

On the Fractional Mixed Fractional Brownian Motion

Yu Miao

College of Mathematics and Information Science
Henan Normal University, 453007 Henan, China
yumiao728@yahoo.com.cn

Weiyin Ren

Xin Xiang University, 453000 Henan, China

Zongxiu Ren

College of Mathematics and Information Science
Henan Normal University, 453007 Henan, China

Abstract

In this paper, we present some stochastic properties and characteristics of the fractional mixed fractional Brownian motion, and we study the α -differentiability of its sample paths.

Mathematics Subject Classification: 60J65

Keywords: Fractional mixed fractional Brownian motion; α -differentiability; Hölder continuity

1 Introduction

Let $W^H = \{W_t^H, t \geq 0\}$ be a fractional Brownian motion with Hurst index H defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is a continuous Gaussian process with the following properties:

(I) $W_0^H = 0$, \mathbb{P} -almost surely;

(II) $\mathbb{E}W_t^H = 0$, $\mathbb{E}W_t^H W_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |s - t|^{2H})$ for all $s, t \geq 0$;

- (III) the increments of W^H are stationary and self-similar with order H , and the trajectories of W^H are almost surely continuous and not differentiable.

Note the standard Brownian motion W is a fractional Brownian motion with Hurst index $H = 1/2$. Let us take a and b as two real constants such that $(a, b) \neq (0, 0)$.

Definition 1.1. A fractional mixed fractional Brownian motion (FMFBM) of parameters a , b , and $H = (H_1, H_2)$ is a process $Z^H = \{Z_t^H(a, b); t \geq 0\} = \{Z_t^H; t \geq 0\}$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\forall t \in \mathbb{R}_+, \quad Z_t^H = Z_t^H(a, b) = aW_t^{H_1} + bW_t^{H_2}, \quad (1.1)$$

where $(W_t^{H_1})_{t \in \mathbb{R}_+}$ and $(W_t^{H_2})_{t \in \mathbb{R}_+}$ are independent fractional Brownian motions of Hurst parameter H_1 and H_2 .

If $H_1 = 1/2$, the FMFBM become the mixed fractional Brownian motion (MFBM), which is discussed in Zili [9]. The MFBM process has been introduced by Cheridito [2] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. This model is the process

$$X_t^H(a, b) = X_0^H(a, b) \exp\left(\nu t + \sigma Z_t^H(a, b)\right), \quad (1.2)$$

where ν, σ are constants, a is a strictly positive constant, $b = 1$, and $Z^H(a, b)$ is a MFBM of parameters a, b , and H .

That is why many authors have proposed and studied a fractional version of the Samuelson model, which is the particular process $(X_t^{1/2}(0, 1))_{t \in [0, 1]}$, able to account for the possibility of long-run non-periodic statistical dependence in stock price returns. This fractional model is the stochastic process $(X_t^H(0, 1))_{t \in [0, 1]}$ with $H \in (1/2; 1)$ (see Cutland et al. [3]). But this model has also some deficiencies; for example, in 2001, Cheridito [2] has shown that such model admits arbitrage. And we recall that intuitively, the existence of an arbitrage is a sign of lack of equilibrium in the market: no real market equilibrium can exist in the long run if there are arbitrages there (see [7]).

On account of the possibility of long run non periodic statistical dependence in stock price returns, it is necessary to study the properties of FMFBM. In the present paper, we will obtain some stochastic general properties of the fractional mixed fractional Brownian motion and treat the Hölder continuity of the sample paths and α -differentiability of the trajectories.

2 The elementary properties

It is easy to check the following properties by the definition of FMFBM.

Lemma 2.1. *The FMFBM $(Z_t^H(a, b))_{t \in \mathbb{R}_+}$ satisfies the following properties:*

- (i) Z^H is a centered Gaussian process;
- (ii) for any $t \in \mathbb{R}_+$, $\mathbb{E}((Z_t^H(a, b))^2) = a^2 t^{2H_1} + b^2 t^{2H_2}$;
- (iii) for any $s, t \in \mathbb{R}_+$, one has that

$$\text{Cov}(Z_t^H(a, b), Z_s^H(a, b)) \tag{2.1}$$

$$= \frac{1}{2} \{ a^2(t^{2H_1} + s^{2H_1} - |s - t|^{2H_1}) + b^2(t^{2H_2} + s^{2H_2} - |s - t|^{2H_2}) \}$$

- (iv) the increments of the FMFBM are stationary.

Let $(X_t)_{t \in \mathbb{R}_+}$ and $(Y_t)_{t \in \mathbb{R}_+}$ be two processes defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The notation $\{X_t\} \triangleq \{Y_t\}$ will mean that $(X_t)_{t \in \mathbb{R}_+}$ and $(Y_t)_{t \in \mathbb{R}_+}$ have the same law.

Lemma 2.2. *FMFBM satisfies the property of mixed-self-similar, that is to say, for any $h > 0$, $\{Z_{ht}^H(a, b)\} \triangleq \{Z_t^H(ah^{H_1}, bh^{H_2})\}$.*

Proof. Since Z^H is a centered Gaussian process then we only has to prove that $\{Z_{ht}^H(a, b)\}$ and $\{Z_t^H(ah^{H_1}, bh^{H_2})\}$ have the same covariance function. It is obvious that

$$\text{Cov}(Z_{th}^H(a, b), Z_{sh}^H(a, b)) = \mathbb{E}(Z_{th}^H(a, b)Z_{sh}^H(a, b)) \tag{2.2}$$

$$\begin{aligned} &= a^2 \mathbb{E}(W_{th}^{H_1} W_{sh}^{H_1}) + ab [\mathbb{E}(W_{th}^{H_1} W_{sh}^{H_2}) + \mathbb{E}(W_{th}^{H_2} W_{sh}^{H_1})] + b^2 \mathbb{E}(W_{th}^{H_2} W_{sh}^{H_2}) \\ &= \frac{1}{2} \left[(ah^{H_1})^2 (t^{2H_1} + s^{2H_1} - |s - t|^{2H_1}) + (bh^{H_2})^2 (t^{2H_2} + s^{2H_2} - |s - t|^{2H_2}) \right] \\ &= \text{Cov}(Z_t^H(ah^{H_1}, bh^{H_2}), Z_s^H(ah^{H_1}, bh^{H_2})). \end{aligned}$$

That is our desired result. □

Theorem 2.3. *For any given $(a, b) \in \mathbb{R}^2$, $(a, b) \neq (0, 0)$, for every $H = (H_1, H_2) \in (0, 1)^2$, $(Z_t^H(a, b))_{t \in \mathbb{R}_+}$ is not a Markov process unless $(H_1, H_2) = (1/2, 1/2)$.*

Proof. By Lemma 2.1, we know that Z^H is a centered Gaussian process and for any $t \in \mathbb{R}_+$,

$$Cov(Z_t^H(a, b), Z_t^H(a, b)) = \mathbb{E}((Z_t^H(a, b))^2) = a^2 t^{2H_1} + b^2 t^{2H_2}. \tag{2.3}$$

According to Revuz and Yor [8], if Z^H is a Markov process then for any $s < t < u$, we could have

$$Cov(Z_s^H, Z_u^H)Cov(Z_t^H, Z_t^H) = Cov(Z_s^H, Z_t^H)Cov(Z_t^H, Z_u^H).$$

Let us consider the particular case: $s = 1/2, t = 1, u = 3/2$, we will have

$$Cov(Z_{1/2}^H, Z_{3/2}^H)Cov(Z_1^H, Z_1^H) = Cov(Z_{1/2}^H, Z_1^H)Cov(Z_1^H, Z_{3/2}^H). \tag{2.4}$$

From (2.3) and (2.1), (2.4) is equivalent to the following equation

$$\begin{aligned} & \frac{1}{2} \left[a^2 \left(1 + \left(\frac{3}{2}\right)^{2H_1} - \left(\frac{1}{2}\right)^{2H_1} \right) + b^2 \left(1 + \left(\frac{3}{2}\right)^{2H_2} - \left(\frac{1}{2}\right)^{2H_2} \right) \right] \\ &= \left[a^2 \left(\left(\frac{1}{2}\right)^{2H_1} + \left(\frac{3}{2}\right)^{2H_1} - 1 \right) + b^2 \left(\left(\frac{1}{2}\right)^{2H_2} + \left(\frac{3}{2}\right)^{2H_2} - 1 \right) \right] \\ \iff & \frac{1}{2} \left(1 + \left(\frac{3}{2}\right)^{2H_1} - \left(\frac{1}{2}\right)^{2H_1} \right) = \left(\frac{1}{2}\right)^{2H_1} + \left(\frac{3}{2}\right)^{2H_1} - 1 \\ & \text{and } \frac{1}{2} \left(1 + \left(\frac{3}{2}\right)^{2H_2} - \left(\frac{1}{2}\right)^{2H_2} \right) = \left(\frac{1}{2}\right)^{2H_2} + \left(\frac{3}{2}\right)^{2H_2} - 1 \\ \iff & 3 + 3^{2H_1} - 3 \cdot 2^{2H_1} = 0 \quad \text{and} \quad 3 + 3^{2H_2} - 3 \cdot 2^{2H_2} = 0. \end{aligned}$$

However it is easy to check that for all $(H_1, H_2) \neq (1/2, 1/2)$,

$$3 + 3^{2H_1} - 3 \cdot 2^{2H_1} \neq 0 \quad \text{and} \quad 3 + 3^{2H_2} - 3 \cdot 2^{2H_2} \neq 0.$$

Thus we have our result. □

3 Correlation between the increments

Let X and Y be two random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote the correlation coefficient $\rho(X, Y)$ by

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}.$$

From Lemma 2.1, we have the following result

Lemma 3.1. For any $s, t, h \in \mathbb{R}_+, 0 < h \leq t - s$, we have

$$\rho(Z_{t+h}^H - Z_t^H, Z_{s+h}^H - Z_s^H) = \frac{a^2U(H_1) + b^2U(H_2)}{2(a^2h^{2H_1} + b^2h^{2H_2})}$$

where

$$U(H_i) = (t + s - h)^{2H_i} - 2(t - s)^{2H_i} + (t - s - h)^{2H_i}, \quad i = 1, 2.$$

Corollary 3.2. For any given $(a, b) \in \mathbb{R}^2, (a, b) \neq (0, 0)$, the increments of $(Z_t^H(a, b))_{t \in \mathbb{R}_+}$ are positive correlated if $1/2 < H_1, H_2 < 1$, and negative correlated if $0 < H_1, H_2 < 1/2$.

Proof. Note the fact that $\forall x \in \mathbb{R}_+, \forall h > 0$,

$$(x + h)^{2H} - 2x^{2H} + (x - h)^{2H} > 0 (< 0), \text{ if } H > 1/2 (H < 1/2).$$

Then from Lemma 3.1, we have

$$\rho(Z_{t+h}^H - Z_t^H, Z_{s+h}^H - Z_s^H) > 0 (= 0, < 0), \text{ if } H_1, H_2 > 1/2 (H_1, H_2 = 1/2, < 1/2).$$

□

Remarks 3.3. (1) If $H_1, H_2 > 1/2$ (resp. $H_1, H_2 < 1/2$), if $a \neq 0, b_1$ and b_2 are two real constants such that $|b_1| \leq |b_2|$ ($|b_1| \geq |b_2|$), then for any $s, t, h \in \mathbb{R}_+, 0 < h \leq t - s$, it is easy to see that

$$\rho(Z_{t+h}^H(a, b_1) - Z_t^H(a, b_1), Z_{s+h}^H(a, b_1) - Z_s^H(a, b_1)) \tag{3.1}$$

$$\leq \rho(Z_{t+h}^H(a, b_2) - Z_t^H(a, b_2), Z_{s+h}^H(a, b_2) - Z_s^H(a, b_2)).$$

Then, if $H_1, H_2 > 1/2$ (resp. $H_1, H_2 < 1/2$),

(1.1.) the smaller (larger) $|b|$ is, the less correlated the increments of Z^H are;

(1.2.) the larger (smaller) $|b|$ is, the more correlated the increments of Z^H are.

(2) As the same reason, if $H_1, H_2 > 1/2$ (resp. $H_1, H_2 < 1/2$),

(2.1.) the smaller (larger) $|a|$ is, the less correlated the increments of Z^H are;

(2.2.) the larger (smaller) $|a|$ is, the more correlated the increments of Z^H are.

- (3) If $H_1 = 1/2$, $H_2 > 1/2$ (resp. $H_2 < 1/2$), $b \neq 0$,
- (3.1.) the smaller (larger) $|b|$ is, the less correlated the increments of Z^H are;
- (3.2.) the larger (smaller) $|b|$ is, the more correlated the increments of Z^H are.
- (4) If $H_1 = 1/2$, $H_2 > 1/2$ (resp. $H_2 < 1/2$), $b \neq 0$,
- (4.1.) the smaller (larger) $|a|$ is, the more correlated the increments of Z^H are;
- (4.2.) the larger (smaller) $|a|$ is, the less correlated the increments of Z^H are.
- (5) For the case $H_1 < 1/2, H_2 > 1/2$ or $H_1 > 1/2, H_2 < 1/2$, the correlation of the increments is not obviously. However, in the modeling of a certain phenomenon, we can choose H_1, H_2, a , and b suitably in such a manner that $\{Z_t^H(a, b)\}$ permits to obtain a good model, taking the sign and the level of correlation between the increments of this phenomenon into account.

Definition 3.1. Let $\{X_t, t \in \mathbb{R}_+\}$ be a process with stationary trajectories and $(r(n))_{n \in \mathbb{N}^*}$ the sequence defined by

$$r(n) = \mathbb{E}(X_{n+1}X_1), \quad \forall n \in \mathbb{N}^*.$$

The process X is called long-range dependent if and only if

$$\sum_{n \in \mathbb{N}^*} r(n) = +\infty.$$

Note since $\{X_t, t \in \mathbb{R}_+\}$ is a process with stationary trajectories, then

$$r(n) = \mathbb{E}(X_{n+s}X_s), \quad \forall s \in \mathbb{R}_+, \quad \forall n \in \mathbb{N}^*.$$

Lemma 3.4. (1) For all $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R} \setminus \{0\}$, the increments of $(Z_t^H(a, b))_{t \in \mathbb{R}_+}$ are long-range dependent if and only if $H_1 > 1/2$ or $H_2 > 1/2$.

(2) For $a = 0$ and $b \in \mathbb{R} \setminus \{0\}$, the increments of $(Z_t^H(a, b))_{t \in \mathbb{R}_+}$ are long-range dependent if and only if $H_2 > 1/2$.

Proof. For all $n \in \mathbb{N}^*$, by Taylor formula, we have

$$r(n) = \mathbb{E}((Z_{n+1}^H - Z_n^H)Z_1^H) \tag{3.2}$$

$$\begin{aligned}
 &= \frac{1}{2} \left(a^2 [(n+1)^{2H_1} - 2n^{2H_1} + (n-1)^{2H_1}] + b^2 [(n+1)^{2H_2} - 2n^{2H_2} + (n-1)^{2H_2}] \right) \\
 &= a^2 H_1(2H_1 - 1)n^{2H_1-2} + b^2 H_2(2H_2 - 1)n^{2H_2-2} + n^{2H_1-2}\gamma_1(n) + n^{2H_2-2}\gamma_2(n)
 \end{aligned}$$

where $\lim_{n \rightarrow \infty} \gamma_1(n) = 0$ and $\lim_{n \rightarrow \infty} \gamma_2(n) = 0$.

It is easy to see that $\sum_{n \in \mathbb{N}^*} r(n) = \infty$ if and only if $2H_1 - 2 > -1$ or $2H_2 - 2 > -1$, i.e., $H_1 > 1/2$ or $H_2 > 1/2$.

From the above proof, (2) is obvious. □

4 Hölder continuity

Lemma 4.1. *For all $T > 0$ and $\gamma < H_1 \wedge H_2$, the FMFBM has a modification which sample paths have a Hölder continuity with order γ , on the interval $[0, T]$.*

Proof. According to Kolmogorov’s theorem of regularity (see Revuz and Yor [8]), it suffices to prove that $\forall \alpha > 0, \exists C_\alpha, \forall (s, t) \in [0, T]^2$,

$$\mathbb{E}(|Z_t^H - Z_s^H|^\alpha) \leq C_\alpha |t - s|^{\alpha(H_1 \wedge H_2)}. \tag{4.1}$$

Without loss of generality, let $\alpha > 0$ and $s, t \in [0, T], s < t$. Using the stationarity and the mixed-self-similarity of increments of Z^H , we have

$$\mathbb{E}(|Z_t^H - Z_s^H|^\alpha) = \mathbb{E}(|Z_{t-s}^H|^\alpha) = \mathbb{E}(|Z_1^H(a(t-s)^{H_1}, b(t-s)^{H_2})|^\alpha).$$

(Case 1: $H_1 \leq H_2$) There are two positive constants C_1 and C_2 , depending on α , such that

$$\begin{aligned}
 \mathbb{E}(|Z_t^H - Z_s^H|^\alpha) &\leq (t-s)^{\alpha H_1} \mathbb{E}(|Z_1^H(a, b(t-s)^{H_2-H_1})|^\alpha) \\
 &\leq (t-s)^{\alpha H_1} [C_1 |a|^\alpha \mathbb{E}(|W_1^{H_1}|^\alpha) + C_2 |b|^\alpha (t-s)^{\alpha(H_2-H_1)} \mathbb{E}(|W_1^{H_2}|^\alpha)] \\
 &\leq C_\alpha (t-s)^{\alpha H_1}
 \end{aligned}$$

where

$$C_\alpha = C_1 |a|^\alpha \mathbb{E}(|W_1^{H_1}|^\alpha) + C_2 |b|^\alpha T^{\alpha(H_2-H_1)} \mathbb{E}(|W_1^{H_2}|^\alpha).$$

(Case 2: $H_1 > H_2$) There are two positive constants C'_1 and C'_2 , depending on α , such that

$$\begin{aligned}
 \mathbb{E}(|Z_t^H - Z_s^H|^\alpha) &\leq (t-s)^{\alpha H_2} \mathbb{E}(|Z_1^H(a(t-s)^{H_1-H_2}, b)|^\alpha) \\
 &\leq (t-s)^{\alpha H_2} \left[C'_1 |a|^\alpha (t-s)^{\alpha(H_1-H_2)} \mathbb{E}(|W_1^{H_1}|^\alpha) + C'_2 |b|^\alpha \mathbb{E}(|W_1^{H_2}|^\alpha) \right] \\
 &\leq C_\alpha (t-s)^{\alpha H_2}
 \end{aligned}$$

where

$$C_\alpha = C'_1 |a|^\alpha T^{\alpha(H_1-H_2)} \mathbb{E}(|W_1^{H_1}|^\alpha) + C'_2 |b|^\alpha \mathbb{E}(|W_1^{H_2}|^\alpha).$$

□

5 The α -differentiability of the FMFBM

The following notions have been studied by BenAdda and Cresson [1].

Definition 5.1. Let f be a continuous function on $[a, b]$, and let $\alpha \in (0, 1)$. Call a right (resp., left) local fractional α -derivative of f at $t_0 \in [a, b]$ the following quantity:

$$d_\sigma^\alpha f(t_0) = \Gamma(1 + \alpha) \lim_{t \rightarrow t_0^\sigma} \frac{\sigma(f(t) - f(t_0))}{|t - t_0|^\alpha}$$

for $\sigma = +$ (resp., $\sigma = -$), where Γ is the Euler function.

Definition 5.2. Let f be a continuous function on $[a, b]$, and let $\alpha \in (0, 1)$. The function f is α -derivative at $t_0 \in [a, b]$ if and only if $d_+^\alpha f(t_0)$ and $d_-^\alpha f(t_0)$ exist and are equal. In this case, denote by $d^\alpha f(t_0)$ the α -derivative of f at t_0 .

From the previous definition, we obtain the notion of α -velocity introduced by Cherbit [2].

Theorem 5.1. For all $\alpha \in (0, H_1 \wedge H_2)$, the sample paths of the FMFBM are almost surely α -differentiable at every $t_0 \geq 0$, and

$$\forall t_0 \geq 0, \quad \mathbb{P}(d^\alpha Z_{t_0}^H = 0) = 1.$$

Proof. Here we only give the proof for the case $\sigma = +$ and the proof for $\sigma = -$ is similar. Using the stationary and the mixed-self-similarity of increments of Z^H , we have for $0 \leq t_0 < t$,

$$\begin{aligned} \frac{Z_t^H - Z_{t_0}^H}{(t - t_0)^\alpha} &\triangleq \frac{Z_{t-t_0}^H}{(t - t_0)^\alpha} \triangleq \frac{Z_1^H(a(t - t_0)^{H_1} + b(t - t_0)^{H_2})}{(t - t_0)^\alpha} \\ &\triangleq a(t - t_0)^{H_1 - \alpha} W_1^{H_1} + b(t - t_0)^{H_2 - \alpha} W_1^{H_2}. \end{aligned}$$

Therefore, if $\alpha \in (0, H_1 \wedge H_2)$,

$$\begin{aligned} \mathbb{P}(d_+^\alpha Z_{t_0}^H = 0) &= \mathbb{P}\left(\lim_{t \rightarrow t_0} \frac{Z_t^H - Z_{t_0}^H}{(t - t_0)^\alpha} = 0\right) \\ &= \mathbb{P}\left(\lim_{t \rightarrow t_0} a(t - t_0)^{H_1 - \alpha} W_1^{H_1} + b(t - t_0)^{H_2 - \alpha} W_1^{H_2} = 0\right) = 1. \end{aligned}$$

□

Theorem 5.2. For all $\alpha \in (H_1 \wedge H_2, 1)$, the sample paths of the FMFBM are nowhere α -differentiable, almost surely.

Proof. For any $r > 0$, we define the events

$$A(t) = \left\{ \sup_{0 \leq s \leq t} \left| \frac{Z_s^H(a, b)}{s^\alpha} \right| > r \right\}.$$

For any decreasing sequence $t_n \rightarrow 0$, we have $A(t_{n+1}) \subset A(t_n)$, thus

$$\mathbb{P}(\lim_{n \rightarrow \infty} A(t_n)) = \lim_{n \rightarrow \infty} \mathbb{P}(A(t_n)),$$

and by using the mixed-self-similarity of Z^H ,

$$\mathbb{P}(A(t_n)) \geq \mathbb{P}\left(\left|\frac{Z_{t_n}^H(a, b)}{t_n^\alpha}\right| > r\right) = \mathbb{P}(|at_n^{H_1-\alpha}W_1^{H_1} + bt_n^{H_2-\alpha}W_1^{H_2}| > r).$$

(i) If $H_1 < H_2$, (i.e., $\alpha > H_1$)

$$\mathbb{P}(A(t_n)) \geq \mathbb{P}(|aW_1^{H_1} + bt_n^{H_2-H_1}W_1^{H_2}| > rt_n^{\alpha-H_1})$$

then

$$\lim_{n \rightarrow \infty} \mathbb{P}(|aW_1^{H_1} + bt_n^{H_2-H_1}W_1^{H_2}| > rt_n^{\alpha-H_1}) = \mathbb{P}(|aW_1^{H_1}| \geq 0) = 1.$$

(ii) If $H_1 = H_2$, (i.e., $\alpha > H_1 = H_2$)

$$\mathbb{P}(A(t_n)) \geq \mathbb{P}(|aW_1^{H_1} + bW_1^{H_2}| > rt_n^{\alpha-H_1})$$

then

$$\lim_{n \rightarrow \infty} \mathbb{P}(|aW_1^{H_1} + bW_1^{H_2}| > rt_n^{\alpha-H_1}) = \mathbb{P}(|aW_1^{H_1} + bW_1^{H_2}| \geq 0) = 1.$$

(iii) If $H_1 > H_2$, (i.e., $\alpha > H_2$)

$$\mathbb{P}(A(t_n)) \geq \mathbb{P}(|at_n^{H_1-H_2}W_1^{H_1} + bW_1^{H_2}| > rt_n^{\alpha-H_2})$$

then

$$\lim_{n \rightarrow \infty} \mathbb{P}(|at_n^{H_1-H_2}W_1^{H_1} + bW_1^{H_2}| > rt_n^{\alpha-H_2}) = \mathbb{P}(|bW_1^{H_2}| \geq 0) = 1.$$

From the above discussion we see that for all $\alpha \in (H_1 \wedge H_2, 1)$, for all $t_0 \geq 0$,

$$\mathbb{P}\left(\limsup_{t \rightarrow t_0^+} \left| \frac{Z_t^H - Z_{t_0}^H}{(t - t_0)^\alpha} \right| = +\infty\right) = 1,$$

and this is our desired result. □

References

- [1] F. BenAdda and J. Cresson, *About non-differentiable functions*. Journal of Mathematical Analysis and Applications, (2001), **263(2)**: 721-737.
- [2] P. Cheridito, *Mixed fractional Brownian motion*. Bernoulli, (2001), **7(6)**: 913-934.
- [3] N.J. Cutland, P.E. Kopp and W. Willinger, *Stock price returns and the Joseph effect: a fractional version of the Black-Scholes model*. Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1993), Progress in Probability, vol. 36, Birkhäuser, Basel, 1995, pp. 327-351.
- [4] C. El-Nouty, *The fractional mixed fractional Brownian motion*. Statistics & Probability Letters, (2003), **65(2)**: 111-120.
- [5] I. Norros, E. Valkeila and J. Virtamo, *An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions*. Bernoulli, (1999), **5(4)**: 571-587.
- [6] A. Novikov and E. Valkeila, *On some maximal inequalities for fractional Brownian motions*. Statistics & Probability Letters, (1999), **44(1)**: 47-54.
- [7] B. Øksendal, *Stochastic Differential Equations. An Introduction with Applications*. 6th ed., Universitext, Springer, Berlin, 2003.
- [8] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer, Berlin, 1991.
- [9] M. Zili, *On the mixed fractional Brownian motion*. Journal of Applied Mathematics and Stochastic Analysis, (2006), Article ID 32435, Pages 1-9.

Received: December 30, 2007