Pulse Vaccination Strategy in an Epidemic Model with Time Delays and Nonlinear Incidence

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Abstract

In this paper, we formulate an SEIRS epidemic model with two time delays and pulse vaccination. By establishing stroboscopic map, we obtain the exact infection free periodic solution of the impulsive epidemic system. According to the comparison arguments, we obtain the sufficient conditions of global attractivity of the infection free periodic solution. Moreover, our results show that a large vaccination or a short pulse of vaccination or a long latent period or a long immune period will lead to the extinction of the disease. The permanence of the model is also discussed.

Keywords: SEIRS epidemic model, Pulse vaccination, Nonlinear incidence, Time delay, Permanence

1 Introduction

Controlling infectious diseases has been an increasingly complex issue for every countries in recent years. Many scholars have investigated and studied lots of epidemic models of ordinary differential equations. The main aim of studying epidemic models is to help improve our understanding of the global dynamics of the spread of infectious diseases. By using the compartmental approach, the dynamics of the SIR and SIRS epidemic model have been extensively analyzed, where $S(t), I(t)$ and $R(t)$ denote the number of the susceptible, infective and recovered at time $t$, respectively. However, many diseases incubate inside the hosts for a period of time before the hosts become infectious. The resulting models are of SEIR or SEIRS types, respectively, depending on whether the acquired immunity is permanent or otherwise, where $E(t)$ denote the number of exposed at time $t$. Moreover, considering some factors of the spread of

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infectious diseases, many authors in their literatures [Song et al., 2009; Jiang et al., 2009] have investigated some new form of the models. The basic and important research subjects for these models are the existence of the threshold value which determines the dynamics of the population sizes in the cases when the disease dies out and when it is endemic, the local and global stability of the disease-free equilibrium and the endemic equilibrium, the existence of periodic solutions, the persistence and extinction of the disease.

It is well-known that one of strategies to control infectious diseases is vaccination. Pulse vaccination strategy (PVS), the repeated application of vaccine over a defined age range is gaining prominence as a strategy for the elimination of childhood viral infectious such as measles hepatitis, parotitis, smallpox and phthisis [Agur et al., 1993; Ramsay et al., 1994]. Pulse vaccination strategy distinguishes from the traditional constant vaccination. At each vaccination time, a constant fraction of susceptible people is vaccinated. And all the vaccine doses are applied in a very short time with respect to the dynamics of the target disease. Pulse vaccination is an effective way to control the transmission of diseases and was considered in the literatures [Gao et al., 2005; Shulgin et al., 1998; Stone et al., 2000].

The incidence of a disease is the number of new cases per unit time. It plays an important role in the study of mathematical epidemiology. In many epidemic models, bilinear incidence $\beta SI$ and standard incidence $\beta SI/N$ are used, where $\beta$ is the probability of transmission per contact, $N(t)$ denote the number of the total population at time $t$. Bilinear incidence bases on the law of mass action. But as the number of available partners is large, it is unreasonable to consider the bilinear incidence rate because the number of available partners with every infective contact per unit time is limited. Hence, contact rate is assumed as a constant. Standard incidence may be more reasonable in some cases. Between the two previous incidences, if we consider the crowding of infective individuals or the protection measures by the susceptible individuals there is a saturation incidence which is more suitable for our real world. This is important because the number of effective contacts between infective individuals and susceptible individuals may saturate at high infective levels. Capasso and Serio in the literature [Capasso et al., 1978] introduced a saturated incidence rate $g(I)S$ into the epidemic model after studying the cholera epidemic spread in Bari in 1973, which describes the contact between infective individuals and susceptible individuals, where $g(I) = kI/(1+\alpha I)$, $\alpha$ is nonnegative constant, $k$ is the probability of transmission per contact, $kI$ measures the infection force of the disease and $1/(1+\alpha I)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. When the number of infective individuals increase $g(I)$ tends to a saturation level. Later Liu et al. in the literature [Liu et al., 1986] proposed the general incidence
rate \( kI^2S/(1 + \alpha I^2) \), where \( l \) and \( h \) are nonnegative constants. This kind of incidence rate includes the behavioral change and crowding effect of the infective individuals. It was used by many authors, see, for example, [Alexander et al., 2004; Derrick et al., 1993; Hethcote, 2000; Hethcote et al., 1991; Hethcote et al., 1989], etc. In the literature [Ruan et al., 2003], Ruan and Wang studied the global dynamics of an epidemic model with nonlinear incidence rate of saturated mass action, which is \( kI^2S/(1 + \alpha I^2) \). They obtained many dynamical behaviors of the model such as a limit cycle, two limit cycles and homoclinic loop, etc. And in the literature [Xiao et al., 2007] and [Cai et al., 2009], Xiao et al. analysed the epidemic models with nonmonotone incidence rate, which is \( \beta IS/(1 + \alpha I^2) \) and it can be used to interpret “the psychological effects”. Because in the initial stage the number of the infective is small people may ignore the epidemic, lots of effective contacts between the infective individuals and susceptable. And when the number of the infective is getting larger and larger, \( g(I) \) is decreasing since many protection measures could be taken by the susceptable individuals.

Generally, death during a latent period and temporary immunity period should be considered in modeling, which is called the phenomena of “time delay”. Therefore, many authors in their literatures [Beretta et al., 1995; Cooke et al., 1996; Mukherjee et al., 1996; Wang, 2002] have studied epidemic models with time delay. But epidemic models with time delays and pulse vaccination are seldom studied. In the present paper, we formulate and analyze an SEIRS disease transmission model with pulse vaccination and two time delays.

The organization of this paper is following. In the next section, we formulate the SEIRS epidemic model with pulse vaccination and time delays. In Section 3, the global attractivity conditions of the disease-free periodic solution is presented. The sufficient conditions for the permanence of the SEIRS model is obtained in Section 4. In Section 5, we give some discussion. The main purpose of this paper is to obtain sufficient conditions that the disease dies out. The second purpose of this paper is to study the role of incubation in disease transmission.

2 Model Formulation and Preliminary

In this section, we firstly formulate an SEIRS epidemic model with pulse vaccination and time delays. Then we introduce some lemmas which are useful in proving the global attractivity of infection free periodic solution and the permanence of the system.

Comparing with bilinear and standard incidence, saturation incidence may be more suitable for our real world. Zhang and Teng [Zhang et al., 2008] introduced and studied the following pulse vaccination delayed SEIRS epidemic
model with saturation incidence

\[
\begin{align*}
S'(t) &= b - bS(t) - \frac{\beta S(t)I(t)}{1 + mS(t)} + \delta R(t) \\
E'(t) &= \frac{\beta S(t)I(t)}{1 + mS(t)} - \frac{\beta S(t - \omega)I(t - \omega)}{1 + mS(t - \omega)}e^{-b\omega} - bE(t) \\
I'(t) &= \frac{\beta S(t - \omega)I(t - \omega)}{1 + mS(t - \omega)}e^{-b\omega} - (b + \alpha + \gamma)I(t) \\
R'(t) &= \gamma I(t) - \delta \frac{I(t)}{1 + \alpha I(t)} - \delta \frac{I(t - \omega)}{1 + \alpha I(t - \omega)}e^{-\delta \omega} - \delta R(t) \\
S(t^+) &= (1 - \theta)S(t) \\
E(t^+) &= E(t) \\
I(t^+) &= I(t) \\
R(t^+) &= R(t) + \theta S(t) \\
\end{align*}
\]

They investigated the existence of the disease-free periodic solution and obtained its exact expression. Further, they established the sufficient conditions of global attractivity of the disease-free periodic solution and the permanence of disease.

In the model (1), they only considered the latent period of the disease (\(\omega\)). The form of incidence is \(\beta S(t)I(t)/(1 + mS(t))\), which is saturated with the susceptible. However, many epidemic diseases have the latent period. In this paper, we formulate an pulse vaccination SEIRS epidemic model with two time delays which includes the latent period and the latent period. And the incidence rate in our model is \(kI^hS/(1 + \alpha I^h)\). We study the following model:

\[
\begin{align*}
S'(t) &= b - \delta S(t) - \frac{kS(t)I(t)}{1 + \alpha I^h(t)} + \gamma I(t - \omega)e^{-\delta \omega} \\
E'(t) &= \frac{kS(t)I(t)}{1 + \alpha I^h(t)} - \frac{kS(t - \tau)I(t - \tau)}{1 + \alpha I^h(t - \tau)}e^{-\delta \tau} - \delta E(t) \\
I'(t) &= \frac{kS(t - \tau)I(t - \tau)}{1 + \alpha I^h(t - \tau)}e^{-\delta \tau} - (\delta + \gamma)I(t) \\
R'(t) &= \gamma I(t) - \delta \frac{I(t)}{1 + \alpha I(t)} - \delta \frac{I(t - \omega)}{1 + \alpha I(t - \omega)}e^{-\delta \omega} - \delta R(t) \\
S(t^+) &= (1 - \theta)S(t) \\
E(t^+) &= E(t) \\
I(t^+) &= I(t) \\
R(t^+) &= R(t) + \theta S(t) \\
\end{align*}
\]

where parameters \(b, \delta, \gamma, \omega, \alpha, \) and \(\tau\) are positive constants, \(b\) is the recruitment rate of susceptible individuals, \(\delta\) represents the natural death rate of the population, \(\gamma\) is the recovery rate from the infected individuals, \(\omega\) is the immune period of the population, \(\tau\) is the latent period of the disease.

In this paper, we will focus on the case when \(l = 1\). Since the first and third equations of system (2) do not include the variables \(E(t)\) and \(R(t)\) the
dynamics of system (2) is determined by the following subsystem:

\[
\begin{aligned}
S'(t) &= b - \delta S(t) - \frac{kS(t)I(t)}{1 + \alpha I_h(t)} + \gamma I(t - \omega)e^{-\delta \omega} \\
I'(t) &= \frac{kS(t - \tau)I(t - \tau)}{1 + \alpha I_h(t - \tau)}e^{-\delta \tau} - (\delta + \gamma)I(t) \\
S(t^+) &= (1 - \theta)S(t) \\
I(t^+) &= I(t)
\end{aligned}
\]  
\quad t \neq nT, n \in N, \quad nT, n \in N, \quad (3)

Set \( j = \max\{\tau, \omega\} \). The initial conditions of (3) is given as

\[
S(\vartheta) = \phi_1(\vartheta), I(\vartheta) = \phi_2(\vartheta), \vartheta \in [-j, 0], \phi_i(0) > 0, i = 1, 2 \quad (4)
\]

where \( \phi = (\phi_1, \phi_2)^T \in C \) and \( C \) is the space of all piecewise continuous functions \( \phi : [-j, 0] \to R^2_+ \), \( R^2_+ = \{ X \in R^2 : X \geq 0 \} \). \( \phi \) has the first kind of discontinuity points at \(-nT(n \in N)\) which are continuous from the left, i.e., \( \phi(-nT^-) = \phi(-nT) \). \( C \) is the Banach space with uniform norm

\[
\|\phi\| = \sup_{\vartheta \in [-j, 0]} \{|\phi_1|, |\phi_2|\}.
\]

The solution of system (3) is a piecewise continuous function \( Y : R_+ \to R^2_+ \), \( Y(t) \) is continuous on \((nT, (n + 1)T]\) and \( Y(nT^+) = \lim_{t \to nT^+} Y(t) \) exists.

Let \( N(t) = S(t) + E(t) + I(t) + R(t) \). By adding the equations in model (2), we obtain

\[
N'(t) = b - \delta N(t). \quad (5)
\]

So the total population size may vary in time and \( N(t) \) is continuous on \( t \in [0, +\infty) \). From (5), we have \( \lim_{t \to \infty} \sup N(t) \leq \frac{b}{\delta} \). Since \( S'(t)|_{S=0} > 0 \) and \( I'(t)|_{I=0} = 0 \), for \( t \neq nT, n \in N \). Moreover, \( S(nT^+) = (1 - \theta)S(nT), I(nT^+) = I(nT) \) for \( n \in N \). Hence, we obtain the following lemma.

**Lemma 2.1** \( \Omega \) is a positive invariant set of system (3), where \( \Omega = \{(S, I) \in R^2_+ : 0 \leq S + I \leq \mu\} \) and \( \mu = \frac{b}{\delta} + \epsilon, \epsilon > 0 \) is sufficiently small.

**Lemma 2.2** [Gao et al., 2006] Consider the following impulsive differential equation

\[
\begin{aligned}
u'(t) &= a - bu(t), \quad t \neq nT, \\
u(t^+) &= (1 - \theta)u(t), \quad t = nT,
\end{aligned}
\]  
\quad (6)

where \( a > 0, b > 0, 0 < \theta < 1 \). Then there exists a unique positive periodic solution of system (6)

\[
\tilde{u}_e(t) = \frac{a}{b} + (u^* - \frac{a}{b})e^{-bt}e^{-nT}, \quad nT < t \leq (n + 1)T
\]
which is globally asymptotically stable, where 

\[ u^* = \frac{a (1 - \theta)(1 - e^{-bT})}{b 1 - (1 - \theta)e^{-bT}}. \]

**Lemma 2.3** [Kuang, 1993; Xiao et al., 2001] Consider the following delay differential equation:

\[ x'(t) = a_1 x(t - \tau) - a_2 x(t) \]

where \( a_1, a_2 \) and \( \tau \) are all positive constants and \( x(t) > 0 \) for \( t \in [-\tau, 0] \). We have:

(i) If \( a_1 < a_2 \), then \( \lim_{t \to \infty} x(t) = 0 \);

(ii) If \( a_1 > a_2 \), then \( \lim_{t \to \infty} x(t) = +\infty \).

### 3 Global attractivity

In this section, we discuss the global attractivity of infection free periodic solution. The technique of the proofs is to use comparison arguments. Firstly, we analyze the existence of the infection free periodic solution of system (3), in which infective individuals are entirely absent from the population permanently, i.e., \( I(t) = 0 \) for all \( t \geq 0 \). So we know that the growth of the susceptible in the time-interval \( nT < t \leq (n+1)T \) must satisfy

\[
\begin{cases}
S'(t) = b - \delta S(t), & t \neq nT, \\
S(t^+) = (1 - \theta)S(t), & t = nT.
\end{cases}
\]  

(7)

By Lemma 2.2, we derive the periodic solution of system (7)

\[ \tilde{S}_e(t) = \frac{b}{\delta} \left(1 - \frac{\theta}{1 - (1 - \theta)e^{-\delta T}}e^{-\delta(t-nT)} \right), \quad nT < t \leq (n + 1)T \]

which is globally asymptotically stable. Therefore, system (3) has a unique infection free periodic solution \((\tilde{S}_e(t), 0)\).

Now we give the conditions which assure the global attractivity of the infection free periodic solution \((\tilde{S}_e(t), 0)\).

**Theorem 3.1** If \( R^* < 1 \), then the infection free periodic solution \((\tilde{S}_e(t), 0)\) of system (3) is global attractive on \( \Omega \), where

\[ R^* = \frac{ke^{-\delta T}(b + \mu \gamma e^{-\delta \omega})(1 - e^{-\delta T})}{\delta(\delta + \gamma)(1 - (1 - \theta)e^{-\delta T})}. \]

**Proof.** Since \( R^* < 1 \), we can choose \( \epsilon_0 > 0 \) sufficiently small such that

\[ ke^{-\delta T} \xi < \delta + \gamma, \]  

(8)
where \( \xi = \frac{(b + \mu \gamma e^{-\delta \omega})(1 - e^{-\delta T})}{\delta(1 - (1 - \theta)e^{-\delta T})} + \epsilon_0. \)

We derive from the first equation of system (3) that

\[ S'(t) < (b + \mu \gamma e^{-\delta \omega}) - \delta S(t). \]

So we establish the following comparison impulsive system

\[
\begin{cases}
  u'(t) = (b + \mu \gamma e^{-\delta \omega}) - \delta u(t), & t \neq nT, \\
  u(t^+) = (1 - \theta)u(t), & t = nT.
\end{cases}
\]

In view of Lemma 2.2, we obtain the unique periodic solution of system (9)

\[ \tilde{u}_e(t) = \frac{b + \mu \gamma e^{-\delta \omega}}{\delta} \left(1 - \frac{\theta}{1 - (1 - \theta)e^{-\delta T}} e^{-\delta(t-nT)}\right), \ nT < t \leq (n+1)T, \]

which is globally asymptotically stable.

Let \((S(t), I(t))\) be the solution of system (3) and \(S(\vartheta) = \phi_1(\vartheta), \vartheta \in [-j, 0]\), \(u(t)\) be the solution of system (9) with initial condition \(u(\vartheta) = \phi_1(\vartheta), \vartheta \in [-j, 0]\). By the comparison theorem of impulsive differential equations [Bainov et al., 1993; Lakshmikantham et al., 1989], there exists an integer \(n_1 > 0\) such that

\[ S(t) \leq u(t) < \tilde{u}_e(t) + \epsilon_0, \ nT < t \leq (n+1)T, \ n > n_1, \]

which yields

\[ S(t) < \tilde{u}_e(t) + \epsilon_0 \leq \frac{(b + \mu \gamma e^{-\delta \omega})(1 - e^{-\delta T})}{\delta(1 - (1 - \theta)e^{-\delta T})} + \epsilon_0 \Delta = \xi, \ nT < t \leq (n+1)T, \ n > n_1. \]

It follows from the second equation of system (3) that

\[ I'(t) < ke^{-\delta \tau} \xi I(t - \tau) - (\delta + \gamma)I(t) \quad t > nT + \tau, \ n > n_1. \]

Then we consider the following auxiliary system

\[ v'(t) = ke^{-\delta \tau} \xi v(t - \tau) - (\delta + \gamma)v(t), \quad (10) \]

By Lemma 2.3, we have

\[ \lim_{t \to \infty} v(t) = 0. \]

According to the comparison theorem and nonnegative of \(I(t)\), we obtain that

\[ \lim_{t \to \infty} I(t) = 0. \quad (11) \]

Hence, for \(\epsilon_0 > 0\) sufficiently small there is an integer \(n_2 > n_1\) (where \(n_2T > n_1T + \tau\)) such that if \(t > n_2T, I(t) < \epsilon_0\).
From the first equation of system (3), we have that for $t > n_2T + \omega$

$$S'(t) = b - \delta S(t) - \frac{kS(t)I(t)}{1 + I^0(T_i(t))} + \gamma(t - \omega)e^{-\delta\omega} \leq b - (\delta + k\epsilon_0)S(t)$$

and

$$S'(t) < (b + \gamma e^{-\delta\omega}\epsilon_0) - \delta S(t).$$

Then, we consider the following two comparison impulsive differential equations for $t > n_2T + \omega$ and $n > n_2$,

$$\begin{cases}
  u_1'(t) = b - (\delta + k\epsilon_0)u_1(t), & t \neq nT, \\
  u_1(t^+) = (1 - \theta)u_1(t), & t = nT,
\end{cases} \quad (12)$$

and

$$\begin{cases}
  u_2'(t) = (b + \gamma e^{-\delta\omega}\epsilon_0) - \delta u_2(t), & t \neq nT, \\
  u_2(t^+) = (1 - \theta)u_2(t), & t = nT.
\end{cases} \quad (13)$$

In view of Lemma 2.2, we obtain that the unique periodic solutions of system (12)

$$\tilde{u}_1(t) = \frac{b}{k\epsilon_0 + \delta} \left( 1 - \frac{\theta}{1 - (1 - \theta) e^{-(k\epsilon_0 + \delta)T} e^{-(k\epsilon_0 + \delta)(t - nT)}} \right), \quad nT < t \leq (n+1)T,$$

and the unique periodic solutions of system (13)

$$\tilde{u}_2(t) = \frac{b + \gamma e^{-\delta\omega}\epsilon_0}{\delta} \left( 1 - \frac{\theta}{1 - (1 - \theta) e^{-\delta T} e^{-\delta(t - nT)}} \right), \quad nT < t \leq (n+1)T,$$

respectively. And they are globally asymptotically stable.

According to the comparison theorem of impulsive differential equations, there exists an integer $n_3 > n_2$ such that $n_3 > n_2 + \omega$ and

$$\tilde{u}_1(t) - \epsilon_0 < S(t) < \tilde{u}_2(t) + \epsilon_0, \quad nT < t \leq (n + 1)T, n > n_3. \quad (14)$$

Because $\epsilon_0$ can be arbitrarily small, from (14) we obtain that

$$\lim_{t \to \infty} S(t) = \tilde{S}_e(t) \quad (15)$$

It follows from (11) and (15) that the infection free periodic solution $(\tilde{S}_e(t), 0)$ of system (3) is globally attractive. The completes the proof.

According to Theorem 3.1, we can easily obtain the following results.

**Corollary 3.1** If

$$\frac{ke^{-\delta\tau}(b + \mu \gamma e^{-\delta\omega})}{\delta} \leq \delta + \gamma,$$

then the infection free periodic
solution \((\tilde{S}_e(t), 0)\) of system (2.2) is globally attractive.

**Corollary 3.2** Assume that \(\frac{ke^{-\delta r}(b + \mu \gamma e^{-\delta \omega})}{\delta} > \delta + \gamma\). Then the infection free periodic solution \((\tilde{S}_e(t), 0)\) of system (2.2) is globally attractive provided that \(\theta > \theta^*\) or \(T < T^*\) or \(\tau > \tau^*\) where

\[
\theta^* = \left(\frac{ke^{-\delta r}(b + \mu \gamma e^{-\delta \omega})}{\delta(\delta + \gamma)} - 1\right)(e^{\delta T} - 1),
\]
\[
T^* = \frac{1}{\delta} \ln \left(1 + \frac{\theta \delta(\delta + \gamma)}{ke^{-\delta r}(b + \mu \gamma e^{-\delta \omega} - \delta(\delta + \gamma))}\right),
\]
\[
\tau^* = \frac{1}{\delta} \ln \frac{k(b + \mu \gamma e^{-\delta \omega})(1 - e^{-\delta T})}{\delta(1 - (1 - \theta)e^{-\delta T})(\delta + \gamma)}.\]

**Corollary 3.3** If \(\omega > \omega^*\), then the infection free periodic solution \((\tilde{S}_e(t), 0)\) of system (2.2) is globally attractive, where

\[
\omega^* = \frac{1}{\delta} \ln \frac{k\mu \gamma e^{-\delta r}(1 - e^{-\delta T})}{\delta(\delta + \gamma)(1 - (1 - \theta)e^{-\delta T}) - kbe^{-\delta r}(1 - e^{-\delta T})}.\]

### 4 Permanence

In this section we say the disease is endemic if the infectious population persists above a certain threshold level for sufficiently large time. Before starting our theorem, we give the following definitions and lemma.

**Definition 4.1** System (3) is said to be uniformly persistent if there is an \(\eta > 0\)(independent of the initial conditions) such that every solution \(S(t), I(t)\) with initial conditions (4) of system (3) satisfies

\[
\liminf_{t \to \infty} S(t) \geq \eta, \quad \liminf_{t \to \infty} I(t) \geq \eta.
\]

**Definition 4.2** System (3) is said to be permanent if there exists a compact region \(\Omega_0 \in \Omega\) such that every solution with initial conditions (4) of system (3) will eventually enter and remain the region \(\Omega_0\).

Denote two quantities

\[
R_* = \frac{kbe^{-\delta r}(1 - \theta)(1 - e^{-\delta T})}{\delta(\delta + \gamma)(1 + \alpha \mu h)(1 - (1 - \theta)e^{-\delta T})},
\]
\[
I^* = \frac{\delta}{k}(R_* - 1) = \frac{\delta}{k} \left(\frac{kbe^{-\delta r}(1 - \theta)(1 - e^{-\delta T})}{\delta(\delta + \gamma)(1 + \alpha \mu h)(1 - (1 - \theta)e^{-\delta T})} - 1\right).
\]

**Lemma 4.1** If \(R_* > 1\), then for any \(t_0 > 0\), it is impossible that \(I(t) < I^*\) for
all $t \geq t_0$.

**Proof.** Suppose the contrary. Then there is a $t_0 > 0$ such that $I(t) < I^*$ for all $t \geq t_0$. According to any positive solution $(S(t), I(t))$ of system (3), we define

$$V(t) = I(t) + ke^{-\delta \tau} \int_{t-\tau}^{t} \frac{I(u)S(u)}{1 + \alpha I^b(u)} du$$  \hspace{1cm} (16)

The second equation of system (3) can be rewritten as

$$I'(t) = \left(\frac{ke^{-\delta \tau} S(t)}{1 + \alpha I^b(t)} - (\delta + \gamma)\right)I(t) - ke^{-\delta \tau} \frac{d}{dt} \int_{t-\tau}^{t} \frac{I(u)S(u)}{1 + \alpha I^b(u)} du.$$  

We get the derivative of $V$ along the solution of system (3)

$$V'(t) = (\delta + \gamma)I(t)\left(\frac{ke^{-\delta \tau} S(t)}{(1 + \alpha I^b(t))(\delta + \gamma)} - 1\right)$$  \hspace{1cm} (17)

From the first equation of system (3) we have that

$$S'(t) > b - \delta S(t) - kI^*S(t) = b - (\delta + kI^*)S(t), \quad t \geq t_0.$$  

Then we establish the following auxiliary system for $t \geq t_0$:

$$\left\{ \begin{array}{ll}
  w'(t) = b - (\delta + kI^*)w(t), & t \neq nT, \\
  w(t^+) = (1 - \theta)w(t), & t = nT.
\end{array} \right.$$  \hspace{1cm} (18)

By Lemma 2.2, we derive that for $nT < t \leq (n+1)T$,

$$\bar{w}_e(t) = \frac{b}{\delta + kI^*} + \left(\frac{b}{\delta + kI^*} \cdot \frac{(1 - \theta)(1 - e^{-(\delta + kI^*)T})}{1 - (1 - \theta)e^{-(\delta + kI^*)T}} - \frac{b}{\delta + kI^*}\right)e^{-(\delta + kI^*)(t-nT)}$$

is globally asymptotically stable. In view of the comparison theorem of impulsive differential equation, we obtain that there exists an integer $t_1(t_1 > t_0 + \omega)$ and sufficiently small $\epsilon > 0$ such that

$$S(t) > \bar{w}_e(t) - \epsilon, \quad t \geq t_1.$$  

Since

$$\bar{w}_e(t) > \frac{b}{\delta + kI^*} \cdot \frac{(1 - \theta)(1 - e^{-(\delta + kI^*)T})}{1 - (1 - \theta)e^{-(\delta + kI^*)T}},$$

then we get

$$S(t) > \frac{b}{\delta + kI^*} \cdot \frac{(1 - \theta)(1 - e^{-(\delta + kI^*)T})}{1 - (1 - \theta)e^{-(\delta + kI^*)T}} - \epsilon \triangleq \zeta, \quad t \geq t_1.$$  \hspace{1cm} (19)

From(17) and (19), we have

$$V'(t) > (\delta + \gamma)I(t)\left(\frac{ke^{-\delta \tau} \zeta}{(1 + \alpha I^b)(\delta + \gamma)} - 1\right), \quad t \geq t_1.$$  \hspace{1cm} (20)
Since $R_s < 1$, it is easy to get $I^* > 0$. Moreover, there exists sufficiently small $\epsilon > 0$ such that
\[
\frac{ke^{-\delta \tau} \zeta}{(1 + \alpha \mu^b)(\delta + \gamma)} > 1.
\] (21)
From the above results, we obtain that
\[
V'(t) > (\delta + \gamma)I(t)(\frac{ke^{-\delta \tau} \zeta}{(1 + \alpha \mu^b)(\delta + \gamma)} - 1) > 0, \quad t \geq t_1.
\] (22)
Define $m = \min_{t \in [t_1, t_1 + \tau]} I(t)$. Next we want to prove that $I(t) \geq m$ for all $t \geq t_1$. Suppose that this claim is not valid. Then there exists $T^*$ such that $I(t) \geq m$ for $t_1 \leq t \leq t_1 + \tau + T^*$, $I(t_1 + \tau + T^*) = m$ and $I'(t_1 + \tau + T^*) \leq 0$. We can derive from the second equation of system (3) and (19) that
\[
I'(t_1 + \tau + T^*) = \frac{ke^{-\delta \tau}I(t_1 + \tau + T^* - \tau)S(t_1 + \tau + T^* - \tau)}{1 + \alpha h(t_1 + \tau + T^* - \tau)} - (\delta + \gamma)I(t_1 + \tau + T^*) > \left(\frac{ke^{-\delta \tau} \zeta}{(1 + \alpha \mu^b)(\delta + \gamma)} - 1\right)(\delta + \gamma)m.
\]
From (21), we get $I'(t_1 + \tau + T^*) > 0$. This contradicts $I'(t_1 + \tau + T^*) \leq 0$. Therefore, $I(t) \geq m$ for all $t \geq t_1$. From (22), we have
\[
V'(t) > (\delta + \gamma)\left(\frac{ke^{-\delta \tau} \zeta}{(1 + \alpha \mu^b)(\delta + \gamma)} - 1\right)m > 0, \quad t \geq t_1,
\]
which yields that as $t \to \infty$, $V(t) \to \infty$. However, from (16) we obtain that $V(t) \leq \frac{b}{\gamma} + ke^{-\delta \tau} \tau \mu^2$. This is a contradiction. Therefore, for any $t_0 > 0$, it is impossible that $I(t) < I^*$ for all $t \geq t_0$. The proof of Lemma 4.1 is complete.

**Theorem 4.1** If $R_s > 1$, there exists a positive constant $p$ such that any positive solution of system (3) satisfies $I(t) \geq p$, for $t$ large enough.

**Proof.** Based on the result of Lemma 4.1, we will discuss the following two cases:

1. $I(t) \geq I^*$, for $t$ large enough; 2. $I(t)$ oscillates about $I^*$, for $t$ large enough.

Obviously, we only need to consider the second case. We assume that there exists sufficiently large $t_1^*$ and $t_2^*$ such that
\[
I(t_1^*) = I(t_2^*) = I^*, \quad I(t) \leq I^*, \quad t \in [t_1^*, t_2^*], \quad S(t) > \zeta, \quad t \in [t_1^*, t_2^*].
\]
Since the positive solutions of (3) are ultimately bounded and $I(t)$ is not effected by impulses, $I(t)$ is uniformly continuous. Therefore there exists a
positive constant $\beta$ such that $I(t) > \frac{L}{2} t^* \text{ for } t \in [t_1^*, t_1^* + \beta]$, where $\beta$ satisfies $\beta < \tau$ and is independent of the choice of $t_1^*$ and $t_2^*$. Set $p = \min\{\frac{L}{2}, p^*\}$, where $p^* = I^* e^{-(\delta+\gamma)r}$. Now we discuss the following three subcases.

Case I. If $t_2^* - t_1^* \leq \beta$, then it is evident that $I(t) \geq p$ for $t$ large enough.

Case II. If $\beta < t_2^* - t_1^* \leq \tau$, then from the second equation of system (2.2) we have $I'(t) > - (\delta + \gamma) I(t)$. Since $I(t^*_2) = I^*$, by the comparison theorem we obtain that

$$I(t) > I^* e^{-(\delta+\gamma)(t-t^*)} = I^* e^{-(\delta+\gamma)\tau} = p^*, \quad t \in [t_1^*, t_2^*].$$

Considering $I(t) > \frac{L}{2} t^*$, for $t \in [t_1^*, t_1^* + \beta]$, we obtain $I(t) \geq p$, for $t \in [t_1^*, t_2^*]$.

Case III. If $t_2^* - t_1^* > \tau$, then by the same analysis of subcase (2), we have $I(t) \geq p$, for $t \in [t_1^*, t_1^* + \tau]$. Next, we will prove that $I(t) \geq p$ is still valid for $t \in [t_1^* + \tau, t_2^*]$. Suppose the contrary. Then there exists $\tilde{T} \geq 0$ such that $I(t) \geq p$ for $t \in [t_1^*, t_1^* + \tau + \tilde{T}]$, $I(t_1^* + \tau + \tilde{T}) = p$ and $I'(t_1^* + \tau + \tilde{T}) \leq 0$. We can derive from the second equation of system (3) that

$$I'(t_1^* + \tau + \tilde{T}) = \frac{ke^{-\delta \tau} I(t_1^* + \tau + \tilde{T} - \tau) S(t_1^* + \tau + \tilde{T} - \tau)}{1 + \alpha I^h(t_1^* + \tau + \tilde{T} - \tau)}$$

$$- (\delta + \gamma) I(t_1^* + \tau + \tilde{T})$$

$$> \left(\frac{ke^{-\delta \tau} e^{\gamma \delta}}{(1 + \alpha(\frac{2}{\beta})^h)(\delta + \gamma)} - 1\right) (\delta + \gamma) p > 0.$$

This contradicts $I'(t_1^* + \tau + \tilde{T}) \leq 0$. Therefore, $I(t) \geq p$ for $t \in [t_1^*, t_2^*]$. Moreover, in view of Lemma 4.1 we obtain that for any $t_0 > 0$, it is impossible that $I(t) < I^*$ for all $t \geq t_0$. Hence, if $R_0 > 1$, any positive solution of system (3) satisfies $I(t) \geq p$, for $t$ large enough. This completes the proof.

**Theorem 4.2.** If $R_0 > 1$, system (3) is permanent.

**Proof.** Let $(S(t), I(t))$ be any solution of system (3). We derive from the first equation of system (3) that

$$S'(t) \geq b - (\delta + k) S(t).$$

Then we establish the following comparison system

$$\begin{cases}
  x'(t) = b - (\delta + k) x(t) & t \neq nT, \\
  x(t^+) = (1 - \theta) x(t) & t = nT.
\end{cases} \quad (23)$$

By the similar analysis in the proof of Theorem 3.1, we obtain that $\lim_{t \to \infty} S(t) \geq M$, where

$$M = \frac{b}{\delta + k} \left(1 - \theta(1 - \theta e^{-(\delta+k)T})ight) - \epsilon_1$$

($\epsilon_1$ is sufficiently small.)
Define \( \Omega_0 = \{(S,I)|S \geq M, I \geq p, S+I \leq \mu\} \). From the above discuss and Theorem 4.1, we know that \( \Omega_0 \) is global attractor in \( \Omega \) and \( \Omega_0 \) is a positively invariant set. Therefore, system (3) is permanent.

**Corollary 4.1** Assume that \( \theta < \theta^* \) or \( \tau < \tau^* \), then system (3) is permanent, where

\[
\theta^* = 1 - \frac{\bar{\delta}(\bar{\delta} + \gamma)(1 + \alpha \mu^h)}{kb e^{-\bar{\delta}T}(1 - e^{-\delta T}) + \delta(\bar{\delta} + \gamma)(1 + \alpha(a^h)e^{-T})},
\]
\[
\tau^* = \frac{1}{\delta} \ln \frac{\bar{\delta}(\bar{\delta} + \gamma)(1 + \alpha \mu^h)(1 - (1 - \theta)e^{-\delta T})}{kb(1 - \theta)(1 - e^{-\delta T})}.
\]

5 Discussion

In this paper, we have studied the delayed SEIRS epidemic model with pulse vaccination and nonlinear incidence. We obtained two thresholds \( R^* \) and \( R_s \) (see Theorem 3.1 and 4.1) and further discussed if \( R^* < 1 \) then the disease will be extinct, if \( R_s > 1 \) then the disease will be permanent which means that after some period of time the disease will become endemic. Corollary 3.2 implies that the disease will disappear if the pulse vaccination rate is larger than \( \theta^* \) or the length of latent period exceeds \( \tau^* \) or the length of pulse vaccination period is smaller than \( T^* \). From Corollary 3.3, we obtain that if the length of immune period is larger than \( \omega^* \), the disease will will lead to the extinction. And Corollary 4.1 show that the disease will be uniformly persistent if the pulse vaccination rate is smaller than \( \theta_s \) or the length of latent period is smaller than \( \tau_s \).

We have discussed two cases: (1)\( R^* < 1 \), (2)\( R_s > 1 \). When \( R_s \leq 1 \leq R^* \), the dynamical behavior of system (3) has not been clear. These works will be left as our future consideration.

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**References**


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