Eventual Stability of Impulsive Differential System

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Abstract

The notion of eventual stability has been recently discussed. We extend this notion to impulsive differential equations. Our technique depends on Liapunov's direct method.

Keywords: Eventual stability; Uniform eventual stability; asymptotic eventual stability; Impulsive systems

1. INTRODUCTION

In recent years the mathematical theory of impulsive differential equations has been developed by a large number of mathematicians, see e.g. Bainov and Simenov [1,2], Lakshmikantham et al. [5], and Somolienko and Perestyuk [7]. Furthermore these systems are adequate mathematical models for numerous processes and phenomena studied in biology, physics, technology, etc.

The main purpose of this paper is to extend the notion of eventual stability for differential equations with impulsive effect, which has been discussed in [6] for ordinary differential equations without impulse. The motivation of this work is the recent work of Kulev and Bainov [3]. The paper is organized as follows.

In section 2, we introduce some preliminary, definitions and results which will be used throughout the paper. In section 3, we extend the notion of eventual stability for impulsive differential equations.

2. PRELIMINARIES

Let $R^s_H$ be the s-dimensional Euclidean space with a suitable norm $\|\|$. Let $R^+ = [0, \infty), R^+_H = \{x \in R^+ : \|x\| < H\}$.
Consider the system of differential equation with impulses

\[\begin{align*}
x'(t) &= f(t, x(t), y(t), z(t)) + g(t, y(t), z(t)) + h(t, z(t)), \quad t \neq \tau_i(x, y, z) \\
y'(t) &= \phi(t, x(t), y(t), z(t)) + p(t, y(t), z(t)) + q(t, x(t), z(t)), \quad t \neq \tau_i(x, y, z) \\
z'(t) &= \psi(t, x(t), y(t), z(t)), \quad t \neq \tau_i(x, y, z) \\
\Delta x &= A_i(x) + B_i(y) + C_i(z), \quad t = \tau_i(x, y, z) \\
\Delta y &= D_{i_1}(x, y) + D_{i_2}(y, z) + D_{i_3}(x, z), \quad t = \tau_i(x, y, z) \\
\Delta z &= E_i(x, y, z), \quad t = \tau_i(x, y, z)
\end{align*}\]

\[\text{--------- (2.1)}\]

Where

\[x \in R^n, \quad y \in R^m, \quad z \in R^l\]

\[f : R^n \times R^m \rightarrow R^n, \quad g : R^n \times R^m \rightarrow R^n, \quad h : R^n \times R^l \rightarrow R^n\]

\[\phi : R^n \times R^m \times R^l \rightarrow R^n, \quad p : R^n \times R^m \times R^l \rightarrow R^n, \quad q : R^n \times R^m \times R^l \rightarrow R^n\]

\[\psi : R^n \times R^m \times R^l \rightarrow R^l, \quad A_i : R^m \rightarrow R^n, \quad B_i : R^m \rightarrow R^n, \quad C_i : R^l \rightarrow R^n\]

\[D_{i_1} : R_n \times R^m \rightarrow R^n \times R^n, \quad D_{i_2} : R^m \times R^l \rightarrow R^m \times R^l, \quad D_{i_3} : R^m \times R^l \rightarrow R^m \times R^l\]

\[E_i : R^m \times R^l \rightarrow R^l, \quad \tau_i : R_n \times R^m \rightarrow R^l\]

\[\Delta x = x(t) - x(t^-), \quad \Delta y = y(t) - y(t^-), \quad \Delta z = z(t) - z(t^-)\]

Let \(t_0 \in R^+, x_0 \in R^m, y_0 \in R^m, z_0 \in R^l\)

Let \((x(t_0, x_0, y_0, z_0), y(t_0, x_0, y_0, z_0), z(t_0, x_0, y_0, z_0))\) be the solution of system (2.1), satisfying the initial conditions

\[x(t_0, x_0, y_0, z_0) = x_0\]

\[y(t_0, x_0, y_0, z_0) = y_0\]

\[z(t_0, x_0, y_0, z_0) = z_0\]

The solution \((x(t), y(t), z(t))\) of the system (2.1) are piecewise continuous functions with points of discontinuity of the first type in which they are left continuous i.e. at the moment \(t_i\) when the integral curve of the solution \((x(t), y(t), z(t))\) meet the hypersurface

\[\sigma_i = \{(x, y, z) \in R^n \times R^m \times R^l : t = \tau_i(x, y, z)\}\]

The following relations are satisfied

\[x(t_0) = x(t_0^-), x(t) = A_i(x(t)), y(t) = B_i(y(t)), z(t) = C_i(z(t))\]

\[y(t_0^-) = y(t_0), y(t) = D_{i_1}(x(t), y(t)), y(t) = D_{i_2}(y(t), z(t)), z(t) = D_{i_3}(x(t), z(t))\]

\[z(t_0^-) = z(t_0), z(t) = E(x(t), y(t), z(t))\]
Together with system (2.1), we consider the following system with impulses
\begin{align*}
x' &= f(t,x) + g(t,y), t \neq \tau_i(x,y,0) \\
y' &= \phi(t,x,y), t \neq \tau_i(x,y,0) \\
\Delta x &= A_i(x) + B_i(y), t = \tau_i(x,y,0) \\
\Delta y &= D_i(x,y), t = \tau_i(x,y,0)
\end{align*}

Let \( S_i = \{(t,x,y) \in \mathbb{R}^+ \times R_m^n \times R_m^n : t = \tau_i(x,y,0)\} \) \hspace{1cm} \text{(2.2)}

Together with system (2.2), we consider the following system with impulses
\begin{align*}
x' &= f(t,x), t \neq \tau_i(x,0,0) \\
\Delta x &= A_i(x), t = \tau_i(x,0,0)
\end{align*}

Let \( \rho_i = \{(t,x) \in \mathbb{R}^+ \times R_m^n : t = \tau_i(x,0,0)\} \) \hspace{1cm} \text{(2.3)}

**Definition 1**: A function \( b(r) \) is said to belong to the class \( K \) if 
\[
b(0) = 0 \quad \text{and} \quad b(r) \text{ is strictly monotone increasing in } r.
\]

Let \( \tau_0(x,y,z) = 0 \) for \( (x,y,z) \in R_m^n \times R_m^n \times R_m^l \)

We define the sets 
\[
G_i = \{(t,x,y,z) \in \mathbb{R}^+ \times R_m^n \times R_m^n \times R_m^l : \tau_{i-1}(x,y,z) < t < \tau_i(x,y,z)\}
\]
\[
\Omega_i = \{(t,x,y) \in \mathbb{R}^+ \times R_m^n \times R_m^n : \tau_{i-1}(x,y,0) < t < \tau_i(x,y,0)\}
\]
\[
J_i = \{(t,x) \in \mathbb{R}^+ \times R_m^n : \tau_{i-1}(x,0,0) < t < \tau_i(x,0,0)\}
\]

**Definition 2**: We say that function \( V : \mathbb{R}^+ \times R_m^n \times R_m^n \times R_m^l \rightarrow \mathbb{R} \) belongs to the class \( V_0 \) if the following conditions hold

1. The function \( V \) is continuous in \( \bigcup_{i=1}^{\infty} G_i \) and is locally lipschitzian with respect to \( x,y \) and \( z \) in each of the sets \( G_i \).
2. \( V(t,0,0,0) = 0 \) for \( t \in \mathbb{R}^+ \)
3. For each \( i=1,2,3,... \) and for any point \( (t_0, x_0, y_0, z_0) \in \sigma_i \), there exist the finite limits 
\[
V(t_0 - 0, x_0, y_0, z_0) = \lim_{(t,x,y,z) \rightarrow (t_0,x_0,y_0,z_0)} V(t,x,y,z)
\]
\[
(t,x,y,z) \in G_i
\]
\[
V(t_0 + 0, x_0, y_0, z_0) = \lim_{(t,x,y,z) \rightarrow (t_0,x_0,y_0,z_0)} V(t,x,y,z)
\]
\[
(t,x,y,z) \in G_{i+1}
\]

And the equality \( V(t_0 - 0, x_0, y_0, z_0) = V(t_0, x_0, y_0, z_0) \) holds.
4. For any point \((t,x,y,z) \in \sigma_t\), the following inequality holds.
\[
V(t_0 + x + A_t(x) + B_t(y) + C_t(z), y + D_t(x, y) + D_t(y, z) + D_t(z, x, z) + E_t(x, y, z)) \leq V(t, x, y, z)
\]

\[\text{(2.4)}\]

**Definition 3:** We say that the \(W : R^+ \times R^n_+ \times R^m_+ \to R\) belongs to class \(W_0\) if the following conditions hold.

1. The function \(W\) is continuous in \(\bigcup_{i=1}^{\infty} \Omega_i\) and is locally lipschitzian with respect to \(x\) and \(y\) in each of the sets \(\Omega_i\).
2. \(W(t,0,0) = 0\) for all \(t \in R^+\)
3. There exists the finite limits
   \[
   W(t_0 - 0, x_0, y_0) = \lim_{(t,x,y) \to (t_0, x_0, y_0)} W(t, x, y)
   \]
   \(W(t_0 + 0, x_0, y_0) = \lim_{(t,x,y) \to (t_0, x_0, y_0)} W(t, x, y)
   \)
   \(\Omega_i\)

And the equality \(W(t_0 - 0, x_0, y_0) = W(t_0, x_0, y_0)\) holds.
4. For any point \((t,x,y) \in S_t\), the following inequality holds
   \[
   W(t_0 + 0, x + A_t(x) + B_t(y), y + D_t(x, y)) \leq W(t, x, y)
   \]
   \[\text{(2.5)}\]

**Definition 4:** We say that \(U : R^+ \times R^n_+ \to R\) belongs to the class \(U_0\) if the following conditions hold.

1. The function \(U\) is continuous in \(\bigcup_{i=1}^{\infty} J_i\) and is locally lipschitz with respect to \(x\) in each of the sets \(J_i\).
2. \(U(t,0) = 0\) for all \(t \in R^+\)
3. There exists the finite limits
   \[
   U(t_0 - 0, x_0) = \lim_{t \to x} U(t, x)
   \]
Impulsive differential system

\[(t, x) \rightarrow (t_0, x_0)\]  
\[(t, x) \in J_i\]

\[U(t_0 + 0, x_0) = \lim_{t \to t_0} U(t, x)\]

\[(t, x) \rightarrow (t_0, x_0)\]  
\[(t, x) \in J_{i+1}\]

And the equality \[U(t_0 - 0, x_0) = U(t_0, x_0)\] holds.

4. For any point \((t, x) \in \rho_i\) the following inequality holds

\[U(t_0 - 0, x + A_i(x)) \leq U(t, x) \quad \text{(2.6)}\]

Let \(V \in V_0\) and \((x(t), y(t), z(t))\) be solution of (2.1) for \((t, x, y, z) \in \bigcup_{i=1}^{\infty} G_i\), we define

\[V_{2,1}(t, x, y, z) = \lim_{s \to 0} \frac{1}{s} [V(t + s, x + s\{f(t, x) + g(t, y) + h(t, z)\}, y + s\{\phi(t, x, y)

\[+ p(t, y, z) + q(t, x, z)\}, z + s\phi(t, x, y, z)) - V(t, x, y, z)]\]

And \(V_{2,1}(t, x, y, z) = D^+ V(t, x, y, z), t \neq \tau_i(x, y, z)\)

Where \(D^+ V(t, x, y, z)\) is upper dini derivative of the function \(V(t, x, y, z)\)

Similarly

\[W_{2,2}(t, x, y) = \lim_{s \to 0} \frac{1}{s} [W(t + s, x + s\{f(t, x) + g(t, y)\}, y + s\phi(t, x, y)) - W(t, x, y)]\]

and \(U_{2,3}(t, x) = \lim_{s \to 0} \frac{1}{s} [U(t + s, x + s\phi(t, x)) - U(t, x)]\)

**Definition 5**:- The zero solution of system (2.1) is said to be eventually stable if for all \(\varepsilon > 0\) for all \(t_0 \in R^+\), there exist \(\tau_0 > 0\) and \(\delta = \delta(t_0, \varepsilon) > 0\) for all \((x_0, y_0, z_0) \in R^+_H \times R^+_{H} \times R^+_H\) such that \(\|x_0 + y_0 + z_0\| < \delta\) implies

\[\|x(t, t_0, x_0, y_0, z_0) + y(t, t_0, x_0, y_0, z_0) + z(t, t_0, x_0, y_0, z_0)\| < \varepsilon, \quad t \geq t_0 \geq \tau_0\]
Definition 6: we say condition (A) hold if the following conditions are satisfied:

(A1). The function \( f(t, x), g(t, y), h(t, z), \phi(t, x, y), p(t, y, z), q(t, x, z), \varphi(t, x, y, z) \) are continuous in their definition domains \( f(t, 0) = 0, g(t, 0) = 0, h(t, 0) = 0, \phi(t, 0, 0) = 0, p(t, 0, 0) = 0, q(t, 0, 0, 0) = 0 \) and \( \varphi(t, 0, 0, 0) = 0 \) for \( t \in R^+ \)

(A2). There exist a constant \( M_1 > 0, M_2 > 0, M_3 > 0, M_4 > 0 \) such that
\[
\phi(t, x, y) \leq M_1, p(t, y, z) \leq M_2, q(t, x, z) \leq M_3, \varphi(t, x, y, z) \leq M_4
\]

\((t, x, y) \in R^+ \times R_n^+ \times R_n^+, (t, y, z) \in R^+ \times R_n^+ \times R_n^+, (t, x, z) \in R^+ \times R_n^+ \times R_n^+, (t, x, y, z) \in R^+ \times R_n^+ \times R_n^+ \times R_n^+ \)

(A3). There exists a continuous function \( P : I \to I \) such that \( P(0) = 0 \) and \( \| g(t, y) \| \leq P \| y \| \) for \( (t, y) \in R^+ \times R_n^+ \)

(A4). The functions \( A_i, B_i, C_i, D_{i_1}, D_{i_2}, D_{i_3}, E_i \) are continuous in their definition domains and \( A_i(0) = B_i(0) = C_i(0) = D_{i_1}(0, 0, 0) = D_{i_2}(0, 0, 0) = D_{i_3}(0, 0, 0) = E_i(0, 0, 0, 0) = 0 \)

(A5). If \( x \in R_n^+, y \in R_n^+, z \in R_n^+ \) then
\[
\| x + A_i(x) + B_i(y) + C_i(z) \| \leq \| x \|
\]
\[
\| y + D_{i_1}(x, y) + D_{i_2}(y, z) + D_{i_3}(x, z) \| \leq \| y \|
\]
\[
\| z + E_i(x, y, z) \| \leq \| z \|
\]

(A6). The functions \( \tau_i(x, y, z) \) are continuous and for \( (x, y, z) \in R_n^+ \times R_n^+ \times R_n^+ \) the following relations hold
\[
0 < \tau_1(x, y, z) < \tau_2(x, y, z) < \ldots < \lim_{i \to \infty} \tau_i(x, y, z) = \infty
\]

uniformly in \( R_n^+ \times R_n^+ \times R_n^+ \)

\[
\inf \tau_i(x, y, z) - \sup \tau_i(x, y, z) \geq \theta > 0
\]

\( R_n^+ \times R_n^+ \times R_n^+ \) \( R_n^+ \times R_n^+ \times R_n^+ \) \( i = 1, 2, 3, \ldots \)
Impulsive differential system

(A7). For each point \((t_0, x_0, y_0, z_0) \in R^+ \times R^+_H \times R^m_H \times R^I_H\) the solution \(x(t, t_0, x_0), y(t, t_0, x_0, y_0), z(t, t_0, x_0, y_0, z_0)\) of the system (2.1) is unique and defined in \((t_0, \infty)\)

(A8). For each point \((t_0, x_0, y_0) \in R^+ \times R^+_H \times R^m_H\), the solution \(x(t, t_0, x_0), y(t, t_0, x_0, y_0)\) of the system (2.2) is unique and defined in \((t_0, \infty)\)

(A9). For each point \((t_0, x_0) \in R^+ \times R^m_H\), the solution of the system (2.3) satisfying \(x(t_0 + 0, t_0, x_0) = x_0\) is unique and exists for all \(t \in (t_0, \infty)\)

(A10). The integral curve of each solution of system (2.1) meets each of the hypersurface \(\{\sigma_i\}\) at most once

3. MAIN RESULTS

Theorem 1:- Assume that

(H1). Condition (A) holds.

(H2). There exist functions \(V \in V_0, a \in K\) such that
\[
\|x + y + z\| \leq \|V(t, x, y, z)\|, (t, x, y, z) \in R^+ \times R^+_H \times R^m_H \times R^I_H
\]

(H3). \(V_{2,1}(t, x, y, z) \leq 0\) for \((t, x, y, z) \in \bigcup_{i=1}^{\infty} G_i\)

Then the zero solution of the system (2.1) is eventually stable.

Proof:- Let \(0 < \epsilon < H\) and \(t_0 \in R^+\). Assume that \(t_0 \leq \tau_i(x, y, z)\) for \((x, y, z) \in R^+_H \times R^m_H \times R^I_H\).

Since \(V(t, 0, 0, 0) = 0\) and from definition 2 it follows that there exists a \(\delta = \delta(t_0, \epsilon) > 0\) Thus it follows that \(\|x_0 + y_0 + z_0\| < \delta\) implies
\[
\|V(t_0 + 0, x_0, y_0, z_0)\| < a(\epsilon)
\]

Let \(x_0 \in R^+_H, y_0 \in R^m_H, z_0 \in R^I_H, \|x_0 + y_0 + z_0\| < \delta\). Let
\[
x(t) = x(t, t_0, x_0, y_0, z_0), y(t) = y(t, t_0, x_0, y_0, z_0), z(t) = z(t, t_0, x_0, y_0, z_0)
\]
be solution
of (2.1). From (2.4) and (H3) it follows that the function \( V(t, x, y, z) \) is monotone decreasing in \([t_0, \infty)\). Then by (H2) we get
\[
a\|x(t) + y(t) + z(t)\| < V(t, x, y, z) \leq V(t_0 + 0, x_0, y_0, z_0) < a(\varepsilon), \quad t \geq t_0 \geq t_0 \quad \text{for} \ t \in (t_0, \infty)
\]
Therefore \( \|x(t) + y(t) + z(t)\| < \varepsilon \). Hence the zero solution of system (2.1) is eventually stable.

**Theorem 2:** Let all conditions of theorem 1 be satisfied except condition (H2) being replaced by

(H4). \( a\|x + y + z\| \leq V(t, x, y, z) \leq b\|x + y\|, \quad a, b \in K \)

Then the zero solution of system (2.1) is uniformly eventually stable.

**Proof:** Since condition (H4) implies condition (H2), it follows from theorem 1 that the zero solution of the system (2.1) is eventually stable.

Thus for \( \varepsilon > 0 \) let \( \delta = b^{-}[a(\varepsilon)] \) be independent of \( t_0 \) for \( a, b \in K \). Let \( x_0 \in R^n, y_0 \in R^m, z_0 \in R^j, \|x_0 + y_0 + z_0\| < \delta \).

Let \( x(t) = x(t, t_0, x_0, y_0, z_0), y(t) = y(t, t_0, x_0, y_0, z_0), z(t) = z(t, t_0, x_0, y_0, z_0) \) be a solution of system (2.1).

From (2.4) and (H3) it follows that the function \( V(t, x, y, z) \) is monotone decreasing in \([t_0, \infty)\).

Then by using (H4) we get
\[
a\|x(t) + y(t) + z(t)\| \leq V(t, x, y, z) \leq V(t_0 + 0, x_0, y_0, z_0) \leq b\|x_0 + y_0 + z_0\| < b(\delta) < a(\varepsilon)
\]

Thus \( \|x(t) + y(t) + z(t)\| < \varepsilon \) whenever \( \|x_0 + y_0 + z_0\| < \delta \) for \( t \geq t_0 \geq t_0 \).

Thus the zero solution of the system (2.1) is uniformly eventually stable.

**Theorem 3:** Suppose that assumptions of theorem 1 are satisfied except condition (H3) being replaced by the conditions

(H5). \( V_2^*(t, x, y, z) \leq -c\|y + z\|, (t, x, y, z) \in R^+ \times R^n \times R^m \times R^j \)
We further assume that

(H6). There exist functions \( U \in U_0 \) and \( a_i \in K \) such that

\[
a_i \|x\| \leq U(t, x), (t, x) \in R^+ \times R^+_n
\]

(H7). There exist functions \( U \in U_0 \) and \( c_i \in K \) such that

\[
U_{2,3}(t, x) \leq -c_i(U(t, x)), (t, x) \in \bigcup_{i=1}^{\infty} J_i
\]

(H8). \( \|U(t, x_1) - U(t, x_2)\| \leq \|x_1 - x_2\| \)

Then the zero solution of the system is asymptotically eventually stable.

**Proof:** Since by theorem 1, we conclude that the zero solution of the system (2.1) is eventually stable. Then we can choose a number \( \lambda = \lambda(t_0) > 0 \) such that if

\[
\|x_0 + y_0 + z_0\| < \lambda \text{ then } \|x + y + z\| < H, t \geq t_0
\]

Now to prove that the zero solution of system (2.1) is asymptotically eventually stable, we must show that

\[
\lim_{t \to \infty}\|y(t, t_0, x_0, y_0, z_0) + z(t, t_0, x_0, y_0, z_0)\| = 0
\]

Firstly, we show that

\[
\lim_{t \to \infty}\|y(t, t_0, x_0, y_0, z_0) + z(t, t_0, x_0, y_0, z_0)\| = 0
\]

Suppose that this is not true. Then for some \( \varepsilon_0 > 0 \) there exists a sequence \( \{\xi_r\} \), which tends to \( \infty \) for \( r \to \infty \) such that

\[
\|y(\xi_r) + z(\xi_r)\| = \varepsilon_0^r, r = 1, 2, 3, \ldots \ldots
\]

If \( t_i (i = 1, 2, 3, \ldots) \) are the moments when the integral curve of the solution \((x(t), y(t), z(t))\) meet the hypersurfaces \( \sigma_i \), then for \( t \neq t_i \) by \((A2)\), we obtain

\[
\left|\frac{d}{dt}\|y(t) + z(t)\|\right| \leq \left|\frac{d}{dt}\|y(t)\| + \frac{d}{dt}\|z(t)\|\right|
\]

\[
= \frac{d}{dt}\|y(t)\| + \frac{d}{dt}\|z(t)\|
\]

\[
\leq \|y'(t)\| + \|z'(t)\|
\]

\[
= \phi(t, x, y) + p(t, y, z) + q(t, x, z) + \varphi(t, x, y, z)
\]

\[
\leq M_1 + M_2 + M_3 + M_4 = M
\]

We shall prove that \( \|y(t) + z(t)\| \geq \frac{\varepsilon_0^r}{2} \) for \( t \in [\xi_r, \xi_{r+1}] \).
Let \( 0 \leq \varepsilon_r - t \leq \frac{\varepsilon_0}{2M} \). Integrating the above inequality from \( t \) to \( \xi_r \), we get

\[
\int_t^{\xi_r} d\tau \left\| y(t) + z(t) \right\| d\tau \leq M(\xi_r - t) \leq \frac{\varepsilon_0}{2}
\]

On the other hand, each interval \( I_r, r = 1, 2, 3, \ldots \) contains a finite number of points \( \{t_i\} \).

As in theorem 1 of [3], we assume that these points are \( t, t_{s+1}, \ldots, t_{s+p} \). Then by (A5), we obtain

\[
\int_t^{\xi_r} d\tau \left\| y(t) + z(t) \right\| d\tau = \int_t^{\xi_r} d\tau \left\| y(t) + z(t) \right\| d\tau + \sum_{i=1}^{s} \int_{t_i}^{t_{i+1}} d\tau \left\| y(t) + z(t) \right\| d\tau + \int_{t_s}^{t_{s+p}} d\tau \left\| y(t) + z(t) \right\| d\tau
\]

\[
\geq \left\| y(\xi_r) + z(\xi_r) \right\| - \left\| y(t) + z(t) \right\|
\]

Therefore

\[
\varepsilon_0 - \left\| y(t) + z(t) \right\| \leq \left\| y(\xi_r) + z(\xi_r) \right\| - \left\| y(t) + z(t) \right\| \leq \int_t^{\xi_r} d\tau \left\| y(\tau) + z(\tau) \right\| d\tau \leq \frac{\varepsilon_0}{2}
\]

Thus we conclude that

\[
\left\| y(t) + z(t) \right\| \geq \frac{\varepsilon_0}{2}
\]

If we choose a suitable subsequence of the sequence \( \{\varepsilon_r\} \) which we again denoted by \( \{\xi_r\} \). We can assume that the intervals \( I_r \) do not intersect one another and \( t_0 < \xi_r - \frac{\varepsilon_0}{2M} \). Then from (H5) we obtain

\[
V((t, x(t), y(t), z(t))) \leq -c\left\| y(t) + z(t) \right\| \leq -c \frac{\varepsilon_0}{2} \quad (1)
\]

In the interval \( I_r \) and

\[
V((t, x(t), y(t), z(t))) \leq 0
\]

For the remaining values of \( t \) for which \( (t, x(t), y(t), z(t)) \in \bigcup_{i=1}^{\infty} G_i \)

Integrating (2.1) and using (2.4), we obtain

\[
V(\varepsilon_r, x(\varepsilon_r), y(\varepsilon_r), z(\varepsilon_r)) \leq V(t_0 + 0, x_0, y_0, z_0) - c\left( \frac{\varepsilon_0}{M}(\varepsilon_0) \right) \to -\infty \quad \text{as} \quad r \to \infty
\]

which contradicts (H4)

Therefore

\[
\lim_{t \to \infty} \left\| y(t, t_0, x_0, y_0, z_0) + z(t, t_0, x_0, y_0, z_0) \right\| = 0 \quad (2)
\]

Secondly we prove that

\[
\lim_{t \to \infty} \left\| x(t, t_0, x_0, y_0, z_0) \right\| = 0
\]

To do this we shall prove that \( u(t) = U(t, x) \to 0 \) for \( t \to \infty \)

Using (H8), we get
Impulsive differential system

\[ U'_{22}(t,x) \leq U'_{23}(t,x) + d\|g(t,y)\| \]
\[ t \in R^+ , x \in R^n , y \in R^n , t \neq \tau_i(x,y,z), i = 1,2,3, \ldots \]
Thus by (H7) and (A3), we have
\[ U'_{22}(t,x(t)) \leq -c_1(U(t,x(t))) + dh\|y(t)\| , \quad t \neq \tau_i(x(t),y(t),z(t)) \]
(3)

We set \( \limsup_{t \to \infty} u(t) = \alpha \), \( \liminf_{t \to \infty} u(t) = \beta \). Then for the arbitrarily small number \( \mu > 0 \) we can find sequences \( q_n > p \to \infty \) for \( n \to \infty \) such that \( u(p_n) = \beta + \mu \) \( u(q_n) = \alpha - \mu \) and \( \beta + \mu < u(t) < \alpha - \mu \) for \( p < t < q_n \). Since the function \( P \) is continuous, \( P(0) = 0 \) and \( \lim_{t \to \infty} \|y(t)\| = 0 \). Thus there exists a positive integer \( \nu \) such that for \( n \geq \nu \) and \( t \geq p_n \) the following inequality holds:
\[ P(\|y(t)\|) \leq \frac{c_1(\beta + \mu)}{l} \]

Then from (3) we have
\[ U'_{22}(t,x(t)) \leq -c_1(\beta + \mu) + d\frac{c_1(\beta + \mu)}{l} = 0 \]
for \( n \geq \mu \) and \( t \in (p_n,q_n), t \neq \tau_i(x(t),y(t),z(t)) \) which together with (2.6) yields \( u(p_n) \geq u(q_n) \).
Hence \( \beta + \mu \geq \alpha - \mu \), which contradicts the assumption that \( \alpha > \beta \). This shows that there exists the limit
\[ U(t,x(t)) = \gamma \geq 0 \]
\[ t \to \infty \]

Now suppose that \( \gamma > 0 \). Then we can find a number \( T > 0 \) such that the following inequalities hold:
\[ \frac{\gamma}{2} \leq U(t,x(t)) \leq \frac{3\gamma}{2}, \]
\[ P(\|y(t)\|) \leq \frac{1}{2d}c_1(\frac{\gamma}{2}) \]
For all \( t \geq T \). Thus by (3), we obtain
\[ U'_{22}(t,x(t)) \leq -\frac{1}{2}c_1(\frac{\gamma}{2}) < 0 \]
For \( t \geq T \), hence using (2.6) and integrating we obtain
\[ U(t, x(t)) = U(T, x(T)) - \frac{1}{2} c_1 \left( \frac{y}{2} \right) (t - T) \rightarrow -\infty \quad \text{as} \quad t \rightarrow \infty \]

This contradicts (H6), therefore \[ \lim_{t \rightarrow \infty} \|x(t, t_0, x_0, y_0, z_0)\| = 0 \]

Hence the zero solution of the system (2.1) is asymptotically eventually stable.

References


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