) = 16\pi \sin\alpha \sin\beta \sin\theta \sum_{m=0} \frac{\sin(m+1)\alpha \sin(m+1)\beta \sin(m+1)\theta}{m+1},$$

which is identical with (3.4). ■

Remark 3.1 In the above proof we made use of the fact that the measure $\mu_\alpha \star \mu_\beta$ is a conjugation invariant function that can be interpreted as the (defective) density function for the space of solutions c to the equation

$$abc = e, \quad \text{where } \ a \in C_\alpha, \ b \in C_\beta$$

and this density is given by (3.3). For the case of finite groups this formula is well-known ([13], p.68).

Remark 3.2 It is possible to prove Theorem 2.1 without the use of harmonic analysis and by integral formulae similar to those used for the proof of Theorem 2.2 below. However, it appears that the above argument is possibly generalizable to products of conjugacy classes in compact connected semi-simple Lie groups, but the one based on integral formulae is not.

4 Proof of Theorem 2.2

Since the proofs of (B) and (C) are essentially the same computation we prove (A) and (C) only. Introduce coordinates on $SU(2)$ by:

$$(\rho,\varphi,\psi) \longrightarrow \begin{pmatrix} \rho e^{i\varphi} & \sqrt{1-\rho^2}e^{-i\psi} \\ -\sqrt{1-\rho^2}e^{i\psi} & \rho e^{-i\varphi} \end{pmatrix}, \quad (\rho,\varphi,\psi) \in [0,1]\times[0,2\pi]\times[0,2\pi] \tag{4.1}$$

The Haar measure on $SU(2)$ in the (ρ,φ,ψ)- coordinates is easily calculated by computing $g^{-1}dg$, a basis of left invariant 1-forms $\omega_1, \omega_2, \omega_3$ and then taking their wedge product to obtain:

$$\Omega = \omega_1 \wedge \omega_2 \wedge \omega_3 = 2\rho d\rho d\varphi d\psi.$$

With this normalization $\mathrm{vol}(SU(2)) = 4\pi^2$ as before.

Since both f and λ_a are K-bi-invariant, $\lambda_a \star f(x)$ is K-bi-invariant and therefore to compute $\mu_a \star f(x)$ we can assume that x is of the form

$$x = \begin{pmatrix} t & \overline{w} \\ -w & t \end{pmatrix} \tag{4.2}$$

where t is a real number and $w = se^{i\alpha}$ a complex number with $t^2 + s^2 = 1$. In (ρ, φ, ψ)- coordinates on $SU(2)$ we have

$$\lambda_a \star \check{f}(x) = \int_{\mathcal{O}_a} f(yx^{-1})dy = 2r(a)\int_0^{2\pi}\int_0^{2\pi} f(y(\rho,\varphi,\psi)x^{-1})d\varphi d\psi$$

With x represented as in (4.2) we have

$$\begin{aligned} \lambda_a \star \check{f}(x) &= 2r(a)\int_0^{2\pi}\int_0^{2\pi} f\left(\begin{pmatrix} \rho e^{i\varphi} & \sqrt{1-\rho^2}e^{-i\psi} \\ -\sqrt{1-\rho^2}e^{i\psi} & \rho e^{-i\varphi} \end{pmatrix}\begin{pmatrix} t & \overline{w} \\ -w & t \end{pmatrix}\right) d\varphi d\psi \\ &= 2r(a)\int_0^{2\pi}\int_0^{2\pi} f\left(\begin{pmatrix} r(a)te^{i\varphi} + b\sqrt{1-\rho^2}e^{-i\psi} & \star \\ \star & \star \end{pmatrix}\right) d\varphi d\psi \end{aligned}$$

Since f is spherical, it depends only on the norm of the $(1,1)$ entry of the above matrix. The square of the norm of the $(1,1)$ entry is

$$t^2\, r^2(a) + s^2(1-r^2(a)) + 2tsr(a)\sqrt{1-r^2(a)}\cos(\varphi+\psi-\alpha).$$

Substituting $t = r(x)$ and $s = \sqrt{1-r^2(x)}$, the norm of the $(1,1)$ entry becomes

$$|r(a)te^{i\varphi} + b\sqrt{1-\rho^2}e^{-i\psi}| = \sqrt{c_0 + c_1\cos(\varphi+\psi-\alpha)}.$$

Therefore

$$\lambda_a \star \check{f}(x) = 2r(a)\int_0^{2\pi}\int_0^{2\pi} f(\sqrt{c_0 + c_1\cos(\varphi+\psi-\alpha)})\, d\varphi\, d\psi.$$

The change of variable

$$(u, v) = (\sqrt{c_0 + c_1\cos(\varphi+\psi-\alpha)}, \psi),$$

is a 2 to 1 covering. Its Jacobian is given by

$$\frac{\partial(\varphi,\psi)}{\partial(u,v)} = \frac{2u}{c_1\sin(\varphi+\psi-\alpha)} = \frac{2u}{\sqrt{c_1^2 - (u^2-c_0)^2}}$$

Therefore

$$\begin{aligned}\lambda_a \star \check{f}(x) &= 4\pi r(a)\int_0^{2\pi}\int_{\sqrt{c_0-c_1}}^{\sqrt{c_0+c_1}} f(u)\frac{u}{\sqrt{c_1^2-(u^2-c_0)^2}}\,dudv\\ &= 2r(a)\int_{\sqrt{c_0-c_1}}^{\sqrt{c_0+c_1}} f(u)\frac{u}{\sqrt{c_1^2-(u^2-c_0)^2}}\,du.\end{aligned}$$

To compute $\lambda_a \star \lambda_b(f)$ we set $g(x) = \mu_b \star \check{f}(x)$. Then g is spherical and

$$\begin{aligned}\lambda_a \star \lambda_b(f) &= \lambda_a \star (\lambda_b \star \check{f})(e)\\ &= (\lambda_a \star g)(e)\\ &= \int_{\mathcal{O}_a} g(x)d\lambda_a(x)\\ &= g(a)\text{vol.}(\mathcal{O}_a),\end{aligned}$$

Now

$$\text{vol.}(\mathcal{O}_a) = 2r(a)\int_0^{2\pi}\int_0^{2\pi} d\varphi\ d\psi = 8\pi^2 r(a).$$

Therefore

$$\mu_a \star \mu_b(f) = 8\pi^2 r(a)g(a).$$

Substituting from the calculation of $g(a) = \lambda_b \star \check{f}(a)$ above we obtain

$$\lambda_a \star \lambda_b(f) = g(a)\ \text{vol.}(\mathcal{O}_a) = 16\pi^2 r(a)r(b)\int_{\sqrt{c_0-c_1}}^{\sqrt{c_0+c_1}} f(u)\ \frac{u}{\sqrt{c_1^2-(u^2-c_0)^2}}\ du$$

This completes the proof of part (A).

Proof of part (C) - Since both f and λ_a are K-bi-invariant, $\lambda_a \star f(x)$ is K-bi-invariant and therefore to compute $\mu_a \star f(x)$ we can assume that x is of the form $x = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}}\end{pmatrix}$. As before let $\{\theta_n\}$ be a sequence of spherical functions on G converging weakly to the (singular) invariant measure λ_a on the orbit $\mathcal{O}_a$. Applying the polar coordinate decomposition, for the convolution $\lambda_a \star \check{f}(x)$ we have

$$\begin{aligned}\lambda_a \star \check{f}(x) &= \int_{\mathcal{O}_a} f(yx^{-1})dy\\ &= \lim_{n\to\infty}\int_G \theta_n(g)f(gx^{-1})dg\\ &= c\lim_{n\to\infty}\int_K\int_K\int_A \theta_n(k_1a'k_2)f(k_1a'k_2x^{-1})\delta(a')da'dk_1dk_2\\ &= c\lim_{n\to\infty}\int_K\int_A \theta_n(a')f(a'kx^{-1})\delta(a')da'dk\\ &= c\delta(t_1)\int_K f(akx^{-1})dk\end{aligned}$$

Writing $M = akx^{-1} = k_1a_1k_2$, where $k_1, k_2 \in K = SU(2)$ and using the coordinates in (4.1)

$$a_1 = \begin{pmatrix} e^{\frac{r}{2}} & 0 \\ 0 & e^{\frac{-r}{2}} \end{pmatrix}, \quad k = \begin{pmatrix} \rho e^{i\varphi} & \sqrt{1-\rho^2}e^{-i\psi} \\ -\sqrt{1-\rho^2}e^{i\psi} & \rho e^{-i\varphi} \end{pmatrix},$$

we compute r in term of t_1, t_2 and k:

$$2\cosh r = \mathrm{Tr}(a_1^2) = \mathrm{Tr}(k_1 a_1^2 k_1^{-1}) = \mathrm{Tr}(MM^*). \tag{4.3}$$

On the other hand we have

$$M = akx^{-1} = \begin{pmatrix} e^{\frac{1}{2}(t_1+t)}\rho e^{i\varphi} & e^{\frac{1}{2}(t_1-t)}\sqrt{1-\rho^2}e^{-i\psi} \\ -e^{\frac{1}{2}(t-t_1)}\sqrt{1-\rho^2}e^{i\psi} & e^{-\frac{1}{2}(t_1+t)}\rho e^{-i\varphi} \end{pmatrix}$$

Therefore

$$\mathrm{Tr}(MM^*) = 2\rho^2\cosh(t+t_1) + 2(1-\rho^2)\cosh(t-t_1)\ ,$$

Comparing with (4.3) we obtain after a simple calculation

$$\cosh r = \cosh t_1 \cosh t - (2\rho^2 - 1)\sinh t_1 \sinh t$$

Now set

$$c_1 = \cosh t_1 \cosh t,\ \ c_2 = \sinh t_1 \sinh t$$

The function f is spherical so it only depends on the component r and therefore

$$\lambda_a \star \check{f}(x) = c\delta(t_1)\int_K f(akx^{-1})dk = c\delta(t_1)\int_0^{2\pi}\int_0^{2\pi}\int_0^1 f(r)(2\rho)d\rho d\varphi d\psi$$

Now make the change of coordinate $\cosh r = c_1 - c_2(2\rho^2 - 1)$ and note that, assuming that $t > t_1$, r ranges over $I_{t,t_1} = [t - t_1, t + t_1]$ as ρ ranges over $[0, 1]$. Substituting in the above integral for $\lambda_a \star \check{f}(x)$ we obtain

$$\lambda_a \star \check{f}(x) = \frac{2\pi^2 c}{c_2}\delta(t_1)\int_{I_{t,t_1}} f(r)\sinh r dr\ , \tag{4.4}$$

To compute $\lambda_a \star \lambda_b(f)$ we set $g(x) = \mu_b \star \check{f}(x)$. Then g is spherical and

$$\begin{aligned} \lambda_a \star \lambda_b(f) &= \lambda_a \star (\lambda_b \star \check{f})(e) \\ &= (\lambda_a \star g)(e) \\ &= \int_{\mathcal{O}_a} g(x)d\lambda_a(x) \\ &= g(a)\mathrm{vol.}(\mathcal{O}_a), \end{aligned}$$

Using (2.2) one easily obtains

$$\mathrm{vol.}(\mathcal{O}_{a_t}) = c(\mathrm{vol.}(SU(2)))^2 \sinh^2 t = 16\pi^4 \sinh^2 t. \tag{4.5}$$

Substituting from (4.5) we obtain

$$\lambda_a \star \lambda_b(f) = 16c\pi^4 (\sinh t_1)^2 g(a).$$

Equation (4.4) implies

$$\lambda_a \star \lambda_b(f) = 32c^2\pi^6 \sinh t_1 \sinh t_2 \int_{I_{t_1,t_2}} f(r) \sinh r dr \ ,$$

which completes the proof of the theorem. ■

5 Relations with the S-Matrix and the Verlinde Algebra

To understand the physical significance of products of conjugacy classes it is necessary to make use of Kac-Moody Lie algebras. Let $\mathfrak{g}$ be a complex simple Lie algebra and $\hat{\mathfrak{g}}$ the corresponding Kac-Moody Lie algebra (see [10] or [5] for basic theory and notation). It is well-known that $\hat{\mathfrak{g}}$ admits of a decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbb{C}c \oplus \widehat{\mathfrak{g}}_+,$$

where $\mathbb{C}c$ is its center, and the subalgebras $\mathfrak{g}_\pm$ play roles analogous to those of nilpotent radicals for opposite Borel subalgebras in the complex semi-simple case. Let

$$\mathfrak{g} = \mathfrak{a} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha,$$

be the root space decomposition for $\mathfrak{g}$ relative to a (fixed) Cartan subalgebra $\mathfrak{a}$ and R (resp. R_+) the corresponding root system (resp. a fixed positive root system). Denote the weight lattice in $\mathfrak{a}^\star$ by P, and P_+ be the set of dominant weights. According to highest weight theory finite dimensional irreducible representations of $\mathfrak{g}$ are parametrized by P_+ (see e.g. [14] for representation theory of complex semisimple Lie algebras). For $\lambda \in P_+$ we denote by V_λ the corresponding irreducible representation of $\mathfrak{g}$. Denote the highest weight for the adjoint representation by θ and its dual in $\mathfrak{a}$ relative to the Killing form by H_θ. For a Lie algebra $\mathfrak{g}$, $U(\mathfrak{g})$ denotes its universal enveloping algebra. Set

$$P_l = \Big\{ \lambda \in P_+ \mid \lambda(H_\theta) \leq l \Big\}.$$

For $\lambda \in P_+$ and a positive integer l let $\mathcal{V}_{\lambda,l}$ be the Verma module

$$\mathcal{V}_{\lambda,l} = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_\lambda,$$

where $\mathfrak{b} = \mathfrak{g} \oplus \mathbb{C}c \oplus \widehat{\mathfrak{g}}_+$, the action of the central element c is by multiplication by l, and $\hat{\mathfrak{g}}_+$ acts trivially. It is well-known that $\mathcal{V}_{\lambda,l}$ contains a unique maximal submodule $\mathcal{Z}_{\lambda,l}$. The quotient space $\mathcal{V}_{\lambda,l}/\mathcal{Z}_{\lambda,l}$ is a finite dimensional representation for the Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ which we denote by $\mathcal{H}_{\lambda,l}$. Since l is fixed throughout the following we simply write $\mathcal{H}_\lambda$ for $\mathcal{H}_{\lambda,l}$.

Let M be the Riemann sphere, and for each open set $U \subset M$, we denote by $\mathcal{O}(U)$ the ring of holomorphic functions on U, and by $\mathfrak{g}(U)$ the Lie algebra $\mathfrak{g} \otimes \mathcal{O}(U)$. We want to associate a vector space to the data consisting of a finite subset $\overrightarrow{p} = \{p_1, \dots, p_n\}$ of M, $n \geq 3$, and elements λ_i of P_l for $i = 1, \dots, n$.

Let z_i be a local coordinate near p_i, and f_{p_i} be a Laurent series at p_i so that $f \in \mathcal{O}(M - \overrightarrow{p})$. Since $\hat{\mathfrak{g}}$ is a central extension of $\mathfrak{g} \otimes \mathbb{C}((z))$ by $\mathbb{C}$, one obtains for each i a ring homomorphism $\mathcal{O}(M - \overrightarrow{p}) \to \mathbb{C}((z))$, and hence a Lie algebra homomorphism $\mathfrak{g}(M - \overrightarrow{p}) \to \mathfrak{g} \otimes \mathbb{C}((z))$. Now set

$$\mathcal{H}_{\overrightarrow{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \dots \otimes \mathcal{H}_{\lambda_n}$$

and define an action of $\mathfrak{g}(M - \overrightarrow{p})$ on $\mathcal{H}_{\overrightarrow{\lambda}}$ by the formula

$$(X \otimes f).(v_1 \otimes \dots v_n) = \sum_i v_1 \otimes \dots \otimes (X \otimes f_{p_i})v_i \otimes \dots \otimes v_n$$

This is indeed a Lie algebra action. Now we define the vector space $V_M(\overrightarrow{p}, \overrightarrow{\lambda})$ as largest quotient of $\mathcal{H}_{\overrightarrow{\lambda}}$ on which $\mathfrak{g}(M - \overrightarrow{p})$ acts trivially.

These vector spaces do not depend, up to a canonical isomorphism, on the choice of the local coordinate $z_1, \dots, z_n$. On the other hand they depend on the Lie algebra $\mathfrak{g}$, the integer l and the genus g of the Riemann surface M. The dimension of the vector space $V_M(\overrightarrow{p}, \overrightarrow{\lambda})$ is denoted by $N(\lambda)$.

Now let $n = 3$. The Verlinde algebra $\mathcal{R}_l(\mathfrak{g})$ of level l is, as a $\mathbb{Z}$-module, the free $\mathbb{Z}$-module with basis the isomorphism classes $[V_\lambda]$ for $\lambda \in P_l$, the product in $\mathcal{R}_l(\mathfrak{g})$ is defined by

$$[V_\lambda] \ *_l \ [V_\mu] = \sum_{\nu \in P_l} N(\lambda + \mu + \nu^*)[V_\nu].$$

This product is clearly commutative and its associativity is proven in [16]. It is important to note the distinction between the Verlinde or fusion product and the decomposition of the tensor product of representations (e.g. Clebsch-Gordon formula).

As noted in [2] there is an interpretation of the support of the product of two conjugacy classes in terms of the Verlinde product for the ring $\mathcal{R}_l(\mathfrak{g})$ where $\mathfrak{g} = \mathfrak{sl}_r(\mathbb{C})$. Let C_α and C_β be two rational conjugacy classes in $SU(n)$ i.e. $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n$ and $\alpha_1 - \alpha_n \leq 1$ and similarly for the eigenvalues β. Then the conjugacy class C_γ occurs in the product $C_\alpha.C_\beta$ if and only if for some N such that $N\alpha, N\beta, N\gamma \in P^*$ (weight lattice) we have $V_{N\gamma}$ occurs in the decomposition of the fusion product $V_{N\alpha} *_N V_{N\beta}$.

While the support of the product of two conjugacy classes is described in terms of the fusion product, one expects the distribution of the product to require information from the S-matrix of the Kac-Moody Lie algebra as explained below (see [10] and [6]). It is well-known that for each non-negative integer l certain space of characters of the Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ (called Virasoro specialized characters and which we denote by $\mathcal{X}_l$) admits of a $PSL(2,\mathbb{Z})$ action. $PSL(2,\mathbb{Z})$ is generated by two elements $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The action of S on on $\mathcal{X}_l$ is given by a symmetric unitary matrix (S_{jk}) called the S-matrix. It is a fact that the fusion product can also be described in terms of products of the matrix elements S_{jk} as

$$N_{ijk} = \sum_n \frac{S_{in} S_{jn} S_{kn}}{S_{0n}}.$$

We make the following conjecture:

Conjecture - The distribution of the product of two conjugacy classes in $SU(n)$ can be described as the limiting distribution of the first column of the S-matrix properly normalized as the level l goes to ∞ and is also equal to the square root of that measure of density in the Weyl integration formula.

Proposition 5.1 *The conjecture is valid for $SU(2)$.*

Proof - The S-matrix for the the Kac-Moody Lie algebra $\widehat{\mathfrak{sl}_2}$ is given by (see [4] and [17])

$$S_{ij} = \sqrt{\frac{2}{l+2}} \sin\left(\frac{(i+1)(j+1)}{l+2}\pi\right).$$

Multiplying the first column by $c_{\alpha,\beta}\sqrt{l+2}$, where the constant $c_{\alpha,\beta}$ depends only on the conjugacy classes C_α and C_β, taking the weak limit as $l \to \infty$ and comparing with Theorem 2.1 the required result follows. ■

6 Numerical Results

By the discretization of products of conjugacy classes $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ one means the choice of N points on each and the determination of the corresponding empirical measure of products of these points. If these N points are chosen randomly according to the invariant measures on the conjugacy classes then the empirical measure of the products converges weakly to the density function (2.1) by Theorem 2.1 and is numerically demonstrated in Figures 1 for a typical choice with $Np = 2172$ points.

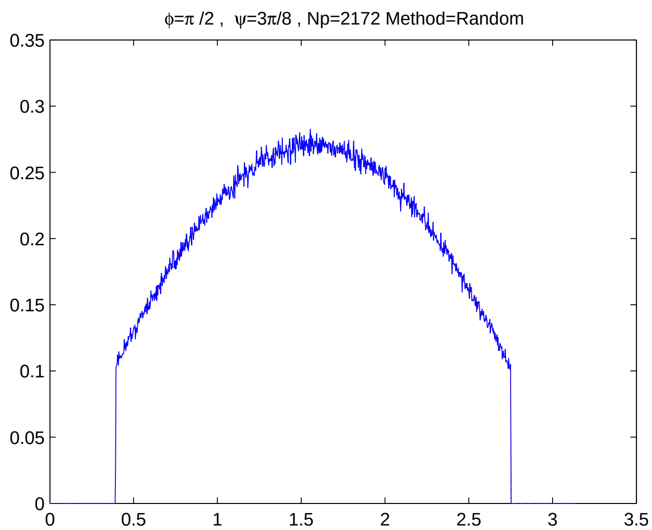

Figure 1.

Each conjugacy class $\mathcal{C}_\alpha$ is naturally equivalent to a copy of S^2. It is therefore reasonable to investigate the weak convergence of the empirical measure if the points on S^2 are chosen according to the requirements of Thomson's Problem of distributing points on the sphere (see [11]). This means that the points should be distributed so that the Coulomb potential

$$\sum_{i<j}^{N} \frac{1}{|z_i - z_j|^\alpha}$$

is minimized. A variation of this problem for $\alpha = 1$ was originally posed by J. J. Thomson in connection with his investigations of the structure of the atom in 1904. It remains unsolved except for a few small values of N, and it has also attracted attention for applications to complexity theory [15]. The lattice point method makes use of the natural embedding of the icosahedron in S^2 and distributes $N = 10(mn + m^2 + n^2) + 2$ points on the sphere in such a way that

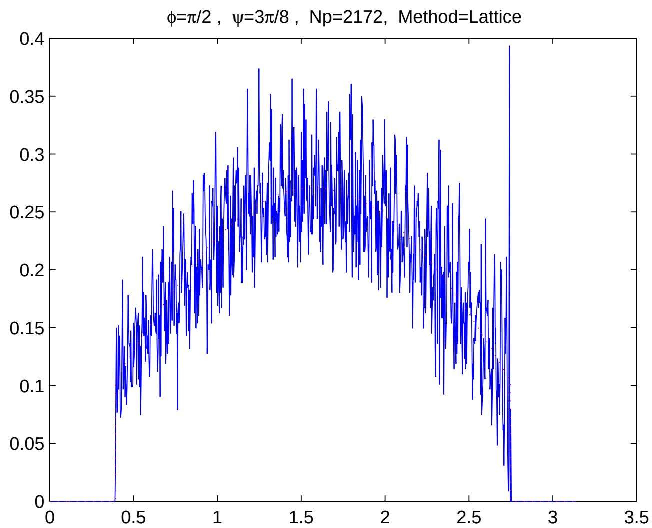

Figure 2.

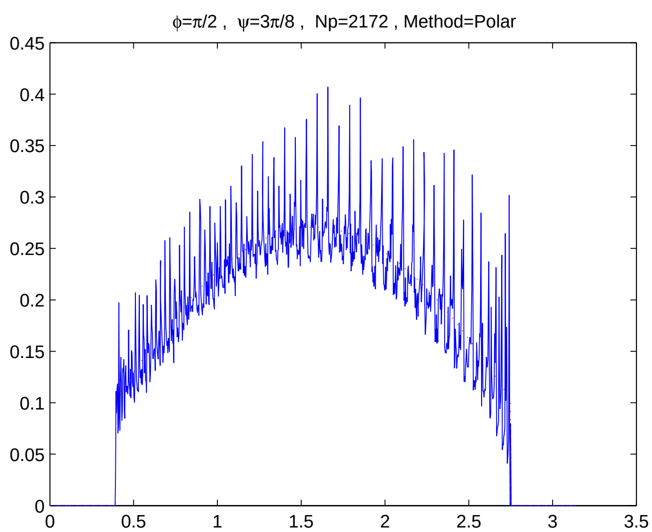

Figure 3.

the distribution exhibits a high degree of symmetry and the points appear to be "evenly" spaced. It was conjectured in [1] that this distribution will provide the solution to Thomson's Problem for $\alpha = 1$. The polar coordinates method was devised in [11] to achieve the minimum required by Thomson's Problem and numerical tests disproved the conjecture in [1] by showing that the (local) minimum achieved by the polar coordinates method (where there was symmetry breakdown) was in fact smaller. It is therefore natural to test the convergence of the empirical measure of products if the discretization is done according the polar coordinates or the lattice point methods. In Figures 1 the convergence of the empirical measure to the density as predicted by Theorem 2.1 is exhibited for $Np = 2172$ points. The angles ϕ and ψ in the captions refer to the conjugacy classes $\mathcal{C}_\phi$ and $\mathcal{C}_\psi$ respectively. In Figure 2 the corresponding empirical measure is calculated for the lattice point method and it is noticed that even if the measure converges to the density given by (2.1), the convergence is significantly slower. Similar conclusion is applicable to the polar coordinates method as shown in Figures 3.

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