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Comment and Improved Results on "Mathematical Analysis of Typhoid Model with Saturated Incidence Rate"

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Abstract

This paper points out some technical problems in the theorems and their proofs of [2]. Furthermore, a corrected stability criterion is also presented.

Keywords: Jacobian matrix; eigenvalues

1 Introduction

Stability analysis of a mathematical model describing the dynamics of a problem in biology requires a knowledge of the eigenvalues of the Jacobian matrix associated with the matrix [1]. The Routh-Hurwitz Criteria [4] give necessary and sufficient condition for the eigenvalues to lie in the left half of the complex plane. However, recent literature in mathematical biology contains instances of authors establishing stability of the Jacobian matrix by using erroneous results concerning eigenvalues of a matrix. Some of the results are stated below.

1. *Eigenvalues of a matrix are invariant under elementary row [or column] operations.*

2. *If det $A > 0$ and trace $A < 0$, then the eigenvalues of A are all negative.*

   See, for example, Altaf et al. [2].

The purpose of the present note is to caution against such pitfalls into which an unwary researcher may fall.

2 Falseness of the above statements

The example occurs in the proof of the following theorems of Altaf et al. [2].

The Theorems states:

1. *If $R_0 < 1$, then the (DFE) of the system (5) at E_0 is locally asymptotically stable.*

2. *If $R_0 > 1$, then the endemic equilibrium E_1 of the system (5) is locally asymptotically stable.*

In order to prove these results, they performed elementary row operation for the Jacobian matrix J_0 (17) (similarly, by elementary row operation for the Jacobian matrix J_* (23)). Then, they analyse the eigenvalues of the matrix obtained, after elementary row transformation, from (17) and (23) to show all eigenvalues are negative from which they conclude the above assertions of their theorem. This, argument, unfortunately is incorrect as the following example shows.

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Matrix B has been obtained from A by adding the first row to the second. Now, eigenvalues of A are $\lambda_1 = \frac{1}{2}(3 + \sqrt{5})$ and $\lambda_2 = \frac{1}{2}(3 - \sqrt{5})$, whereas B has repeated eigenvalues $\lambda_{1,2} = 1$. Clearly, the eigenvalues may change after an elementary row transformation. But the determinant remains. Here the determinant of both A and B is unity. The above statement may hold in special cases but is false in general [3].

Finally, it should be pointed out that, from $detJ^* > 0$ and $traceJ^* < 0$, it can not be concluded that all the eigenvalues of J^* are negative as the following example shows. It leads to a misleading statement in Theorem 4.1 in [2].

$$C = \begin{pmatrix} 0 & 0 & -2 \\ 3 & -2 & -3 \\ 6 & -6 & 1 \end{pmatrix}.$$

Now, eigenvalues of C are $\lambda_1 = -2 < 0$, $\lambda_2 = -2 < 0$ and $\lambda_3 = 3 > 0$. Clearly, $detC = 12 > 0$ and $traceC = -1 < 0$. Hence, from $detC > 0$ and $traceC < 0$, we can not conclude that all the eigenvalues of C are negative.

3 Correction

Now, we present the corrected results.

Theorem 3.1. *If $R_0 < 1$, then the (DFE) of the system (5) in [2] at E_0 is locally asymptotically stable.*

Proof: The characteristic equation of the Jacobian matrix (17) in [2] is given by

$$(\lambda + d)(\lambda^2 + a\lambda + b) = 0, \tag{1}$$

where,

$$\begin{aligned} a &= Q_1 - (1-q)\beta(1-k)\tfrac{\pi}{d} + Q_2, \\ b &= Q_1 Q_2 (1 - R_0). \end{aligned} \tag{2}$$

One of the roots of the characteristic equation (1) $\lambda_1 = -d$, has negative real part. The other two roots can be determined from the quadratic term in (1). Since, R_0 is sum of two positive terms, $R_0 < 1$ implies both $\frac{\eta q\beta(1-q)\pi}{dQ_1Q_2}$ and $\frac{(1-q)\beta(1-k)\pi}{dQ_2}$ are less than unity. From $\frac{(1-q)\beta(1-k)\pi}{dQ_2} < 1$, we can easily see that $a > 0$. Thus, $a > 0$ and $b > 0$ if $R_0 < 1$. Therefore, by Routh Hurwitz criteria, the DFE of the system (5) in [2] is locally asymptotically stable about the point E_0 if $R < 1$.

For the following theorem, we only sketch the proof.

Theorem 3.2. *If $R_0 > 1$, then the endemic equilibrium E_1 of the system (5) is locally asymptotically stable.*

Proof: The characteristic equation of the Jacobian matrix (23) in [2] has the following form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0. \tag{3}$$

It can be easily seen, after a laborious calculations, that $a_1a_2 - a_3 > 0$ if $R_0 > 1$. Thus, by Routh Hurwitz criteria, the endemic equilibrium E_1 of the system (5) in [2] is locally asymptotically stable if $R > 1$.

4 Conclusion

This paper has pointed out some technical problems in the results in [2], and has presented the corrected version of the corresponding result. Studies of mathematical models of the spread of Typhoid have great impact on health authorities' planning and allocation of funds to control the spread of the disease. The effective control decisions of the disease have an important role in the combat of the disease and will be very useful for the public as well as the funding agencies. However, such resources are likely to go waste if scientific studies which purport to guide them are based on faulty theoretical basis. The conclusion based on the model propose by Altaf et al. [2] may not be valid and Typhoid may still be far from reaching its equilibrium from the community.

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