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Exact Solutions of Einstein's Field Equations Associated to a Point-Mass Delta-Function Source

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Abstract

We explicitly solve the static spherically symmetric Einstein field equations due to a delta function point mass source at $r = 0$ and explain why our solutions are *not* diffeomorphic to the textbook solution. It is shown that the Euclidean action (in $\hbar$ units) *is* precisely *equal* to the black hole entropy (in Planck area units). This result holds in any dimensions $D \geq 3$. Instead of a black-hole solution with a horizon at $r = 2GM$ one has a spacetime *void* surrounding the singularity.

1 The Difference between a Point Mass Source and the Vacuum Solutions

We begin by writing down the class of static spherically symmetric (SSS) vacuum solutions of Einstein's equations [1] studied by [5] given by a $infinite$ family of solutions parametrized by a family of admissible radial functions $R(r)$ (in $c = 1$ units)

$$(ds)^2 \ = \ (1 - \frac{2G_N \ M_o}{R}) \ (dt)^2 \ - (1 - \frac{2G_N \ M_o}{R})^{-1} \ (dR)^2 \ - \ R^2(r) \ (d\Omega)^2. \quad (1.1)$$

where the solid angle infinitesimal element is

$$(d\Omega)^2 = (d\phi)^2 + sin^2(\phi)(d\theta)^2, \quad (1.2)$$

This expression of the metric in terms of the radial function $R(r)$ (a radial gauge) does *not* violate Birkoff's theorem since the metric (1.1, 1.2) expressed in terms of the radial function $R(r)$ has exactly the same functional form as that required by Birkoff's theorem and $0 \leq r \leq \infty$. In this letter we will solve the SSS solutions when a point mass delta function source is present at the location $r = 0$. Notice that the vacuum SSS solutions of Einstein's equations, with and without a cosmological constant, do *not* determine the form of the radial function $R(r)$. In [24] we were able to show why the cosmological constant is *not* zero and why it *is so tiny* based on a judicious choice of the radial function and also on a Weyl geometric extension of the Jordan-Brans-Dicke scalar-tensor theory gravity [24] In the appendix we construct the Schwarzschild-like solutions in any dimensions $D > 3$ and show that the radial function $R(r)$ is completely *arbitrary* [32].

There are two interesting cases to study based on the boundary conditions obeyed by $R(r)$: (i) the Hilbert textbook (black hole) solution [4] based on the choice $R(r) = r$ obeying $R(r = 0) = 0$, $R(r \to \infty) \to r$. And : (ii) the Abrams-Schwarzschild radial gauge based on choosing the cutoff $R(r = 0) = 2GM$ such that $g_{tt}(r = 0) = 0$ which apparently seems to "eliminate" the horizon and $R(r \to \infty) \to r$. The choice $R^3 = r^3 + (2GM)^3$ was the original solution of 1916 found by Schwarzschild.

However, the choice $R(r = 0) = 2GM$ has a serious $flaw$ and is : How is it *possible* for a point-mass at $r = 0$ to have a non-zero area $4\pi(2GM)^2$ and a *zero* volume *simultaneously* ? so it seems that one is forced to choose the Hilbert gauge $R(r = 0) = 0$. Nevertheless we will show in this letter how by choosing a *judicious* choice of $R(r)$ (not contemplated before to our knowledge), one can cure the flaw and have the correct boundary condition $R(r = 0) = 0$ while displacing the horizon from $r = 2GM$ to a location *arbitrarily* close to $r = 0$ as one desires, $r_h \to 0$, and where stringy geometry and Quantum Gravitational effects begin to take place. Many authors [5], [6], [7], [8], [9], [13], [10], among many others, have explored the gauge choice obeying $R(r = 0) = 2GM$, after Brillouin [3], Schwarzschild [2] found that possibility long ago. Unfortunately the solution to this serious problem was never found.

In this work we will propose a very straightforward solution to this cut-off problem by finding a solution to the vacuum SSS solutions of Einstein equations which *is not* $diffeomorphic$ to the standard Hilbert textbook solution (based on the choice $R(r) = r$) by choosing for a radial function $R(r) = r + 2GM\Theta(r)$, where the Heaviside Step function [1] is defined $\Theta(r) = 1$ when $r > 0$, $\Theta(r) = -1$ when $r < 0$, and $\Theta(r = 0) = 0$ (the arithmetic mean of the values at $r > 0$ and $r < 0$). The reason why a metric solution $g_{\mu\nu}(R(r))$ based on the choice $R(r) = r + 2GM\Theta(r)$ *is not* $diffeomorphic$ to the Hilbert textbook solution $g_{\mu\nu}(r)$ is due to the *discontinuity* of the step function at

[1] We thank Michael Ibison for pointing out the importance of the Heaviside step function and the use of the modulus $|r|$ to account for point mass sources at $r = 0$. To be more precise one should write $R = r + 2G|M|\Theta(r)$ so that solutions with $r < 0, M > 0$ correspond to solutions with $r > 0, M < 0$ (white hole).

$r = 0$.

There are many *fundamental* difference (besides others) with the Fronsdal-Kruskal-Szekeres analytical extension of the Hilbert textbook metric into the interior region of the black-hole horizon $r = 2GM$. The Fronsdal-Kruskal-Szekeres metric is *no* longer *static* in the interior region $r < 2GM$, whereas in our case the metric $g_{\mu\nu}(R(r))$ when $R(r) = r + 2GM\Theta(r)$ *is static* for *all* values of r. The asymptotic value is $R \sim r$ for $r >> 2GM$ and one recovers the correct Newtonian limit in the asymptotic regime. It is now, via the Heaviside step function, that we may maintain the correct behaviour $R(r) = r = 0$, when $r = 0$, such that we can satisfy the required condition $R(r = 0) = r = 0$, consistent with our intuitive notion that the spatial area and spatial volume of a point $r = 0$ *has* to be *zero*.

Solutions with point mass delta function sources are physically *distinct* from the SSS vacuum solutions. In order to generate the required $\delta(r)$ terms in the right hand side of Einstein's equations, one must replace everywhere $r \to |r|$ as required when point-mass sources are inserted. The Newtonian gravitational potential due to a point-mass source at $r = 0$ is given by $-GM/|r|$ and is consistent with Poisson's law which states that the Laplacian of the Newtonian potential $-GM/|r|$ is $4\pi G\rho$ where $\rho = (M/4\pi r^2)\delta(r)$ in Newtonian gravity. However, the Laplacian in spherical coordinates of $(1/r)$ is *zero*.

For this reason, there is a *fundamental* difference in dealing with expressions involving absolute values $|r|$ like $1/|r|$ from those which depend on r like $1/r$ [11]. Therefore the radial gauge must be chosen by $|R(r) = |r + 2GM\ \Theta(r)|$. Had one *not* use $|r|$ in the expression for the metric, one will not generate the desired $\delta(r)$ terms in the right hand side of Einstein's equations $\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G\ T_{\mu\nu} \neq 0$, and one would get an expression *identically* equal to *zero* which is consistent with the vacuum solutions $\mathcal{R}_{\mu\nu} = \mathcal{R} = 0$ in the absence of matter [25].

The metric associated with point mass delta function sources $g_{\mu\nu}(|R(r)|)$ is smooth and differentiable for all $r > 0$ and $\mathcal{R}_{\mu\nu} = \mathcal{R} = 0$ when $r > 0$. The metric $g_{\mu\nu}[|R(r)|]$ is *discontinuous only* at the location of the point mass singularity $r = 0$ whose world-line which may be thought of as the boundary of spacetime or transition region to the white hole solution. The scalar curvature contains now delta function terms (instead of being zero as in the vacuum case) due to the delta function point mass source at $r = 0$; i.e. the scalar curvature jumps from zero to infinity at $r = 0$.

For instance, $g_{tt}(r) = 1 - (2GM/|r + 2GM\Theta(r)|)$ is well defined and smooth for all $r > 0$ and tends to zero when r tends to $0 + \epsilon$. It tends to minus infinity when $r = 0$ and there is a discontinuity at the location of the point mass $r = 0$, a boundary, where the singularity lies. Colombeau's theory of Nonlinear Distributions (and Nonstandard Analysis) is the proper way to deal with point-mass sources in nonlinear theories like Gravity where one may rigorously solve the problem without having to introduce a boundary at $r = 0$ [14]. Thus the metric is continuous at *all* points *except* at the location of the point mass singularity $r = 0$, which is to be expected when an infinitely compact point mass source is present at $r = 0$.

Active diffs must not be confused with passive diffs. One defines an active diffs by mapping $r \rightarrow R(r)$ such that the metric $g_{\mu\nu}[R(r)]$ is diffeomorphic to $g_{\mu\nu}(r)$ if, and only if, the $R(r)$ is a *smooth* and invertible mapping. However, in order to recover the field due to a point mass delta function source at $r = 0$ one must use the *modulus* function $R(r) = |r|$ instead of r. Since the derivative of the function $|r|$ has a discontinuity at $r = 0$, the right and left derivatives are ± 1 respectively, the second derivative yields a $\delta(r)$ term, therefore, the function $R(r) = |r|$ strictly speaking is *not* smooth in *all* of the points in a domain *enclosing* the singularity $r = 0$. Consequently, the metric $g_{\mu\nu}(|r|)$ is *not* diffeomorphic to the Hilbert textbook metric $g_{\mu\nu}(r)$. The former leads to a scalar curvature involving delta function terms related to the point mass delta function sources, whereas the latter vacuum solutions yields an *identically* zero expression for the Ricci tensor and scalar curvature $\mathcal{R}_{\mu\nu} = \mathcal{R} = 0$. The same reasoning applies to the metric $g_{\mu\nu}[|R(r)|]$ when $|R(r) = |r + 2GM\Theta(r)|$ which is *neither* diffeomorphic to the Hilbert textbook metric $g_{\mu\nu}(r)$ *nor* diffeomorphic to the metric $g_{\mu\nu}(|r|)$ due to the fact that the step function $\Theta(r)$ is discontinuous at $r = 0$.

To sum up, by using $|R(r) = |r + 2GM\ \Theta(r)|$, we will have a metric $g_{\mu\nu}(|R(r)|)$ that *is not diffeomorphic* to the Hilbert textbook metric $g_{\mu\nu}(r)$. Because $R(r) = r + 2G_N M$ when $r > 0$, the horizon can the be displaced from $r = 2GM$ to a location as arbitrarily close to $r = 0$ as desired $r_{Horizon} \rightarrow 0$. To be more precise, the horizon actually never forms since at $r = 0$ one hits the singularity. After performing the mapping from r to $R(r)$, a *void* (hole) surrounding $r = 0$ forms; i.e. a void in the region $0 < R < 2GM$ with the singularity remaining at the center $r = 0 = R(r = 0) = 0$ and a ring extending from $R = 2G_N M$ to $R = r = \infty$ (when $M = finite$). Due to the infinite mass density of the point-mass located at $r = 0$, it will rip and tear apart the fabric of spacetime in its neighborhood and create a *void* in spacetime in the region $0 < R < 2GM$. Thus, instead of a Black Hole, we really have a *void* bulk region $0 < R < 2GM$ *empty* of spacetime while leaving the point-mass singularity at $r = 0 = R(r = 0) = 0$ behind. For further details see [23].

2 The Gaussian as a smeared delta function and Wave-Particle Duality

Our aim is to solve the field equations with a delta function point mass source at $r = 0$ in $D = 3+1$ dimensions. There are two ways of solving this problem. In this section we will smear the point mass delta function distribution using a Gaussian of finite width and calculate the metric, curvature, and stress enrrgy tensor. At the end of the calculations, when one goes back and computes the Einstein-Hilbert action, one will get the same answer as if one had started with a pure delta function point mass source as shown in the next section. This is a consequence of the semi-classical properties inherent in the Gaussian matter distribution and associated with the the wave-particle duality in QM; namely, the mass and charge density distributions (in the case of EM interactions) are just the square of the Gaussian wave function amplitude (times a mass and charge factor) as shown by [19]

Delta-function point sources for general-relativisitic gravity in $1+1$ dimensions yields a rich variety of solutions. Exact solution for 2 point sources on a line, 3 point sources on a line and N point sources on a circle have been found. For 3 point sources the system is chaotic and is a simple model where to study relativistic chaos [16] besides the Kasner-Misner mixmaster chaotic cosmological models. Before doing so, we shall model the mass distribution by a smeared delta function ρ [17], by starting with the following equations

$$G_{00} = \mathcal{R}_{00} - \frac{1}{2} g_{00} \mathcal{R} = 8\pi G_N T_{00} = g_{00} 8\pi G_N \rho(r)$$

$$\mathcal{R}_{ij} - \frac{1}{2} g_{ij} \mathcal{R} = 8\pi G_N T_{ij} \tag{2.1}$$

where $\rho(r)$ is a smeared delta function given by the Gaussian and the T_{ij} elements are comprised of a radial and tangential pressures of a self-gravitating *anisotropic* fluid [17]

$$\rho(r) = M_o \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}}, \quad p_r = -\rho(r), \quad p_{tan} = p_\theta = p_\phi = -\rho(r) - \frac{r}{2}\frac{d\rho}{dr}. \tag{2.2}$$

The components of the mixed stress energy tensor are $T^\mu_\nu =$ *diagonal* $(-\rho(r), p_r, p_\theta, p_\phi)$. The radial pressure $p_r = -\rho$ is negative pointing towards the center $r = 0$ consistent with the self-gravitating picture of the droplet. The radial dependence of the mass distribution is explicitly given in terms of the incomplete Gamma function $\gamma[a, r]$ as

$$M(r,\sigma) = M_o \int_0^r \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} 4\pi r^2 \, dr = \frac{2M_o}{\sqrt{\pi}} \gamma[\frac{3}{2}, \frac{r^2}{4\sigma^2}]. \tag{2.3}$$

In the limit $\sigma^2 \to 0$ one recovers the delta function

$$lim_{\sigma\to 0} \ \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \ \to \frac{\delta(r)}{4\pi r^2}. \tag{2.4}$$

and the incomplete Gamma function reduces to the ordinary Gamma function $\Gamma(\frac{3}{2}) = (\sqrt{\pi}/2)$ and $M(r, \sigma \to \infty)$ tends to M_o. The line element which solves Einstein' s equations in the presence of the smeared delta function $\rho(r)$ distribution can be obtained by a direct application of Birkoff's theorem by evaluating the (variable) mass $M(r,\sigma)$ enclosed by a radius r [17]

$$(ds)^2 \ = \ (1 - \frac{2G_N \ M(r,\sigma)}{r}) \ (dt)^2 \ - (1 - \frac{2G_N \ M(r,\sigma)}{r})^{-1} \ (dr)^2 \ - \ r^2 \ (d\Omega)^2. \tag{2.5}$$

In the Appendix we check that the line element (2.5) based on the radial mass distribution $M(r,\sigma)$ given by the incomplete Gamma function solves Einstein's equations (2.1).

The scalar curvature is given by $8\pi G_N \ trace \ (T_{\mu\nu})$

$$\mathcal{R} \ = \ - \ 8\pi G_N \ 2\rho(r) \ [\ 2 - \frac{r^2}{4\sigma^2} \ ] = \ - \ 8\pi G_N \ 2M_o \ \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \ [\ 2 - \frac{r^2}{4\sigma^2} \ ] \tag{2.6}$$

At $r = 0$ one has $\mathcal{R}(r = 0, \sigma) = -4G_N M_o \ /\sqrt{\pi} \ \sigma^3$ and blows up when $\sigma = 0$. At $r = \infty, \ \mathcal{R} = 0$.

The $\sigma \to 0$ limit must be taken after, and only after, performing the calculations. For instance, the Einstein-Hilbert action in the domain bounded by $[0, r]$ will contain the incomplete gammas $\gamma[\frac{5}{2}, \frac{r^2}{\sigma^2}]$ and $\gamma[\frac{3}{2}, \frac{r^2}{\sigma^2}]$

$$S \ = \ - \ \frac{1}{16\pi G_N} \int \ \mathcal{R} \ \sqrt{|det \ g|} \ d^4x =$$

$$\frac{1}{16\pi G_N} \int \int_0^r \ 8\pi G_N \ 2M_o \ \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \ [\ 2 - \frac{r^2}{4\sigma^2} \ ] \ (4\pi r^2 \ dr \ dt) \ =$$

$$\frac{2M_o}{\sqrt{\pi}} \ [\ 2 \ \gamma(\frac{3}{2}, \frac{r^2}{\sigma^2}) \ - \gamma(\frac{5}{2}, \frac{r^2}{\sigma^2}) \ ] \int \ dt. \tag{2.7a}$$

In the $\sigma \to 0$ limit, the incomplete gammas become the ordinary Euler $\Gamma[\frac{5}{2}], \Gamma[\frac{3}{2}]$ giving

$$\frac{2M_o}{\sqrt{\pi}} \ [\ 2 \ \frac{1}{2}\sqrt{\pi} \ - \frac{3}{2} \ \frac{1}{2}\sqrt{\pi} \ ] \ \int \ dt \ = \ \frac{M_o}{2} \ \int dt. \tag{2.7b}$$

Thus, the limit $\sigma \to 0$ has the *same* effect as if one took the upper r limit of the action integral from $r = finite$ all the way to $r \to \infty$ irrespective of the value assigned to σ (as long as $\sigma \neq \infty$) ; ie. we may evaluate the action *all* over space (as one should) by fixing the value of σ and integrating r from 0 to ∞. The incomplete gammas will turn into the Euler gammas when $r = \infty$ and one arrives at the same answer (2.7b). Whether or not this has a relationship to the holographic principle is worth investigating.

The Euclideanized Einstein-Hilbert action associated with the scalar curvature in the limit $\sigma \to 0$ is obtained after a compactification of the temporal direction along a circle S^1 giving an Euclidean time coordinate interval of $2\pi t_E$ and which is defined in terms of the Hawking temperature T_H (in the limit $\sigma \to 0$) and Boltzman constant k_B as $2\pi t_E = (1/k_B T_H) = 8\pi G_N M_o$. The Euclidean action becomes

$$S_E = (\frac{M_o}{2})\ (2\pi t_E)\ =\ 4\pi\ G_N\ M_o^2 = \frac{1}{4}\ \frac{4\pi(2G_N M_o)^2}{G_N}\ =\ \frac{Area}{4\ L_P^2}. \qquad (2.7c)$$

which is the Black Hole Entropy in Planck area units $G_N = L_P^2$ ($\hbar = c = 1$). We will show below in eq-(2.7c) that the Euclidean action associated with the scalar curvature corresponding to the delta function point mass source given by

$$\mathcal{R}\ =\ -\ \frac{2G_N M_o\ \delta(r)}{r^2}. \qquad (2.8)$$

yields precisely the *same* value for the action and entropy (2.7c) in the limit $\sigma \to 0$.

After, and only after, having solved Einstein's equations, one may take the $\sigma \to 0$ limit of those equations and not before. The order in taking this limiting procedure is essential. It is when this limit is properly taken when one recaptures the true solution due to a point mass delta function source M_o at $r = 0$. Upon performing the $\sigma = 0$ limit in the right order we get

$$\rho(r,\sigma=0) = -p_r = \frac{M_o\delta(r)}{4\pi r^2},\ p_\theta(r,\sigma=0) = p_\phi(r,\sigma=0)$$
$$=\ -\ \frac{M_o\delta(r)}{4\pi r^2} - \frac{r}{2}\ \partial_r\ (\frac{M_o\delta(r)}{4\pi r^2}). \qquad (2.9)$$

such

$$\mathcal{R}(r,\sigma=0)\ =\ -\ 8\pi G_N M_o\ [\ \frac{4\delta(r)}{4\pi r^2}\ +\ r\ \partial_r\ (\frac{\delta(r)}{4\pi r^2})\ ]\ \Rightarrow$$
$$-\frac{1}{16\pi G_N}\ \int\int\ \mathcal{R}(r,\sigma=0)\ (4\pi r^2\ dr\ dt)$$
$$=\ -\ \frac{1}{16\pi G_N}\ \int\int\ \frac{-2G_N M_o\ \delta(r)}{r^2}\ (4\pi r^2\ dr\ dt). \qquad (2.10)$$

after an integration by parts since at $r = \infty$, $r\delta(r) \rightarrow 0$. The relation between the action expressed in terms of $\mathcal{R}(r, \sigma = 0)$ and the point mass source delta function case is one of the most important results of this work.

At any given value of σ, the location of the horizon $r_s(\sigma)$ is now shifted to a new location dependent now on the Gaussian width σ parameter

$$1 - \frac{2G_N \ M(r_s, \sigma)}{r_s} \ = \ 0 \ \Rightarrow r_s \ = \ 2G_N M_o \ \frac{2}{\sqrt{\pi}} \ \gamma[\frac{3}{2}, \ \frac{r_s^2}{4\sigma^2}] \tag{2.11}$$

The solution of this transcendental equation (1.13) yields the new location $r_s = r_s(2G_N M_o, \sigma)$ of the horizon . In the $\sigma \rightarrow 0$ limit, $r_s \rightarrow 2G_N M_o$ as expected. The authors [17] plotted the function $g_{00}(r, 2G_N M_o/\sigma)$ and found that there are one, two and no horizons depending on the values of the ratio $G_N M_o/\sigma$. The critical value of the mass below which no horizon forms was $G_N M_o \sim 1.9\ \sigma$ (our notation differs from [17]) which corresponds to $r_s(2G_N M_o, \sigma) \sim 3\ \sigma$. Similar findings have been found in the Renormalization-Group improved Schwarzchild solutions by [31] based on the running flow of the Newtonian constant $G(r)$ with a non-Gaussian ultraviolet fixed point $G(r = 0) = 0$ (asymptotic freedom).

After having shown how to construct SSS solutions of Einstein's equations in the presence of a delta function mass source at $r = 0$, as a limiting procedure $\sigma \rightarrow 0$ of a smeared-delta function mass distribution, we can return now to the introduction of a particular σ-dependent radial gauge, $R(r, \sigma) = r + r_s(\sigma)\ \Theta(r)$ (notice the presence of $r_s(\sigma)$) and replace the metric (2.5) by :

$$(ds)^2 \ = \ (1 - \frac{2G_N \ M(R, \sigma)}{R}) \ (dt)^2 - (1 - \frac{2G_N \ M(R, \sigma)}{R})^{-1} \ (dR)^2 - R^2(r) \ (d\Omega)^2. \tag{2.12}$$

with the upshot that $R(r = 0) = 0$ as required (the area and volume of the point $r = 0$ has to be zero) and such that the location of the horizon $r_h \rightarrow 0$ can be *shifted* to a location arbitrarily close to $r = 0$, since $R(r_h = 0 + \epsilon) = \epsilon + r_s \sim r_s$ where r_s is the solution (if any) to the prior transcendental equation (2.11) depending on the ratio $G_N M_o/\sigma$ [17]. However, when the radial gauge $R(r) = r + r_s \Theta(r)$ is chosen, r_s is *no* longer equal to $r_h \rightarrow 0$. Once again, to be more precise, the horizon actually never forms at $r = 0$ when $\sigma = 0$ (one hits the singularity). When $\sigma \neq 0$, there is one, two and no horizons depending on the values of $G_N M_o/\sigma$ [17]. When there is no horizon then $R(r) = r$. When there is one horizon $R(r) = r + r_s\ \Theta(r)$. When there is an outer r_s^+ and inner horizon r_s^-, by choosing $R(r) = r + r_s^-\ \Theta(r)$ then $R \rightarrow r_s^-$ as one approaches $r = 0 + \epsilon$ and $R(r = 0) = 0$; while $R = r_s^+$ when $r = r_s^+ - r_s^-$.

At $r = 0 \rightarrow R = 0$, the mass is $M(R = 0) = 0$ (when $\sigma \neq 0$). Therefore, the behaviour of the metric component $g_{00}(R = 0) = limit\ _{R \rightarrow 0}\ (1 - 2G_N M(R)/R)$

requires a very careful evaluation due to the $\frac{0}{0}$ ratio of $\frac{M(R)}{R}$ at $R = 0$. Using the properties of the incomplete gammas, the ratio $\frac{M(R)}{R} \sim \gamma[\frac{3}{2}, \frac{R^2}{4\sigma^2}]/R$ for very small values of R behaves as $R^3/R = R^2 \to 0$. Hence, when $\sigma \neq 0$, the metric component $g_{00}(R = 0) = 1$ in agreement with the diagrams of [17].

The scalar curvature corresponding to the metric (2.12) is

$$\mathcal{R} \; = \; -8\pi G_N \; 2\rho \; (R(r), \sigma) \, [\, 2 - \frac{R^2(r)}{4\sigma^2} \,] \; = \; - \, 16\pi \; G_N M_o \; \frac{e^{-R(r)^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \, [\, 2 - \frac{R^2(r)}{4\sigma^2} \,]. \tag{2.13a}$$

in the $\sigma \to 0$ limit, the scalar curvature contribution to the action may be evaluated once again by simply inserting

$$\mathcal{R} \to -\frac{2 \; G_N \; M_o \; \delta(r)}{R^2(dR/dr)}. \tag{2.13b}$$

into the action in the same manner described by eq-(1.12).

Therefore, the Euclideanized Einstein-Hilbert action associated with the scalar curvature delta function is obtained after a compactification of the temporal direction along a circle S^1 giving an Euclidean time coordinate interval of $2\pi t_E$ and which is defined in terms of the Hawking temperature T_H (when $\sigma \to 0$) and Boltzman constant k_B as $2\pi t_E = (1/k_B T_H) = 8\pi G_N M_o$. The measure of integration is $4\pi R^2 \; dR \; dt_E$, leading to :

$$S_E = -\frac{1}{16\pi G_N} \int\int \; (\; - \frac{2G_N M_o}{R^2(dR/dr)} \delta(r) \;) \; (4\pi R^2 \; dR \; dt) \; =$$

$$-\frac{1}{16\pi G_N} \int\int \; (\; - \frac{2G_N M_o}{r^2} \delta(r) \;) \; (4\pi r^2 \; dr \; dt) \; =$$

$$\frac{4\pi (G_N M_o)^2}{L^2_{Planck}} \; = \; \frac{4\pi \; (2G_N M_o)^2}{4 \; L^2_{Planck}} = \frac{Area}{4 \; L^2_{Planck}} \; =$$

$$\frac{1}{16\pi G_N} \; lim_{\; \sigma \to 0} \int\int_0^R 16\pi \; G_N M_o \; \frac{e^{-R(r)^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \, [\; 2 - \frac{R^2(r)}{4\sigma^2} \,] \, (4\pi R^2 \; dR \; dt) \; =$$

$$\frac{1}{16\pi G_N} \; \int\int_0^\infty 16\pi \; G_N M_o \; \frac{e^{-R(r)^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} \; [\; 2 \; - \; \frac{R^2(r)}{4\sigma^2} \;] \; (4\pi R^2 \; dR \; dt) \tag{2.14}$$

when equating $G_N = L^2_P$ and after performing the integration in terms of the incomplete gammas and taking the $\sigma \to 0$ limit. Once again, the limit $\sigma \to 0$ has the same effect as if one took the upper R limit of the action integral from $R = finite$ all the way to $R \to \infty$ (as one should to define the action) while keeping σ fixed. It is interesting that the Euclidean action (in $\hbar$ units) *is* precisely the *same* as the black hole entropy in Planck area units. This result holds in any dimensions $D \geq 3$. This is not a numerical coincidence and is deeply related to the *thermal* nature of Euclidean time; namely, the conserved global charges associated to the Euclidean Einstein-Hilbert action obey a relation that could be interpreted as a thermodynamical equation of state. Furthermore, the action is *invariant* of the choices of $R(r)$, whether or not it is the Hilbert text book choice $R(r) = r$ or another. The choice of the radial function $R(r)$ amounts to a *radial gauge* that leaves the *action* invariant but it does *not* leave the scalar curvature, nor the measure of integration, invariant. Only the *action* (integral of the scalar curvature) remains invariant.

The physical picture behind the Euclidean action = entropy relation is the following. One can imagine the point mass delta function source at $r = 0$ as being smeared all over spacetime, from $r = 0$ to $r = \infty$ by a Gaussian mass density distribution of a variable width σ. As long as $\sigma \neq \infty$ the answer (2.14) is the same whether one integrates *all* over space, keeping σ fixed, or if one integrates up to a given r and takes the subsequent $\sigma = 0$ limit, consistent with the delta function distribution being the zero width limit of a Gaussian. In the delta function case for the scalar curvature , we know that $\mathcal{R} = 0, r > 0$ thus the contribution to the action is zero for all $r > 0$; only the delta function behaviour of the scalar curvature at $r = 0$ contributes to the action. A detailed study of the *nonzero* σ modifications of the Hawking temperature, entropy and horizon may be found in [17], [28], [18]. However, as we have seen in the last term of eq-(2.14), when one integrates *all* over space for any fixed value of $\sigma \neq \infty$ one always gets the *same* answer in terms of the mass parameter $S_E = 4\pi G M_o^2$, if, and only if, $k_B T_H = (8\pi G_N M_o)^{-1}$. It is only the relationship between M_o and the modified horizon $r_s = r_s(2G_N M_o, \sigma)$ that changes when $\sigma \neq 0$.

The $action - entropy$ connection has been obtained from a different argument, for example, by Padmanabhan [21] by showing how it is the $surface$ term added to the action which is related to the entropy, interpreting the horizon as a boundary of spacetime. The surface term is given in terms of the trace of the *extrinsic* curvature of the boundary. The surface term in the action is directly related to the observer-dependent-horizon entropy, such that its variation, when the horizon is moved infinitesimally, is equivalent to the change of entropy $d\mathcal{S}$ due to the virtual work. The variational principle is equivalent to the thermodynamic identity $Td\mathcal{S} = dE + pdV$ due to the variation of the matter terms in the right hand side. A bulk and boundary stress energy tensors are required to capture the Hawking thermal radiation flux seen by an asymptotic observer at infinity as the black hole evaporates.

3 Why the use of the modulus $|r|$ is necessary to yield delta function terms

The direct approach in solving Einstein's equation in the presence of a point mass delta function source requires replacing everywhere $r \rightarrow |r|$ in the radial gauge $R(|r|) = |r| + 2GM\ \Theta(|r|)$. If one does not properly use $|r|$ (instead of r) in the metric one will get an *identically zero expression* for the Einstein tensor as in the vacuum case. To illustrate how relevant it is to take the proper absolute values, we recall (in flat space) that the Laplacian in spherical coordinates of $1/|r|$ is

$$\frac{1}{r^2}(d/dr)[\ r^2(d/dr)(1/|r|)\ ] = \frac{1}{r^2}(d/dr)[\ r^2(-1/|r|^2)\ sign(r)\ ] =$$

$$-\frac{1}{r^2}(d/dr)sign(r) = -(1/r^2)\ \delta(r) \tag{3.1}$$

since $r^2 = |r|^2$, which is consistent with Poisson's law which states that the Laplacian of the Newtonian potential $-GM/|r|$ is $4\pi G\rho$. This is true here if, and only if, $\rho = (M/4\pi r^2)\delta(r)$ that is indeed the case in Newtonian gravity. To reiterate once more, the Laplacian in spherical coordinates of $(1/r)$ is *zero*. For this reason, there is a fundamental difference in dealing with expressions involving absolute values $|r|$ like $1/|r|$ from those which depend on r like $1/r$ [11].

Let us try to solve Einstein's equations for a point mass, firstly, by writing the components of $T_{\mu\nu}$ associated with a point mass particle which is moving in its own gravitational background (neglecting the *back reaction* on the particle) in terms of the appropriately defined *covariantized* delta function The worldline of the point mass source is parametrized by the four functions

$$X^0 = t(\tau), \quad X^1 = r(\tau); \quad X^2 = \theta(\tau); \quad X^3 = \phi(\tau) \tag{3.2}$$

The matter action is

$$S_{matter} = -M_o \int d\tau = -M_o \int \sqrt{g_{\mu\nu}(dX^\mu/d\tau)(dX^\nu/d\tau)}\ d\tau =$$

$$-M_o \int \sqrt{g}\ d^n x \int \frac{\delta^n(x^\mu - X^\mu(\tau))}{\sqrt{|g|}} \sqrt{g_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau)}\ d\tau. \tag{3.3}$$

From which we can deduce the expression for the stress energy tensor density

$$T^{\mu\nu} \ = \ -\ 2\frac{\delta S_{matter}}{\delta g_{\mu\nu}} =$$

$$M_o \int \frac{(dx^\mu/d\tau)(dx^\nu/d\tau)}{\sqrt{(dx^\sigma/d\tau)(dx_\sigma/d\tau)}} \frac{1}{\sqrt{|g|}} \delta(r-r(\tau))\, \delta(\theta-\theta(\tau))\, \delta(\phi-\phi(\tau))\delta(t-x^0(\tau))\, d\tau. \tag{3.4}$$

The worldline of an inert point mass (ignoring the back reaction of the gravitational field) at *fixed* values of

$$r = r_o = constant \neq 0; \quad \theta = \theta_o = constant, \quad \phi = \phi_o = constant \tag{3.5a}$$

is determined by the temporal function $x^0 = t = x^0(\tau)$ such that

$$(d\tau)^2 = g_{00}(dt)^2 \ \Rightarrow \int \tau = \int \sqrt{g_{00}}\ dt \Rightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{g_{00}}}$$

$$(dx^0/d\tau)(dx^0/d\tau) = \frac{1}{g_{00}} = g^{00}. \tag{3.5b}$$

For this particular timelike worldline history (on-shell so $(dx^\sigma/d\tau)(dx_\sigma/d\tau) = 1$) the only non-vanishing component of the stress energy tensor is

$$T_{00} = M_o \int \frac{(dx_0/d\tau)^2}{\sqrt{|g|}} \delta(r-r(\tau))\, \delta(\theta-\theta(\tau))\, \delta(\phi-\phi(\tau)) \frac{\delta(t-x^0(\tau))}{\sqrt{(dx^\sigma/d\tau)(dx_\sigma/d\tau)}}\, d\tau =$$

$$T_{00} = M_o \int \frac{g_{00}(\ |\vec{r}-\vec{r}_o|\ )}{\sqrt{|g|}}\ \delta(r-r_o)\ \delta(\theta-\theta_o)\ \delta(\phi-\phi_o)\ \delta(t-x^0(\tau))\ d\tau =$$

$$T_{00} = M_o\ \frac{g_{00}(\ |\vec{r}-\vec{r}_o|\ )}{\sqrt{|g|}}\ \delta(r-r_o)\ \delta(\theta-\theta_o)\ \delta(\phi-\phi_o). \tag{3.6}$$

As expected, we have found that the T_{00} component is just related to the mass density ρ in spherical ccordinates for a point mass source located at $\vec{r}_o = (x_o, y_o, z_o) \neq 0$. If the point mass source is located at the *origin* of the spherical coordinates system $\vec{r}_o = 0$, the Jacobian in this case becomes $\sqrt{|g|} = R^2(dR/dr)\ sin\phi$, and $g_{00}(|\vec{r}-\vec{r}_o|) = g_{00}(|r|)$. However, since the angles are *degenerate* at $r = 0$ (the angles are not well defined at the origin) to cure this ambiguity one can perform the *average* over all solid angle directions (from 0 to 4π) and which furnishes a crucial $(1/4\pi)$ factor that is deeply connected to the ubiquitous $2M$ term, as follows

$$\frac{1}{4\pi} \int T_{00}\ sin(\phi)\ d\phi\ d\theta$$

$$= \frac{M_o}{4\pi} \int \frac{g_{00}(r)}{R^2(dR/dr)\ sin\phi}\ \delta(r)\ \delta(\theta-\theta_o)\ \delta(\phi-\phi_o)\ sin(\phi)\ d\phi\ d\theta =$$

$$< T_{00} >_{solid\ angle} = g_{00}(r)\ \frac{M_o}{4\pi R^2(dR/dr)}\ \delta(r). \tag{3.7}$$

Using the rules of differentiation outlined in eqs-(3.9, 3.10), the Einstein tensor, obtained by replacing $r \longrightarrow |r|$ in the solutions (1.1), has non-vanishing diagonal elements involving a stress energy tensor with both *pressure* and *density* terms proportional to $\delta(r)$ [25]. However, despite this unexpected finding, we found that the *integral* of $8\pi G$ times the *trace* of the stress energy tensor does satisfy the condition [25]

$$\int 8\pi G_N \ trace \ T_{\mu\nu} = \int 8\pi G_N \ g^{\mu\nu} \ T_{\mu\nu} = \int 8\pi G_N \ g^{00} < T_{00} > =$$

$$- \int \mathcal{R} = \int \frac{2G_N M_o}{R^2(r) \ (dR/dr)} \ \delta(r) \ (4\pi R^2) \ dR \ dt = \int \frac{2G_N M_o}{r^2} \ \delta(r) \ (4\pi r^2) \ dr \ dt. \tag{3.8}$$

in accordance to the results involving the integrals in the $\sigma \longrightarrow 0$ limit of (2.7a, 2.7b).

The scalar curvature associated with radial gauges involving the modulus $|r|$ (instead of r) generates the sought after $\delta(r)$ terms only in those expressions involving *second* derivatives of the metric. This is a consequence of

$$\frac{d \ g_{\mu\nu}(|r|)}{dr} = \frac{d \ g_{\mu\nu}(|r|)}{d|r|} \ \frac{d|r|}{dr} = \frac{d \ g_{\mu\nu}(|r|)}{d|r|} \ sign \ (r)$$

$$\Rightarrow \left(\frac{d \ g_{\mu\nu}(|r|)}{dr}\right)^2 = \left(\frac{d \ g_{\mu\nu}(|r|)}{d|r|}\right)^2. \tag{3.9}$$

$$\frac{d^2 \ g_{\mu\nu}(|r|)}{dr^2} = \frac{d^2 \ g_{\mu\nu}(|r|)}{d|r|^2} + \frac{d \ g_{\mu\nu}(|r|)}{d|r|} \ \delta \ (r), \ since \ sign(r)^2 = 1, \ \frac{dsign(r)}{dr} = \delta(r). \tag{3.10}$$

Writing the metric components

$$g_{00} = 1 - \frac{2GM}{|r|} = 1 - \frac{2GM}{r} \ \frac{r}{|r|} = 1 - \frac{2GM}{r} \ f(r); \quad f(r) \ \equiv \ \frac{r}{|r|}. \tag{3.11a}$$

$$g_{rr} = - \ \frac{1}{g_{00}}. \tag{3.11b}$$

such that the derivatives

$$f'(r) = \frac{df(r)}{dr} = \delta(r); \quad f''(r) = \frac{d^2 f(r)}{dr^2} = \delta'(r). \tag{3.12}$$

reveals that the *nonvanishing* $\mathcal{R}$ is given by :

$$\mathcal{R} = - 2GM \ [\ \frac{f''(r)}{r} + 2 \ \frac{f'(r)}{r^2} \ ] =$$

$$- \ 2GM \ [\ \frac{\delta'(r)}{r} \ + \ 2 \ \frac{\delta(r)}{r^2} \ ]. \tag{3.13}$$

the signature chosen is $(+, -, -, -)$.

Therefore, the Einstein-Hilbert action involving both density and pressure terms is exactly equal to an integral involving $2GM\delta(r)/r^2$:

$$S = \ - \frac{1}{16\pi G} \int \mathcal{R} \ 4\pi r^2 \ dr \ dt = \ \frac{1}{16\pi G} \int \ 2GM \ [\ \frac{\delta'(r)}{r} \ + 2 \ \frac{\delta(r)}{r^2} \ ] \ 4\pi r^2 \ dr \ dt. \tag{3.14}$$

Integrating by parts yields

$$\frac{1}{16\pi G} \int \ 8\pi GM \ [\ 2\delta(r) - \delta(r) \ ] \ dr \ dt \ = \frac{1}{16\pi G} \int \ 8\pi G \ (\frac{M \ \delta(r)}{4\pi r^2}) \ 4\pi r^2 \ dr \ dt \ =$$

$$\frac{1}{16\pi G} \int \ 8\pi G \ \rho(r) \ 4\pi r^2 \ dr \ dt \ = \ \frac{1}{2} \int \ M \ dt \ \Rightarrow \ \rho(r) \ \equiv \ \frac{M \ \delta(r)}{4\pi r^2}. \tag{3.15}$$

which is precisely the same result as the integral in eq-(2.10) . Notice that the authors [15] chose a very $different$ function $f(r) = r^\lambda$ than the one chosen above $f(r) = r/|r|$, and in the limit $\lambda \rightarrow 0$, arrived at similar results for the distribution-valued scalar curvature. A different approach based on Colombeau's nonlinear distributional calculus was undertaken by [14].

In showing why the *integral* of the *trace* of Einstein's equations coresponding to both density *and* pressure terms given by (3,13) yields the *same* integral corresponding to the scalar curvature associated to a pure density term $8\pi G \ \rho = 8\pi G \ (M \ \delta(r)/4\pi r^2)$, is basically a similar exercise as *integrating* the Schroedinger equation in the presence of a delta function potential :

$$\int_{-\infty}^{+\infty} \ [\ - \frac{\hbar^2}{2m} \ \frac{d^2 \ \Psi}{dx^2} \ + \ \lambda \ \delta(x) \ \Psi(x) \ ] \ dx \ = \ \int_{-\infty}^{+\infty} \ E \ \Psi \ dx \ \Rightarrow$$

$$\lambda \ \Psi \ (x = 0) \ = \ E \int_{-\infty}^{+\infty} \ \Psi(x) \ dx. \tag{3.16}$$

since the wave function and its derivative is required to obey the boundary conditions $\Psi(x = \pm\infty) = \Psi'(x = \pm\infty) = 0$ and normalized $\int \ |\Psi|^2 = 1$. The solution to (3.16) is $\Psi \ \sim \ e^{-k|x|}$ where $\lambda = 2E/k$. The matter density $\rho(r, \sigma)$ chosen by [17] is just the QM analog of the square $|\Psi|^2$ of a spherically symmetric Gaussian wave function $\Psi(r, \sigma(t))$ in 3-dim. The $\sigma = 0$ limit corresponds to a delta function localized source at $r = 0$.

The black hole horizon can be *displaced* from $r_h = 2GM$ to a location arbitrarily close to $r = 0$ by simply choosing the proper radial function $R = r + 2GM\Theta(r)$ with the correct behavior at $r = 0$ given by $R(r = 0) = 0$.

The novel metric which is *not* diffeomorphic to the metric in (3.11) due to the *discontinuity* at $r = 0$, resulting from the definition of the Heaviside step function $\Theta(r) = 1, r > 0$; $\Theta(r = 0) = 0$ and $\Theta(r) = -1, r < 0$, has for components

$$g_{00} = 1 - \frac{2GM}{|R|} = 1 - \frac{2GM}{R} \frac{R}{|R|} = 1 - \frac{2GM}{R} f(R); \quad f(R) \equiv \frac{R}{|R|}. \quad (3.17a)$$

$$g_{RR} = - \frac{1}{g_{00}} \quad (3.17b)$$

such that the *nonvanishing* $\mathcal{R}$ is given by :

$$\mathcal{R} = -2GM \left[\frac{f''(R)}{R} + 2 \frac{f'(R)}{R^2} \right] =$$

$$-2GM \left[\frac{\delta'(R)}{R} + 2 \frac{\delta(R)}{R^2} \right]. \quad (3.18)$$

The Einstein-Hilbert action involving both density and pressure terms is exactly equal to an integral involving $2GM\delta(R)/R^2$:

$$S = -\frac{1}{16\pi G} \int \mathcal{R}\, 4\pi R^2\, dR\, dt = \frac{1}{16\pi G} \int 2GM \left[\frac{\delta'(R)}{R} + 2 \frac{\delta(R)}{R^2} \right] 4\pi R^2\, dR\, dt. \quad (3.19)$$

Integrating by parts yields

$$\frac{1}{16\pi G} \int 8\pi GM\, [\, 2\delta(R) - \delta(R)\,]\, dR\, dt = \frac{1}{16\pi G} \int 8\pi G \left(\frac{M\ \delta(R)}{4\pi R^2}\right) 4\pi R^2\, dR\, dt =$$

$$\frac{1}{16\pi G} \int 8\pi G\, \rho(R)\, 4\pi R^2\, dR\, dt = \frac{1}{2} \int M\, dt \Rightarrow \rho(R) \equiv \frac{M\ \delta(R)}{4\pi R^2}. \quad (3.20)$$

One learns also that the *integrals* are equal despite that the *integrands* and integration measures are not equal

$$\frac{1}{16\pi G} \int 8\pi G \left(\frac{M\ \delta(R)}{4\pi R^2}\right) 4\pi R^2\, dR\, dt = \frac{1}{16\pi G} \int 8\pi G \left(\frac{M\ \delta(r)}{4\pi r^2}\right) 4\pi r^2\, dr\, dt. \quad (3.21)$$

Only for the Schwarzschild radial gauge $R^3 = r^3 + (2GM)^3$, as a result of the condition $R^2\, dR = r^2\, dr$, and $\delta(R) = \delta(r)/(dR/dr)$ one can see that the *integrands* and measures in (3.21) are also the same. However, the Schwarzschild radial gauge is not correct since it leads to a contradiction due to the fact that it imposes a finite non-zero area condition for the point mass

$A(r=0) = 4\pi R(r=0)^2 = 4\pi(2GM)^2$, while having a *zero* volume simultaneously. The point mass location is the center of spherical symmetry, and as such, cannot have a finite non-zero area. To cure this problem one must choose a radial gauge like $R = r + 2GM\Theta(r)$ to insure that $R(r=0) = 0$ and such that the black hole horizon can be *displaced* from $r_h = 2GM$ to a location arbitrarily close to $r = 0$. The metric is static *everywhere*, contrary to the Hilbert text-book solution that ceases to be static inside the horizon, and is discontinuous only at the location of the point mass singularity $r = 0$ as expected.

Many still argue that the initial assumption of a point mass at $r = 0$ is not physical. This is why we have modeled the delta function by a smeared Gaussian distribution and such that in the zero width limit one recover the same effects of the point mass source. A Gaussian wave function that begins as a delta function source localized at $r = 0$, then it diffuses all over space but the center of the Gaussian remains fixed (static) at $r = 0$ (zero group velocity). The point mass at $r = 0$ behaves as if it were delocalized all over space consistent with the wave-particle duality property in QM. To summarize : solutions invloving the modulus $|r|$ like $g_{\mu\nu}(|r|)$ are not diffeomorphic to those found by Hilbert $g_{\mu\nu}(r)$ and have a clear physical interpretation within the context of wave-particle duality in QM. A Gaussian mass distribution is inherently Quantum Mechanical when the mass density ρ is the square of the wave function.

We conclude this work with some important remarks. Modifications of the standard thermo-dynamical properties of black holes (logarithmic corrections to the black hole entropy) based on the solutions of (2.5) within the context of Noncommutative geometry and stringy uncertainty relations have been studied by several authors. See [28] and references therein. In [26] a derivation of the logarithmic corrections to the entropy was found based on a generalized p-Loop oscillator in Clifford spaces and an upper limiting Planck temperature was obtained where Black Hole evaporation stops at the Planck scale.

In [30] a natural cut-off of the form $R(r=0) = 2G_N M$ was interpreted from the standard Noncommutative spacetime coordinates algebra $[x^\mu, x^\nu] = i\Sigma^{\mu\nu}$, $[p^\mu, p^\nu] = 0$, $[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}$ where $\Sigma^{\mu\nu}$ are c-numbers of $(Planck\ length)^2$ units. A change of coordinates in phase space $x'^\mu = x^\mu + \frac{1}{2}\Sigma^{\mu\nu}\ p_\nu$ leads to commuting coordinates x'^μ and allows to define $r'(r) = \sqrt{(x^i + \frac{1}{2}\ \Sigma^{i\rho}\ p_\rho)\ (x_i + \frac{1}{2}\ \Sigma_{i\tau}\ p^\tau)}$. One can select $\Sigma^{\mu\nu}$ such that $r'(x^i = 0) = r'(r=0) = 2G_N M_o$, upon using $p_\mu p^\mu = M_o^2$ in the static case $p^\mu = (M_o, 0, 0, 0)$ [30] which is precisely the cut-off corresponding to the Abrams-Schwarzschild radial gauge. Noncommutative Finsler gravity (Lagrange-Finsler manifolds) associated with the spacetime tangent bundle and the Hamilton-Cartan geometry of Noncommutative phase spaces, *is* the arena where properly one can study the Noncommutative Gravity of the spacetime tangent (co-tangent) bundle [20].

Since coordinates and velocities (momenta in phase spaces) are treated on equal footing, Lagrange-Finsler (Hamilton-Cartan) geometry is the backdrop where one may achieve a Geometrization of matter and have a spacetime-matter unity. The natural group acting in phase spaces is $U(1,3) = SU(1,3) \times U(1)$ to account for acceleration and boosts transformations. The maximal proper force (acceleration) postulated by Max Born many years ago that a $fundamental$ particle may experience is $m_{Planck}\ c^2/L_{Planck}$. The group $SU(1,3)$ is not the same as the conformal group $SU(2,2)$. Finsler spaces have $torsion$ which is the hallmark of *spin*. An entirely different approach to treat point mass delta function source can be found in [29].

A Planck scale cut-off can be derived in terms of noncommutative Moyal star products $f(x) * g(x)$ simply by replacing $r \to r_* = \sqrt{r * r}$ $= \sqrt{r^2 + \Sigma^{ij} x_i x_j / r^2 + ...}$ so $r_*(x^i = 0) \neq 0$, and receives Planck scale corrections. A point is fuzzy and delocalized, henceforth it has a non-zero fuzzy area and fuzzy volume. A p-Adic norm naturally attains the feature of delocalization since a p-Adic disc has no center. Every point is the center. Notice the difference between the latter Planck scale cut-off in ordinary spacetime with the former mass (momentum) dependent cut-off $2G_N M_o$ in the spacetime tangent (co-tangent) bundle. Yang's Noncommutative algebra in phase space, has both an ultraviolet and infrared cut-off related to the minimal (Planck) and maximal (Hubble) scale. A Moyal-Fedosov-Kontsevich star products deformations of p-branes were constructed in [22] based on Yang's algebra. Other relevant work can be found in [27], [28], [29], [30], [31], [32], [33], [34],[36], [37].

4 Appendix A: Schwarzschild-like solutions in any dimension $D > 3$

In this Appendix we follow closely our prior calculations [32]. Let us start with the line element

$$ds^2 = -e^{\mu(r)}(dt_1)^2 + e^{\nu(r)}(dr)^2 + R^2(r)\tilde{g}_{ij} d\xi^i d\xi^j. \qquad (A.1)$$

Here, the metric $\tilde{g}_{ij}$ corresponds to a homogeneous space and $i, j = 3, 4, ..., D-2$ and the temporal and radial indices are denoted by $1, 2$ respectively. In our text we denoted the temporal index by 0. The only non-vanishing Christoffel symbols are

$$\begin{array}{lll} \Gamma^1_{21} = \frac{1}{2}\mu', & \Gamma^2_{22} = \frac{1}{2}\nu', & \Gamma^2_{11} = \frac{1}{2}\mu' e^{\mu-\nu}, \\ \Gamma^2_{ij} = -e^{-\nu} R R' \tilde{g}_{ij}, & \Gamma^i_{2j} = \frac{R'}{R}\delta^i_j, & \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}, \end{array} \qquad (A.2)$$

and the only nonvanishing Riemann tensor are

$$\mathcal{R}^1_{212} = -\tfrac{1}{2}\mu'' - \tfrac{1}{4}\mu'^2 + \tfrac{1}{4}\nu'\mu', \qquad \mathcal{R}^1_{i1j} = -\tfrac{1}{2}\mu' e^{-\nu} RR'\tilde{g}_{ij},$$

$$\mathcal{R}^2_{121} = e^{\mu-\nu}(\tfrac{1}{2}\mu'' + \tfrac{1}{4}\mu'^2 - \tfrac{1}{4}\nu'\mu'), \qquad \mathcal{R}^2_{i2j} = e^{-\nu}(\tfrac{1}{2}\nu' RR' - RR'')\tilde{g}_{ij},$$

$$\mathcal{R}^i_{jkl} = \tilde{R}^i_{jkl} - R'^2 e^{-\nu}(\delta^i_k \tilde{g}_{jl} - \delta^i_l \tilde{g}_{jk}). \tag{A.3}$$

The field equations are

$$\mathcal{R}_{11} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\mu'\nu' + \frac{(D-2)}{2}\mu'\frac{R'}{R}) = 0, \tag{A.4}$$

$$\mathcal{R}_{22} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu' + (D-2)(\frac{1}{2}\nu'\frac{R'}{R} - \frac{R''}{R}) = 0, \tag{A.5}$$

and

$$\mathcal{R}_{ij} = \frac{e^{-\nu}}{R^2}(\frac{1}{2}(\nu' - \mu')RR' - RR'' - (D-3)R'^2)\tilde{g}_{ij} + \frac{k}{R^2}(D-3)\tilde{g}_{ij} = 0, \tag{A.6}$$

where $k = \pm 1$, depending if $\tilde{g}_{ij}$ refers to positive or negative curvature. From the combination $e^{-\mu+\nu}\mathcal{R}_{11} + \mathcal{R}_{22} = 0$ we get

$$\mu' + \nu' = \frac{2R''}{R'}. \tag{A.7}$$

The solution of this equation is

$$\mu + \nu = \ln R'^2 + a, \tag{A.8}$$

where a is a constant.

Substituting (A.7) into the equation (A.6) we find

$$e^{-\nu} \; (\; \nu' RR' - 2RR'' - (D-3)R'^2 \;) \;\; = \;\; - k(D-3) \tag{A.9}$$

or

$$\gamma' RR' + 2\gamma RR'' + (D-3)\gamma R'^2 = k(D-3), \tag{A.10}$$

where

$$\gamma = e^{-\nu}. \tag{A.11}$$

The solution of (A.10) for an ordinary D-dim spacetime (one temporal dimension) corresponding to a $D-2$-dim sphere for the homogeneous space can be written as

$$\gamma = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2}R^{D-3}}) \; (\frac{dR}{dr})^{-2} \Rightarrow$$

$$g_{rr} = e^{\nu} = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} R^{D-3}})^{-1} \; (\frac{dR}{dr})^2. \qquad (A.12)$$

where Ω_{D-2} is the appropriate solid angle in $D-2$-dim and G_D is the D-dim gravitational constant whose units are $(length)^{D-2}$. Thus $G_D M$ has units of $(length)^{D-3}$ as it should. When $D = 4$ as a result that the 2-dim solid angle is $\Omega_2 = 4\pi$ one recovers from eq-(A.12) the 4-dim Schwarzchild solution. The solution in eq-(A.12) is consistent with Gauss law and Poisson's equation in $D-1$ spatial dimensions obtained in the Newtonian limit.

For the most general case of the $D-2$-dim homogeneous space we should write

$$-\nu = \ln(k - \frac{\beta_D G_D M}{R^{D-3}}) - 2\ln R'. \qquad (A.13)$$

where β_D is a constant. Thus, according to (A.8) we get

$$\mu = \ln(k - \frac{\beta_D G_D M}{R^{D-3}}) \; + \; constant. \qquad (A.14)$$

we can set the constant to zero, and this means the line element (A.1) can be written as

$$ds^2 = -(k - \frac{\beta_D G_D M}{R^{D-3}})(dt_1)^2 \; + \; \frac{(dR/dr)^2}{(k - \frac{\beta_D G_D M}{R^{D-3}})}(dr)^2 \; + \; R^2(r)\tilde{g}_{ij} d\xi^i d\xi^j. \qquad (A.15)$$

One can verify, taking for instance (A.5), that the equations (A.4)-(A.6) do *not* determine the form $R(r)$. It is also interesting to observe that the only effect of the homogeneous metric $\tilde{g}_{ij}$ is reflected in the $k = \pm 1$ parameter, associated with a positive (negative) constant scalar curvature of the homogeneous $D-2$-dim space. $k = 0$ corresponds to a spatially flat $D-2$-dim section.

The stress energy tensor for a point mass source is given explicitly by the zero-width limit of the Gaussian in the right hand side of eqs-(2.1, 2.2), as shown explicitly in eqs-(2.4), (2.9) and (2.10). Let us now verify that the line element (2.5) is a solution of Einstein's equations (2.1) in the presence of a mass distribution density $\rho(r)$. The temporal components of (2.1) yield

$$\mathcal{R}_{00} \; - \frac{1}{2} \; g_{00} \; \mathcal{R} \; = \; e^{\mu-\nu} \; [\; \frac{\nu'}{r} \; - \frac{1}{r^2} \;] \; + \; \frac{e^{\mu}}{r^2}. \qquad (A.16)$$

Defining the new solutions corresponding to the mass distribution $M(r,\sigma)$ by

$$e^{\mu} \; = \; 1 - \frac{2 \; G_N \; M(r,\sigma)}{r}; \quad e^{\nu} = e^{-\mu}; \quad \mu \; = \; -\nu \; = \; log \; (1 - \frac{2 \; G_N \; M(r,\sigma)}{r}). \qquad (A.17)$$

inserting eqs-(A.16, A.17) into the temporal components of eq-(2.1), and after *factoring* out the metric component $g_{00}(r) = e^{\mu}$, it becomes

$$\frac{2G_N}{r^2}\left(\frac{d\ M(r,\sigma)}{dr}\right) = \frac{2G_N}{r^2}\ 4\pi\ r^2\ \frac{M_o\ e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}} = 8\pi\ G_N\ \rho(r) = -8\pi G_N\ T_{00}. \tag{A.18}$$

as expected. Notice that the sign change in (A.18) compared to eq-(2.1) is due to the choice of signature $(-,+,+,+)$ in this appendix. Similarly, one can verify that

$$\mathcal{R}_{ij} - \frac{1}{2}\ g_{ij}\ \mathcal{R} = -\ 8\pi G_N\ T_{ij}. \tag{A.19}$$

by solving the covariant conservation equation of the stress energy tensor [17]

$$\nabla_\nu\ T^{\mu\nu} = 0 \Rightarrow \partial_r\ T^r_r = -\frac{1}{2}g^{00}(\partial_r g_{00})\ (T^r_r - T^0_0) - g^{\theta\theta}(\partial_r g_{\theta\theta})\ (T^r_r - T^\theta_\theta). \tag{A.20}$$

for

$$T^\mu_\nu = diagonal\,(-\rho,\ p_r,\ p_\theta,\ p_\phi),\quad p_r = -\rho,\ p_\theta = p_\phi = -\rho - \frac{r}{2}\partial_r\rho = -\rho\,(1 - \frac{r^2}{4\sigma^2}). \tag{A.21}$$

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